EVERY GROUP IS A MAXIMAL SUBGROUP OF A NATURALLY OCCURRING FREE IDEMPOTENT GENERATED SEMIGROUP

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ABSTRACT. The study of the free idempotent generated semigroup $\mathrm{IG}(E)$ over a biordered set E has recently received a deal of attention. Let G be a group, let $n \in \mathbb{N}$ with $n \geq 3$ and let E be the biordered set of idempotents of the wreath product $G \wr \mathcal{T}_n$. We show, in a transparent way, that for $e \in E$ lying in the minimal ideal of $G \wr \mathcal{T}_n$, the maximal subgroup of e in $\mathrm{IG}(E)$ is isomorphic to G.

It is known that $G \wr \mathcal{T}_n$ is the endomorphism monoid End $F_n(G)$ of the rank n free G-act $F_n(G)$. Our work is therefore analogous to that of Brittenham, Margolis and Meakin for rank 1 idempotents in full linear monoids. As a corollary we obtain the result of Gray and Ruškuc that *any* group can occur as a maximal subgroup of *some* free idempotent generated semigroup. Unlike their proof, ours involves a natural biordered set and very little machinery.

Dedicated with gratitude and affection to the memory of John Howie

1. INTRODUCTION

Let S be a semigroup and denote by $\langle E(S) \rangle$ the subsemigroup of S generated by the set of idempotents $E(S)$ of S. If $S = \langle E(S) \rangle$, then we say that S is *idempotent generated.* In a landmark paper, Howie [16] showed that every semigroup may be embedded into one that is idempotent generated, thus making transparent the importance of the role played by such semigroups. In the same article, Howie showed that the semigroup of non-bijective endomorphisms of a finite set to itself is idempotent generated. This latter theorem was quickly followed by a 'linearised' version due to J.A. Erdös [7], who proved that the multiplicative semigroup of singular square matrices over a field is idempotent generated. Fountain and Lewin [11] subsumed these results into the wider context of endomorphism monoids of independence algbras. Indeed, Howie's work can be extended in many further ways: see, for example, [10, 21]. We note here that sets and vector spaces over division rings are examples of independence algebras, as are free (left) G-acts over a group G.

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For any set of idempotents $E = E(S)$ there is a free object $\mathrm{IG}(E)$ in the category of semigroups that are generated by E, given by the presentation

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle,$$

where here $\overline{E} = \{\bar{e} : e \in E\}$.[1] We say that $\mathrm{IG}(E)$ is the *free idempotent generated semigroup over* E. The relations in the presentation for $\mathrm{IG}(E)$ correspond to taking *basic products* in E, that is, products between $e, f \in E$ where e and f are comparable under one of the quasi-orders $\leq_{\mathcal{L}}$ or $\leq_{\mathcal{R}}$ defined on S. In fact, E has an abstract characterisation as a *biordered set*, that is, a partial algebra equipped with two quasi-orders satisfying certain axioms. A celebrated result of Easdown [6] shows every biordered set E occurs as $E(S)$ for some semigroup S, hence we lose nothing by assuming that our set of idempotents is of the form $E(S)$ for a semigroup S.

The semigroup $\mathrm{IG}(E)$ has some pleasant properties. It follows from the definition that the natural map $\phi : \mathrm{IG}(E) \to S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E(S) \rangle$. Since any morphism preserves $\mathcal{L}$-classes and $\mathcal{R}$-classes, certainly so does ϕ. More remarkably, we have the following lemma, taken from [8, 20, 6, 2, 14].

Proposition 1.1. *Let $S, S', E = E(S)$, the free idempotent generated semigroup* $\mathrm{IG}(E)$ *and ϕ be as above, and let $e \in E$.*

(i) *The restriction of ϕ to the set of idempotents of* $\mathrm{IG}(E)$ *is a bijection onto E (and an isomorphism of biordered sets).*
(ii) *The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in* $\mathrm{IG}(E)$ *and the corresponding set in $\langle E(S) \rangle$.*
(iii) *The restriction of ϕ to $H_{\bar{e}}$ is a morphism onto H_e.*

Biordered sets were introduced by Nambooripad [20] in his seminal work on the structure of regular semigroups, as was the notion of free idempotent generated semigroups $\mathrm{IG}(E)$. A significant and longstanding conjecture (which nevertheless seems not to have appeared formally until 2002 [19]), purported that all maximal subgroups of $\mathrm{IG}(E)$ were free. This conjecture was disproved by Brittenham, Margolis and Meakin [1]. The result motivating our current paper (and its forerunner [13], which does not emphasise the use of wreath products) is the main theorem of [14], in which Gray and Ruškuc show that *any* group occurs as the maximal subgroup of some $\mathrm{IG}(E)$. Their proof involves machinery developed by Ruškuc to handle presentations of maximal subgroups, and, given a group G, a careful construction of E. A recent article [5] of Dolinka and Ruškuc, shows that any group occurs as $\mathrm{IG}(E)$ for some *band* E - again, the same machinery is used and the band must be delicately chosen.

The question remained of whether a group G occurs as a maximal subgroup of some $\mathrm{IG}(E)$ for a 'naturally occurring' E. The signs for this were positive, given recent work in [2] and [15] showing (respectively) that the multiplicative group of non-zero elements of any division ring Q occurs as a maximal subgroup of a rank 1 idempotent in $\mathrm{IG}(E)$,

[1] It is more usual to identify elements of E with those of $\overline{E}$, but it helps the clarity of our later arguments to make this distinction.

where E is the biordered set of idempotents of $M_n(Q)$ for $n \geq 3$, and that any symmetric group $\mathcal{S}_r$ occurs as a maximal subgroup of a rank r idempotent in IG(F), where F is the biordered set of idempotents of a full transformation monoid $\mathcal{T}_n$ for some $n \geq r+2$. Note that in both these cases, $H_{\bar{e}} \cong H_e$ for the idempotent in question.

As pointed out above, sets and vector spaces are examples of *independence algebras*, as is any rank n free (left) G-act $F_n(G)$. The endomorphism monoid $\operatorname{End} F_n(G)$ is known to be the wreath product $G \wr \mathcal{T}_n$ (see, for example, [18, Theorem 6.8]). Elements of $\operatorname{End} F_n(G)$ are endowed with rank (simply the size of a basis of the image) and the maximal subgroups of rank 1 idempotents are isomorphic to G. We elaborate on the structure of $\operatorname{End} F_n(G)$ in Section 2, but stress that much of what we write can be extracted from known results for independence algebras. Once these preliminaries are over, Section 3 demonstrates in a very direct manner (without appealing to presentations) that for $E = E(\operatorname{End} F_n(G))$ and $e \in \operatorname{End} F_n(G)$ with $n \geq 3$ and $\operatorname{rank} e = 1$ we have $H_{\bar{e}} \cong H_e$, thus showing that any group occurs as a maximal subgroup of some *natural* IG(E). However, although none of the technicalities involving presentations appear here explicitly, we nevertheless have made use of the essence of some of the arguments of [4, 14], and more particularly earlier observations from [3, 22] concerning sets of generators for subgroups.

2. Preliminaries: free G-acts, rank-1 $\mathcal{D}$-classes and singular squares

Let G be a group and let $F_n(G) = \bigcup_{i=1}^{n} Gx_i$ be a rank n free left G-act with $n \in \mathbb{N}, n \geq 3$. We recall that, as a set, $F_n(G)$ consists of the set of formal symbols $\{gx_i : g \in G, i \in [1,n]\}$, and we identify x_i with $1x_i$, where 1 is the identity of G; here $[1,n] = \{1,\ldots,n\}$. For any $g,h \in G$ and $1 \leq i,j \leq n$ we have that $gx_i = hx_j$ if and only if $g = h$ and $i = j$; the action of G is given by $g(hx_i) = (gh)x_i$. For our main result, it is enough to take $n = 3$, but for the sake of generality we proceed with arbitrary $n \geq 3$. Let End $F_n(G)$ denote the endomorphism monoid of $F_n(G)$ (with composition left-to-right). As pointed out above, End $F_n(G)$ is isomorphic to $G \wr \mathcal{T}_n$. Properties of the latter tend to be folklore and we prefer here to prove directly the requisite results. The image of $a \in \operatorname{End} F_n(G)$ being a (free) G-subact, we can define the *rank* of a to be the rank of $\operatorname{im} a$.

Since $F_n(G)$ is an independence algebra, a direct application of Corollary 4.6 [12] gives a useful characterization of Green's relations on End $F_n(G)$.

Lemma 2.1. [12] *For any $a,b \in \operatorname{End} F_n(G)$, we have the following:*
- (i) $\operatorname{im} a = \operatorname{im} b$ *if and only if* $a \,\mathcal{L}\, b$;
- (ii) $\ker a = \ker b$ *if and only if* $a \,\mathcal{R}\, b$;
- (iii) $\operatorname{rank} a = \operatorname{rank} b$ *if and only if* $a \,\mathcal{D}\, b$.

Elements $a,b \in \operatorname{End} F_n(G)$ depend only on their action on the free generators $\{x_i : i \in [1,n]\}$ and it is therefore convenient to write

$$a = \begin{pmatrix} x_1 & \ldots & x_n \\ a_1 x_{1\bar{a}} & \ldots & a_n x_{n\bar{a}} \end{pmatrix} \text{ and } b = \begin{pmatrix} x_1 & \ldots & x_n \\ b_1 x_{1\bar{b}} & \ldots & b_n x_{n\bar{b}} \end{pmatrix}.$$

Let

$$D = D_1 = \{a \in \operatorname{End} F_n(G) \mid \operatorname{rank} a = 1\}.$$

Clearly $a, b \in D$ if and only if $\overline{a}, \overline{b}$ are constant, and from Lemma 2.1 we have $a \,\mathcal{L}\, b$ if and only if $\operatorname{im} \overline{a} = \operatorname{im} \overline{b}$.

Lemma 2.2. *Let $a, b \in D$ be as above. Then $\ker a = \ker b$ if and only if $(a_1, \dots, a_n)g = (b_1, \dots, b_n)$ for some $g \in G$.*

Proof. Suppose $\ker a = \ker b$. For any $i, j \in [1, n]$ we have $(a_i^{-1}x_i)a = x_{i\overline{a}} = (a_j^{-1}x_j)a$ so that by assumption, $(a_i^{-1}x_i)b = (a_j^{-1}x_j)b$. Consequently, $a_i^{-1}b_i = a_j^{-1}b_j = g \in G$ and it follows that $(a_1, \dots, a_n)g = (b_1, \dots, b_n)$.

Conversely, if $g \in G$ exists as given then for any $u, v \in G$ and $i, j \in [1, n]$ we have

$$(ux_i)a = (vx_j)a \Leftrightarrow ua_i = va_j \Leftrightarrow ua_ig = va_jg \Leftrightarrow ub_i = vb_j \Leftrightarrow (ux_i)b = (vx_j)b.$$

□

We index the $\mathcal{L}$-classes in D by $J = [1, n]$, where the image of $a \in L_j$ is Gx_j, and we index the $\mathcal{R}$-classes of D by I, so that by Lemma 2.2, the set I is in bijective correspondence with G^{n-1}. From [12, Theorem 4.9] we have that D is a completely simple semigroup. We denote by H_{ij} the intersection of the $\mathcal{R}$-class indexed by i and the $\mathcal{L}$-class indexed by j, and by e_{ij} the identity of H_{ij}. For convenience we also suppose that $1 \in I$ and let

$$e_{11} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_1 \end{pmatrix}.$$

Clearly, for any given $i \in I, j \in J$ we have

$$e_{1j} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_j & \cdots & x_j \end{pmatrix} \text{ and } e_{i1} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & a_{2i}x_1 & \cdots & a_{ni}x_1 \end{pmatrix},$$

where $a_{2i}, \cdots, a_{ni} \in G$.

Lemma 2.3. *Every $\mathcal{H}$-class of D is isomorphic to G.*

Proof. By standard semigroup theory, we know that any two group $\mathcal{H}$-classes in the same $\mathcal{D}$-class are isomorphic, so we need only show that H_{11} is isomorphic to G. By Lemma 2.2 an element $a \in \operatorname{End} F_n(G)$ lies in H_{11} if and only if $a = a_g = \begin{pmatrix} x_1 & \cdots & x_n \\ gx_1 & \cdots & gx_1 \end{pmatrix}$, for some $g \in G$. Clearly $\psi : H_{11} \to G$ defined by $a_g\psi = g$ is an isomorphism. □

Since D is a completely simple semigroup, it is isomorphic to some Rees matrix semigroup $\mathcal{M} = \mathcal{M}(H_{11}; I, J; P)$, where $P = (p_{ji}) = (q_jr_i)$, and we can take $q_j = e_{1j} \in H_{1j}$ and $r_i = e_{i1} \in H_{i1}$. Since the q_j, r_i are chosen to be idempotents, it is clear that $p_{1i} = p_{j1} = e_{11}$ for all $i \in I, j \in J$.

Lemma 2.4. *For any $a_{g_2}, \dots, a_{g_n} \in H_{11}$, we can choose $k \in I$ such that the kth column of P is $(e_{11}, a_{g_2}, \dots, a_{g_n})^T$.*

Proof. Choose $k \in I$ such that $e_{k1} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & g_2x_1 & \dots & g_nx_1 \end{pmatrix}$ (note that if $g_2 = \dots = g_n = 1$ then $k = 1$). □

Let E be a biordered set; from [6] we can assume that $E = E(S)$ for some semigroup S. An *E-square* is a sequence (e, f, g, h, e) of elements of E with $e \mathcal{R} f \mathcal{L} g \mathcal{R} h \mathcal{L} e$. We draw such an E-square as $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$.

Lemma 2.5. *The elements of an E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ form a rectangular band (within S) if and only if one (equivalently, all) of the following four equalities holds: $eg = f$, $ge = h$, $fh = e$ or $hf = g$.*

Proof. The necessity is clear. To prove the sufficiency, without loss of generality, suppose that the equality $eg = f$ holds. We need to prove $ge = h$, $fh = e$ and $hf = g$. Notice that $gege = gfe = ge$, so ge is idempotent. But, as $f \in L_g \cap R_e$ [17, Proposition 2.3.7] gives that $ge \in R_g \cap L_e$, which implies $ge = h$. Furthermore, $fh = fge = fe = e$ and $hf = heg = hg = g$, and so $\{e, f, g, h\}$ is a rectangular band. □

We will be interested in rectangular bands in completely simple semigroups. The following lemma makes explicit ideas used implicitly elsewhere. We remark that the notation for idempotents used in the lemma fits exactly with that above.

Lemma 2.6. *Let $\mathcal{M} = \mathcal{M}(G; I, J; P)$ be a Rees matrix semigroup over a group G with sandwich matrix $P = (p_{ji})$. For any $i \in I, j \in J$ write e_{ij} for the idempotent (i, p_{ji}^{-1}, j). Then an E-square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix}$ is a rectangular band if and only if $p_{ji}^{-1}p_{jk} = p_{li}^{-1}p_{lk}$.*

Proof. We have

$$e_{ij}e_{kl} = e_{il} \Leftrightarrow (i, p_{ji}^{-1}, j)(k, p_{lk}^{-1}, l) = (i, p_{li}^{-1}, l) \Leftrightarrow p_{ji}^{-1}p_{jk}p_{lk}^{-1} = p_{li}^{-1} \Leftrightarrow p_{ji}^{-1}p_{jk} = p_{li}^{-1}p_{lk}.$$

The result now follows from Lemma 2.5. □

An E-square (e, f, g, h, e) is *singular* if, in addition, there exists $k \in E$ such that either:

$$\begin{cases} ek = e,\ fk = f,\ ke = h,\ kf = g \text{ or} \\ ke = e,\ kh = h,\ ek = f,\ hk = g. \end{cases}$$

We call a singular square for which the first condition holds an *up-down singular square*, and that satisfying the second condition a *left-right singular square.*

Lemma 2.7. *If an E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is singular, then $\{e, f, g, h\}$ is a rectangular band.*

Proof. Suppose that $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is singular. If $k = k^2 \in E$ is such that $ek = e$, $fk = f$, $ke = h$ and $kf = g$ then $eg = ekf = ef = f$. By Lemma 2.5, $\{e, f, g, h\}$ is a rectangular band. Dually for a left-right singular square. □

3. Free idempotent generated semigroups

Continuing the notation of the previous section, the rest of this paper is dedicated to proving that the maximal subgroup $H_{\overline{e_{11}}}$ is isomorphic to $H_{e_{11}}$ and hence by Lemma 2.3 to G. For ease of notation we denote $H_{\overline{e_{11}}}$ by $\overline{H}$ and $H_{e_{11}}$ by H.

As remarked earlier, although we do not directly use the presentations for maximal subgroups of semigroups developed in [3] and [22] and adjusted and implemented for free idempotent generated semigroups in [14], we are nevertheless making use of ideas from those papers. In fact, our work may be considered as a simplification of previous approaches, in particular [4], in the happy situation where a $\mathcal{D}$-class is completely simple, our sandwich matrix has the property of Lemma 2.4, and the next lemma holds.

Lemma 3.1. *An E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ in D is singular if and only $\{e, f, g, h\}$ is a rectangular band.*

Proof. One direction follows immediately from Lemma 2.7.

Suppose that $\{e, f, g, h\}$ is a rectangular band. If $e\,\mathcal{L}\,f$, then our E-square becomes $\begin{bmatrix} e & e \\ g & g \end{bmatrix}$ and taking $k = g$ we see this is an up-down singular square.

Without loss of generality we therefore suppose that $\{1\} = \operatorname{im}\overline{e} = \operatorname{im}\overline{h} \neq \operatorname{im}\overline{f} = \operatorname{im}\overline{g} = \{2\}$. Following standard notation we write $x_ie = e_ix_1$, $x_ih = h_ix_1$, $x_if = f_ix_2$ and $x_ig = g_ix_2$. As e, f, h and g are idempotents, it is clear that $e_1 = h_1 = 1$ and $f_2 = g_2 = 1$. Since $\{e, f, g, h\}$ is a rectangular band, we have $eg = f$ and so $x_1eg = x_1f$, that is, $g_1 = f_1$. Similarly, from $ge = h$, we have $e_2 = h_2$. Now we define $k \in \operatorname{End} F_n(G)$ by

$$x_ik = \begin{cases} x_1 & \text{if } i = 1; \\ x_2 & \text{if } i = 2; \\ g_ix_2 & \text{else.} \end{cases}$$

Clearly k is idempotent and since $\operatorname{im} e$ and $\operatorname{im} f$ are contained in $\operatorname{im} k$ we have $ek = e$ and $fk = f$. Next we prove that $ke = h$. Obviously, $x_1ke = x_1h$ and $x_2ke = x_2h$ from $e_2 = h_2$ obtained above. For other $i \in [1, n]$, we use the fact that from Lemma 2.2, there is an $s \in G$ with $h_i = g_is$, for all $i \in [1, n]$. Since

$$h_ie_2^{-1} = g_ish_2^{-1} = (g_is)(g_2s)^{-1} = g_ig_2^{-1} = g_i$$

we have $x_ike = (g_ix_2)e = g_ie_2x_1 = h_ix_1 = x_ih$ so that $ke = h$. It remains to show that $kf = g$. First, $x_1kf = f_1x_2 = g_1x_2 = x_1g$ and $x_2kf = x_2 = x_2g$. For other i, since $x_2f = x_2$ the definition of k gives $x_ikf = x_ig$. Hence $kf = g$ as required. Thus, by definition, $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is a singular square. □

Lemma 3.2. *For any idempotents $e, f, g \in D$, $ef = g$ implies $\overline{e}\,\overline{f} = \overline{g}$.*

Proof. Since D is completely simple, we have $e\,\mathcal{R}\,g\,\mathcal{L}\,f$ and since every $\mathcal{H}$-class in D contains an idempotent, there exists some $h^2 = h \in D$ such that $h \in L_e \cap R_f$. We therefore obtain an E-square $\begin{bmatrix} e & g \\ h & f \end{bmatrix}$, which by Lemma 2.5 is a rectangular band. From Lemma 3.1 it is a singular square, so that from Proposition 1.1 $\begin{bmatrix} \overline{e} & \overline{g} \\ \overline{h} & \overline{f} \end{bmatrix}$ is also a singular square. By Lemma 2.7, $\overline{e}\,\overline{f} = \overline{g}$. □

We now locate a set of generators for $\overline{H}$. Observe first that for any $i \in I$ and $j \in J$

$$(\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}})(\overline{e_{1j}}\ \overline{e_{i1}}) = \overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{1j}}\ \overline{e_{i1}} = \overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{i1}} = \overline{e_{11}}\ \overline{e_{i1}} = \overline{e_{11}}$$

so that $\overline{e_{1j}}\ \overline{e_{i1}}$ is the inverse of $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}}$ in $\overline{H}$; certainly then $e_{1j}e_{i1}$ is the inverse of $e_{11}e_{ij}e_{11}$ in H.

In view of Lemma 1 of [9], which itself uses the techniques of [8], the next result follows from [3, Theorem 2.1]. Note that the assumption that the set of generators in [3] is finite is not critical.

Lemma 3.3. *Every element in $\overline{H}$ is a product of elements of the form $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}}$ and $(\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}})^{-1}$, where $j \in J$ and $i \in I$.*

The next result is immediate from Lemma 3.2 and the observation preceding Lemma 3.3.

Lemma 3.4. *If $e_{1j}e_{i1} = e_{11}$, then $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}$.*

Lemma 3.5. *Let $i, l \in I$ and $j, k \in J$.*
(i) *If $e_{1j}e_{i1} = e_{1j}e_{l1}$, that is, $p_{ji} = p_{jl}$ in the sandwich matrix P, then $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{lj}}\ \overline{e_{11}}$.*
(ii) *If $e_{1j}e_{i1} = e_{1k}e_{i1}$, that is, $p_{ji} = p_{ki}$ in the sandwich matrix P, then $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{ik}}\ \overline{e_{11}}$.*

Proof. (i) Notice that $p_{1i}^{-1}p_{1l} = e_{11} = p_{ji}^{-1}p_{jl}$, so that from Lemma 2.6 we have that the elements of $\begin{bmatrix} e_{i1} & e_{ij} \\ e_{l1} & e_{lj} \end{bmatrix}$ form a rectangular band. Thus $e_{ij} = e_{i1}e_{lj}$ and so from Lemma 3.2 we have that $\overline{e_{ij}} = \overline{e_{i1}}\,\overline{e_{lj}}$. So, $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{i1}}\ \overline{e_{lj}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{lj}}\ \overline{e_{11}}$.

(ii) Here we have that $p_{ji}^{-1}p_{j1} = p_{ki}^{-1}p_{k1}$, so that $\begin{bmatrix} e_{ij} & e_{ik} \\ e_{1j} & e_{1k} \end{bmatrix}$ is a rectangular band and $\overline{e_{ij}} = \overline{e_{ik}}\ \overline{e_{1j}}$. So, $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{ik}}\ \overline{e_{1j}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{ik}}\ \overline{e_{11}}$. □

Lemma 3.6. *For any $i, i' \in I$, $j, j' \in J$, if $e_{1j}e_{i1} = e_{1j'}e_{i'1}$, then $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}\ \overline{e_{i'j'}}\ \overline{e_{11}}$.*

Proof. Let $a = e_{1j}e_{i1} = e_{1j'}e_{i'1}$. By Lemma 2.4 we can choose a $k \in I$ such that the kth column of P is $(e_{11}, a, \ldots, a)$. Then $p_{ji} = p_{jk}$ and $p_{j'k} = p_{j'i'}$ (this is true even if j or j' is 1) and our hypothesis now gives that $p_{jk} = p_{j'k}$. The result now follows from three applications of Lemma 3.5. □

In view of Lemma 3.6, we can define $w_a = \overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}}$ where $a = e_{11}e_{ij}e_{11} = (e_{1j}e_{i1})^{-1}$. Of course, $a = a_g$ for some $g \in G$, and from Lemma 2.4, w_a is defined for any $a \in H$.

Lemma 3.7. *With the notation given above, for any $u, v \in H$, we have $w_u w_v = w_{uv}$ and $w_u^{-1} = w_{u^{-1}}$.*

Proof. By Lemma 2.4, P must contain columns $(e_{11}, u^{-1}, v^{-1}u^{-1}, \cdots)^T$ and $(e_{11}, e_{11}, v^{-1}, \cdots)^T$. For convenience, we suppose that they are the i-th and l-th columns, respectively. So, $p_{2i} = e_{12}e_{i1} = u^{-1}$, $p_{3i} = e_{13}e_{i1} = v^{-1}u^{-1}$, $p_{2l} = e_{12}e_{l1} = e_{11}$ and $p_{3l} = e_{13}e_{l1} = v^{-1}$. It is easy to see that $p_{2i}^{-1}p_{2l} = p_{3i}^{-1}p_{3l}$. Then $\begin{bmatrix} e_{i2} & e_{i3} \\ e_{l2} & e_{l3} \end{bmatrix}$ is a rectangular band by Lemma 2.6. In the notation given above, we have $w_u = \overline{e_{11}}\ \overline{e_{i2}}\ \overline{e_{11}}$, $w_v = \overline{e_{11}}\ \overline{e_{l3}}\ \overline{e_{11}}$ and $w_{uv} = \overline{e_{11}}\ \overline{e_{i3}}\ \overline{e_{11}}$. By Lemma 3.2, $\overline{e_{12}}\ \overline{e_{l1}} = \overline{e_{11}}$. We then calculate

$$\begin{aligned} w_u w_v &= \overline{e_{11}}\ \overline{e_{i2}}\ \overline{e_{11}}\ \overline{e_{11}}\ \overline{e_{l3}}\ \overline{e_{11}} \\ &= \overline{e_{11}}\ \overline{e_{i2}}\ \overline{e_{12}}\ \overline{e_{l1}}\ \overline{e_{l3}}\ \overline{e_{11}} \\ &= \overline{e_{11}}\ \overline{e_{i2}}\ \overline{e_{l3}}\ \overline{e_{11}} \\ &= \overline{e_{11}}\ \overline{e_{i3}}\ \overline{e_{11}} \quad \left(\text{since } \begin{bmatrix} e_{i2} & e_{i3} \\ e_{l2} & e_{l3} \end{bmatrix} \text{ is a rectangular band}\right) \\ &= w_{uv}. \end{aligned}$$

Finally, we show $w_u^{-1} = w_{u^{-1}}$. This follows since $\overline{e_{11}} = w_{e_{11}} = w_{u^{-1}u} = w_{u^{-1}}w_u$. □

It follows from Lemma 3.3 and Lemma 3.7 that any element of $\overline{H}$ can be expressed as $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}}$ for some $i \in I$ and $j \in J$.

Theorem 3.8. *Let $F_n(G) = \bigcup_{i=1}^{n} Gx_i$ be a finite rank n free (left) group act with $n \geq 3$, and let* End $F_n(G)$ *the endomorphism monoid of $F_n(G)$. Let e be an arbitrary rank 1 idempotent. Then the maximal subgroup of* IG(E) *containing $\overline{e}$ is isomorphic to G.*

Proof. Without loss of generality we can take $e = e_{11}$. We have observed in Lemma 2.3 that $H = H_{e_{11}}$ is isomorphic to G.

We consider the restriction of the natural morphism $\phi : \mathrm{IG}(E) \to \mathrm{End}\, F_n(G)$ as defined before the statement of Proposition 1.1; from (iii) of that result, $\overline{\phi} = \phi|_{\overline{H}} \to H$ is an onto morphism, where $\overline{H} = H_{\overline{e_{11}}}$. We have observed that every element of $\overline{H}$ can be written as $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}}$ for some $i \in I$ and $j \in J$. If $e_{11} = e_{11}e_{ij}e_{11} = (\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}})\phi'$, then $\overline{e_{11}}\ \overline{e_{ij}}\ \overline{e_{11}} = \overline{e_{11}}$ by Lemma 3.4. Thus $\overline{\phi}$ is an isomorphism and $\overline{H}$ is isomorphic to H. □

Corollary 3.9. *Any (finite) group G is a maximal subgroup of some free idempotent generated semigroup over a (finite) biordered set.*

We remark that in [14] it is proven that if G is *finitely presented*, then G is a maximal subgroup of IG(E) for some *finite* E: our construction makes no headway in this direction.

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