Sufficient Conditions For A Group Of Homeomorphisms Of The Cantor Set To Be 2-Generated

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The Cantor Set, Cones, Open Sets, Clopen Sets

- By the Cantor set we mean the set of functions from $\mathbb{N}$ to $\{0,1\}$ which we will denote $\{0,1\}^{\mathbb{N}}$.
- If u is a partial function from $\mathbb{N}$ to $\{0,1\}$ with domain finite and closed downwards then we will use $\bar{u}$ for the set $\left\{f \in \{0,1\}^{\mathbb{N}} \;\middle|\; u \subseteq f\right\}$. We will call such $\bar{u}$ cones.
- The open subsets of $\{0,1\}^{\mathbb{N}}$ are the arbitrary unions of cones.
- The clopen subsets of $\{0,1\}^{\mathbb{N}}$ are the finite unions of cones.
- We will use H for the group of homeomorphisms of $\{0,1\}^{\mathbb{N}}$ that is the group bijections on $\{0,1\}^{\mathbb{N}}$ such that both the image and pre-image of each open set is also open.

Thompson's Group V

- V is a subgroup of H intruduced by R. Thompson circa 1965.
- We will define elements of V piecewise with the domains and ranges of pieces being cones.
- For cones $\bar{u}$ and $\bar{v}$ the only bijection $f : \bar{u} \to \bar{v}$ which is allowed to be a piece of an element of V is defined by $(uw)f := (vw)$ where here we are thinking of uw and vw as infinite strings instead of as functions.
- V is simple and 2-generated.

Important Properties

Definition

If G is a group of homeomorphisms of the Cantor set we will say G is *vigorous* if for any proper clopen subset A of $\{0,1\}^{\mathbb{N}}$ and any B, C non-empty proper clopen subsets of $\{0,1\}^{\mathbb{N}} \setminus A$ there exists $g \in \mathrm{pstab}_G(A)$ with $Bg \subseteq C$.

Definition

If G is a group of homeomorphisms of the Cantor set we will say G is *flawless* if the set

$$\{[a,b] \mid a, b \in \mathrm{pstab}_G(A) \text{ for some } A \text{ non-empty and clopen}\}$$

generates G.

Lemma

If G is a vigorous subgroup of H then G is flawless exactly if G is simple.

Statement Of Theorem

Theorem

If G is a vigorous simple subgroup of H and E is a finitely generated subgroup of G then there exists F a 2-generated subgroup of G containing E.

Corollary

If G is a finitely generated vigorous simple subgroup of H then G is 2-generated.

Examples

Definition

We will use K for the set of vigorous simple (or equivalently flawless) finitely generated (and therefore 2-generated) subgroups of H. Note that H acts on K by conjugation.

Example

Our first example is V though it has been known to be 2-generated for a while.

Lemma

If G, H are in K then $\langle G \cup H \rangle$ is also in K.

Lemma

If $G \in K$ and $g \in G$ and $h \in H$ and A is a non-empty clopen set with $A \cap \operatorname{supp}(g) = \emptyset$ and $\operatorname{supp}(h) \cap \operatorname{supp}(h^g) = \emptyset$ then $\langle G \cup \{[g, h]\} \rangle$ is in K.

Sketch Of Proof Of Theorem

- If $g, h \in G$ are such that $\mathrm{supp}(g) \cap \mathrm{supp}(h)$ is setwise stabilised by both g and h then $\mathrm{supp}([g, h]) \subseteq \mathrm{supp}(g) \cap \mathrm{supp}(h)$.
- We can find $(u_i)_i$ and $(v_i)_i$ lists over G and $(A_i)_i$ a list of non-empty clopen sets with all lists of length j and $u_i, v_i \in \mathrm{pstab}_G(A_i)$ for each i and with $E \leq \langle [u_1, v_1], \cdots, [u_j, v_j] \rangle$.
- For each $n \geq 2$ and X a non-empty proper clopen subset of $\{0, 1\}^{\mathbb{N}}$ we can construct a partition of $\{0, 1\}^{\mathbb{N}}$ into n bits with one of the bits being X and an element $\sigma \in G$ with σ nearly cyclically permuting the components of the partition.
- There exists $f : \mathbb{N} \to \mathbb{N}$ (independent of n) unbounded such that for any set of $(n)f$ commutators of elements of G with supports contained in $X \cup X\sigma$ there exists $\xi \in G$ such that that $\langle \sigma, \xi \rangle$ contains the set of $(n)f$ commutators.

Sketch Of A Proof Continued

- There exists a, b, c in G with the supports of a and b contained in $X \cup X\sigma$ and c in $\left\langle \left\{ [a,b]^{(\sigma^n)} \;\middle|\; n \in \mathbb{N} \right\} \right\rangle$ with $Xc \supseteq \{0,1\}^{\mathbb{N}} \setminus X$.
- We can choose σ and ξ such that for each $i \leq j$ there is t_i in $\langle \sigma, \xi \rangle$ with the supports of ${u_i}^{t_i}$ and ${v_i}^{t_i}$ contained in X.
- In fact we can choose σ and ξ such that for each $i \leq j$ there exists t_i in $\langle \sigma, \xi \rangle$ and $p, q \leq j$ with $\xi^{\sigma^p t_i}$ agrees with u_i on $\mathrm{supp}(u_i)$ and $\mathrm{supp}(\xi^{\sigma^q t_i})$ agrees with v_i on $\mathrm{supp}(v_i)$ and
$\left(\mathrm{supp}(\xi^{\sigma^p t_i}) \setminus \mathrm{supp}(u_i)\right) \cap \left(\mathrm{supp}(\xi^{\sigma^q t_i}) \setminus \mathrm{supp}(v_i)\right) = \varnothing$.

Question

- Do there exist finitely presented simple groups which are not 2-generated?
- By the classification of finite simple groups all finite simple groups are 2-generated so if there are finitely presented simple groups which are not 2-generated they must be infinite.
- One possible approach to proving that no such group exists would be to prove the main theorem holds even if $\{0,1\}^{\mathbb{N}}$ is replaced by another space from some nice class and show that all finitely presented simple groups act vigorously on some space from this class.
- The proof can be easily modified to work for the circle and we are checking if it works for arbitrary manifolds.