Congruence Lattices of Graph Inverse Semigroups

Marina Anagnostopoulou-Merkouri

Joint work with James Mitchell and Zak Mesyan
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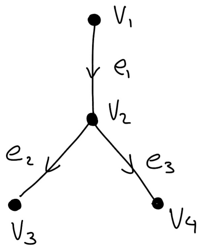
Digraphs

A digraph is a tuple $(E^0, E^1, \mathbf{s}, \mathbf{r})$, where E^0 and E^1 are sets called the vertices and edges respectively, and $\mathbf{s} : E^1 \to E^0$ and $\mathbf{r} : E^1 \to E^0$ are functions from E^1 to E^0 that we call the source and range respectively.



Inverse Semigroups

We say that a semigroup S is an inverse semigroup if for every $x \in S$ there exists unique $x^{-1} \in S$ such that

$$xx^{-1}x = x \text{ and } x^{-1}xx^{-1} = x^{-1}.$$

Now, given a digraph we can define an inverse semigroup. A graph inverse semigroup (GIS) $G(E)$ is an inverse semigroup with zero adjoined, generated by E^0, E^1, and a third set E^{-1} that corresponds to the inverses of edges. The semigroup $G(E)$ must satisfy the following four axioms for all $u, v \in E^0$ and $e, f \in E^1$:

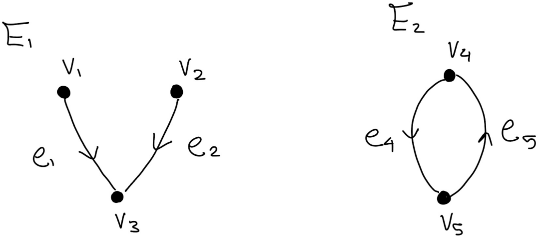
(V) $vu = \delta_{v,u} v$,

(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,

(E2) $\mathbf{r}(e)e^{-1} = e^{-1}\mathbf{s}(e) = e^{-1}$,

(CK1) $e^{-1}f = \delta_{e,f}\mathbf{r}(e)$.

Example



In this case,

- $G(E_1) = \{0, v_1, v_2, v_3, e_1, e_2, e_1^{-1}, e_2^{-1}, e_1 e_1^{-1}, e_2 e_2^{-1}, e_1 e_2^{-1}, e_2 e_1^{-1}\}$.
- $G(E_2) = \{0, v_4, v_5, e_4, e_5, e_4^{-1}, e_5^{-1}, e_4 e_5, e_5 e_4, (e_4 e_5)^2, (e_5 e_4)^2, \dots\}$

Remark: If a digraph E has cycles or loops then $G(E)$ is infinite. For most of this talk our digraphs will be finite, acyclic, and will have no loops.

Congruences on Graph Inverse Semigroups

A congruence ρ of a semigroup S is an equivalence relation that satisfies the following property: If $(s,t),(u,v) \in \rho$, then $(su,tv) \in \rho$.

Question

Is there a way to describe congruences of a a graph inverse semigroup in terms of its digraph?

We call a subset $H \subseteq E^0$ hereditary if it is closed under reachability, i.e. if $v \in H$ then all the out-neighbours of v also lie in H.
For any $v \in E^0$ we denote by $\mathbf{s}^{-1}(v)$ the set of edges of $G(E)$ whose source is v. Finally, we denote the subset of edges in $\mathbf{s}^{-1}(v)$ whose range does not lie in $V \subseteq E^0$ by $\mathbf{s}^{-1}_{E\setminus V}(v)$.

An example

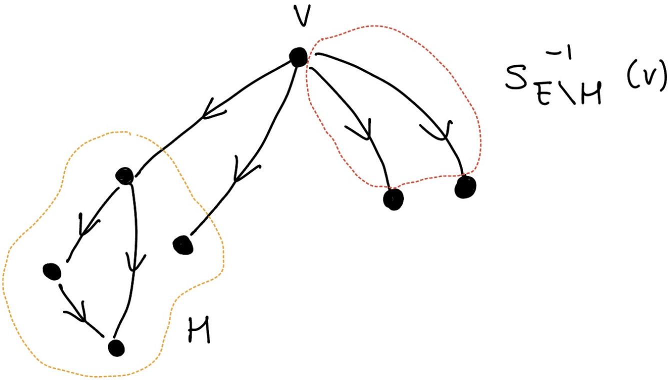


Figure: An example

Representing congruence in terms of the graph

Definition

Let $H \subseteq E^0$ be a hereditary subset and W be any subset of $E^0 \setminus H$ such that $|\mathbf{s}^{-1}_{E\setminus H}(w)| = 1$ for all $w \in W$. Then we call (H, W) a *Wang pair*.

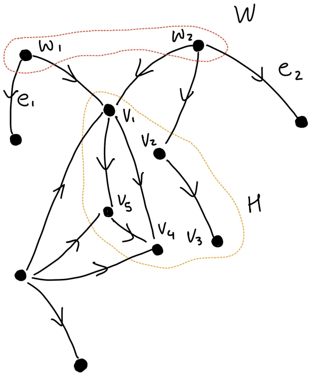
If (H, W) is a Wang pair then we define

$$\rho(H, W) = \big((H \times \{0\}) \cup \{(ee^{-1}, w) : w \in W, \mathbf{s}(e) = w, \mathbf{r}(e) \notin H\}\big)^{\sharp}.$$

Theorem (Wang, '19)

There is a one-to-one correspondence between the set $\mathcal{W}$ of Wang pairs and congruences of $G(E)$.

Another example



Here (H, W) is a Wang pair and

$$\rho(H, W) = \left(\{(v_i, 0) \mid 1 \leq i \leq 5\} \cup \{(w_1, e_1 e_1^{-1}), (w_2, e_2 e_2^{-1})\}\right)^{\sharp}.$$

Lattices

A lattice L is a partially ordered set where for every $a, b \in L$ there exists a unique greatest lower bound $a \wedge b$, called the meet of a and b, and a unique least upper bound $a \vee b$, called the join. The congruences of a semigroup form a lattice. We denote the congruence lattice of $G(E)$ by $L(G(E))$.

Question

Can we use Wang's congruence representation to describe $L(G(E))$?

Theorem (Wang, '19)

We can define a partial order $\leq$ on the set of Wang pairs.

Theorem (Luo, Wang, '21)

The partially ordered set $(\mathcal{W}, \leq)$ and $L(G(E))$ are order isomorphic, and consequently $(\mathcal{W}, \leq)$ forms a lattice.

Meets and joins, and covers

Luo and Wang also showed how to compute meets and joins of Wang pairs, which completely reduces the question of working with congruences of $G(E)$ to working with Wang pairs.
Let $(L, \leq)$ be a lattice and $a, b \in L$. Then we say that b covers a and write $a \prec b$ if $a < b$ and there is no $c \in L$ such that $a < c < b$.

Lemma (AM, Mesyan, Mitchell, ?)

If $G(E)$ is finite and $(H, W), (H', W') \in \mathcal{W}$, then $(H, W) \prec (H', W')$ if and only if $|(H' \cup W') \setminus (H \cup W)| = 1$.

If L is finite, we say that L is upper-semimodular if $a \wedge b \prec a, b$ implies $a, b \prec a \vee b$, and we say that L is lower-semimodular if $a, b \prec a \vee b$ implies $a \wedge b \prec a, b$. We say that L is modular if $a \leq b$ implies $a \vee (x \wedge b) = (a \vee x) \wedge b$ and finally we say that L is distributive if it satisfies the distributive law.

Upper-semimodularity

Question

What properties does $L(G(E))$ possess?

Theorem (Luo, Wang, '21)

The lattice $L(G(E))$ is upper-semimodular, but not necessarily lower-semimodular.

Question

When is $L(G(E))$ lower-semimodular?

Lower-semimodularity

If $u, v \in E^0$ we write $u \geq v$ if there is a path from u to v. We say that $v \in E^0$ is a forked vertex if there exist $e, f \in \mathbf{s}^{-1}(v)$ such that

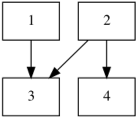
- $\mathbf{r}(g) \not\geq \mathbf{r}(e)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e\}$;
- $\mathbf{r}(g) \not\geq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f\}$.

Theorem (AM, Mesyan, Mitchell, ?)

If $G(E)$ is finite, then the following are equivalent:

- $L(G(E))$ is lower-semimodular;
- $L(G(E))$ is modular;
- $L(G(E))$ is distributive;
- E has no forked vertices.

An example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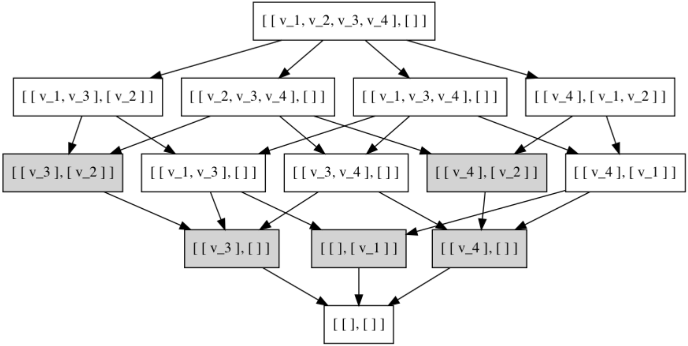
[[v_1, v_2, v_3, v_4], [ ]]
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Other properties

If 0 is the least element of L, we say that $a \in L$ is an atom if $0 \prec a$, and we call L atomistic if it can be generated by atoms. We call a connected digraph a tree if there is at most one path between any two vertices.

Theorem (AM, Mesyan, Mitchell, ?)

If $G(E)$ is finite, then the following are equivalent:

- $L(G(E))$ is atomistic;
- $L(G(E))$ is isomorphic to the lattice $(\mathcal{P}(E^0), \subseteq)$;
- $L(G(E))$ is a disjoint union of trees with a unique sink.

Generating $L(G(E))$

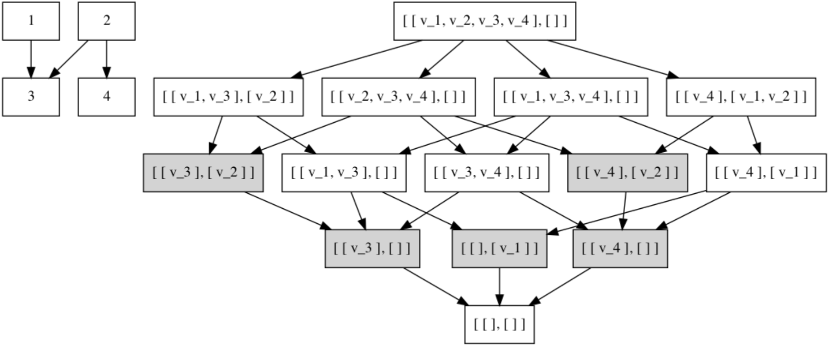
Theorem (AM, Mesyan, Mitchell, ?)

Let E be a finite simple acyclic digraph, and let $\mathcal{A}$ be a collection of congruences of $G(E)$. Then $\mathcal{A}$ is a minimal generating set (with respect to containment) for $L(G(E))$ if and only if $\mathcal{A}$ contains all the congruences of the following types:

- $\rho(\{v\}, \emptyset)$ for every $v \in E^0$ such that $|\mathbf{s}^{-1}(v)| = 0$ (i.e. v is a sink);
- $\rho(H, \{v\})$ for every $v \in E^0$ such that $|\mathbf{s}^{-1}(v)| > 0$ and where H is any minimal (with respect containment) hereditary subset of E^0 such that $|\mathbf{s}^{-1}_{E\setminus H}(v)| = 1$.

This gives us a way to generate the congruence lattice using only the graph and not the semigroup.The graph has significantly smaller size than the semigroup, and hence we can use this to construct a much faster algorithm to compute $L(G(E))$.

Back to our example



Computation

We can use the generation result to compute $L(G(E))$.

- Step 1: **Compute generating Wang pairs.** Iterate over the vertices v of E. If v is a sink add $(v, \emptyset)$ to the generating set. Otherwise, use DFS compute minimal hereditary subsets H such that (H, v) is a Wang pair and add (H, v) to the generating set for every such H.
- Step 2: **Compute joins.** Create a method that computes joins of Wang pairs using Luo and Wang's method for computing joins.
- Step 3: **Compute $L(G(E))$.** Use already existing algorithms to compute the lattice.

Methods that carry out this computation are available in the GAP semigroups package.

Complexity

It can be derived from the axioms that the elements of $G(E)$ are of the form xy^{-1} where x and y are paths of edges. Computing an upper bound for the number of such paths we deduce that $|G(E)|$ is $O(2^n n(n-1))$ where $n = |E^0|$.
Using semigroup-theoretic algorithms to compute $L(G(E))$ we can have at best a polynomial time algorithm on $|G(E)|$. Using the structure of the digraph instead together with the generation result we can compute $L(G(E))$ in time $O(n^3 + n^2 m)$, where $m = |E^1|$.

Extending results to the infinite case

Theorem (AM, Mesyan, Mitchell, ?)

If $G(E)$ is a graph inverse semigroup, then $L(G(E))$ is lower-semimodular if and only if E has no forked vertices.

Theorem (AM, Mesyan, Mitchell, ?)

Every congruence in $L(G(E))$ is the join of a (possibly infinite) collection of atoms if and only if for every $v \in E^0$ one of the following holds:

- $|\mathbf{s}^{-1}(v)| = 0$;
- $|\mathbf{s}^{-1}(v)| = 1$, v does not belong to a cycle, and $v > u$ for some $u \in E^0$ such that $|\mathbf{s}^{-1}(u)| \neq 1$;
- $|\mathbf{s}^{-1}(v)| \geq 2$, and $\mathbf{r}(e) \geq v$ for all $e \in \mathbf{s}^{-1}(v)$.

Moreover, $L(G(E))$ is atomistic if and only if, in addition to the above conditions on all vertices, E^0 has only finitely many strongly connected components and vertices v such that $|\mathbf{s}^{-1}(v)| = 1$.