Amalgams and HNNs of Inverse Semigroups

York Semigroup External Talk

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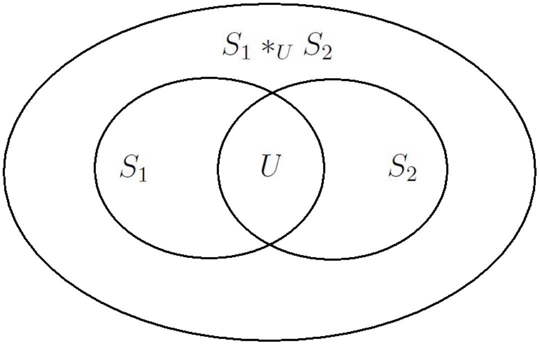
Bali

Amalgams and HNNs, 1997–2019: Italians et al.

- Sandra Cherubini, Emanuele Rodaro and many others.

Amalgams of Inverse Semigroups

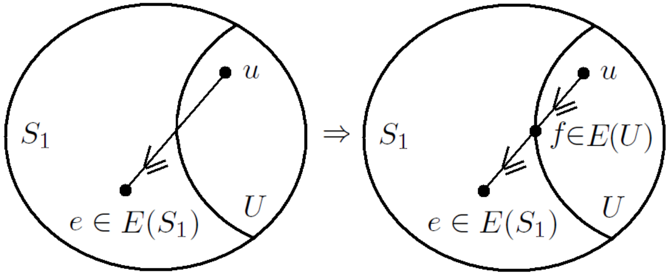


- S_1, S_2, U inverse semigroups, $S_1 \cap S_2 = U$.
- Hall, 1975: $S_1 \cup S_2 \hookrightarrow S_1 *_U S_2$.

Literature on $S_1 *_U S_2$

- Haataja, Margolis, Meakin, 1996.
- Cherubini, Meakin, Piochi, 1997–2005.
- B., 1997.
- Stephen, 1998.
- Cherubini, Jajcayová, Rodaro et al. 2008–2015.

Definition (B., 2020): U lower bounded in S_1



- U lower bounded in S_2, similar.

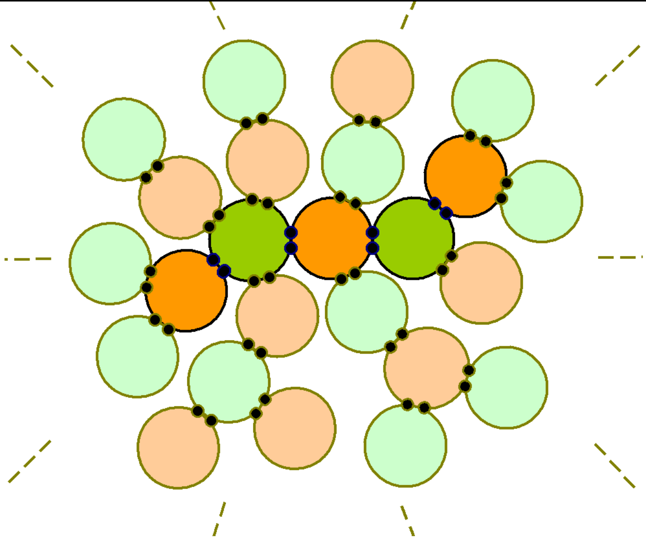
Lower bounded case

Theorems (B., 2020)
If U is lower bounded in S_1 and S_2 then, for $S_1 *_U S_2$, we have:

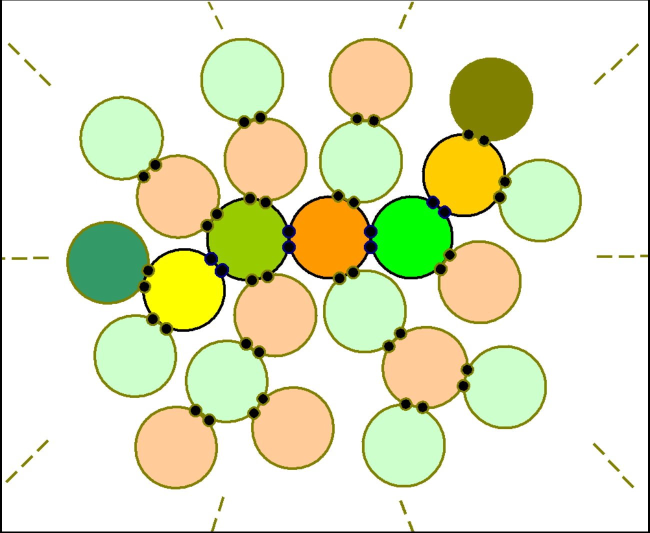
- Schützenberger automata descriptions.
- Structure of maximal subgroups (Bass-Serre theory).
- Preservational properties (e.g. completely semisimple).
- Conditions for decidable word problem (e.g. finite U).

Opuntia 'Prickly Pear' Cacti

Schützenberger 'Opuntoid' graphs of $S_1 *_U S_2$



Hosts and Parasites



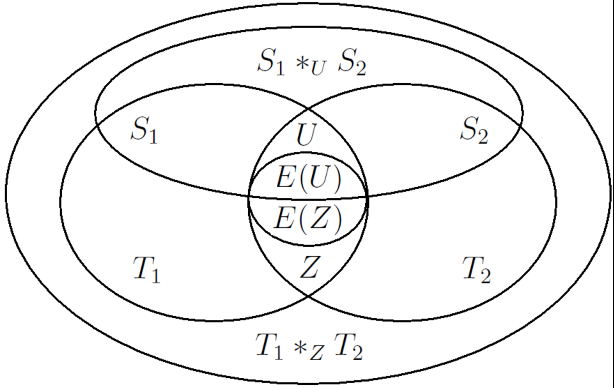
Finite case

Theorems (Italians et al., 2008-2015)
If S_1 and S_2 are finite then, for $S_1 *_U S_2$, we have:

- Schützenberger graph descriptions.
- Structure of maximal subgroups.
- Preservational properties.
- Decidable word problem.

Finite case overlaps with lower bounded case.

General case: a new approach



- Construct a new amalgam $[T_1, T_2; Z]$.
- Show Z lower bounded in T_1 and T_2.
- Show $S_1 *_U S_2 \hookrightarrow T_1 *_Z T_2$.

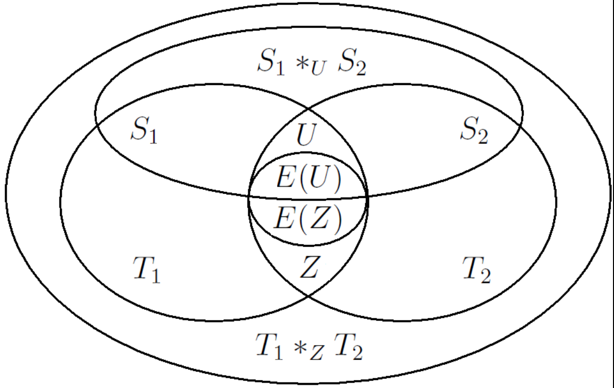
New amalgam $[T_1, T_2; Z]$

- $M(U) =$ semilattice of closed inverse submonoids of U.
- $M_1 \cdot M_2 =$ inverse semigroup closure of $M_1 \cup M_2$ in U.
- $\langle u \rangle =$ closed inverse submonoid of U generated by $u \in U$.
- Construct $S_i *_{E(U)} M(U)$.
- μ_U is the least congruence on $S_i *_{E(U)} M(U)$ with:

$$g\mu_U \leq u\mu_U \Leftrightarrow g\mu_U \leq \langle u \rangle \mu_U$$

  $\forall u \in U, g \in E(S_i *_{E(U)} M(U))$, $i = 1, 2$.
- $T_i = (S_i *_{E(U)} M(U))/\mu_U$, $i = 1, 2$.
- $Z = (U *_{E(U)} M(U))/\mu_U$, similarly.

Theorem (B., 2020)



- $Z \hookrightarrow T_1$, $Z \hookrightarrow T_2$.
- Z is lower bounded in T_1 and T_2.
- $S_1 *_U S_2 \hookrightarrow T_1 *_Z T_2$.

Generalisation 1

Theorem (Cherubini, Meakin and Piochi, 2005)
If S_1 and S_2 are finite then $S_1 *_U S_2$ has decidable word problem.

Theorem (B., 2020)
Suppose U is finite and S_1, S_2 have:

- finite presentations with decidable word problems,
- finite descending chains of idempotents of calculable length,
- finite subgroups of calculable order generated by $\mathcal{H}$-related partial conjugates of U.

Then $S_1 *_U S_2$ has decidable word problem.

Generalisation 2

Theorem (Cherubini, Jajcayová, Rodaro, 2011)
If S_1 and S_2 are finite then the maximal subgroup of $S_1 *_U S_2$ containing an idempotent of S_1 or S_2 has a Bass-Serre description.

Theorem (B., 2020)
The above result extends to when U is finite.

Theorem (B., 2020)
Suppose, in addition, S_1 and S_2 have:

- finite descending chains of idempotents,
- finite subgroups generated by $\mathcal{H}$-rel. partial conjugates of U.

Then any other subgroup of $S_1 *_U S_2$ is a homomorphic image of a subgroup of S_1 or S_2.

Generalisation 3

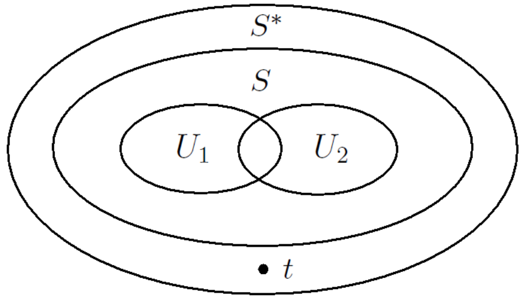
- Define $f \prec_i g \Leftrightarrow f\mathcal{D}h \leq g$ in S_i, for some $h \in E(S_i)$, for all $f, g \in E(U)$ and $i = 1, 2$.
- Define $\prec$ as the transitive closure of $\prec_1$ and $\prec_2$.

Theorem (Rodaro, 2010)
If S_1 and S_2 are finite then $S_1 *_U S_2$ is completely semisimple if and only if $\prec \cap \succ_1 \subseteq \prec_1$ and $\prec \cap \succ_2 \subseteq \prec_2$.

Theorem (B., 2020)
The above result extends to when U is finite and S_1, S_2:

- are completely semisimple,
- have finite descending chains of idempotents.
- have finite $\mathcal{H}$-classes.

HNN Extension S^* of an Inverse Semigroup S

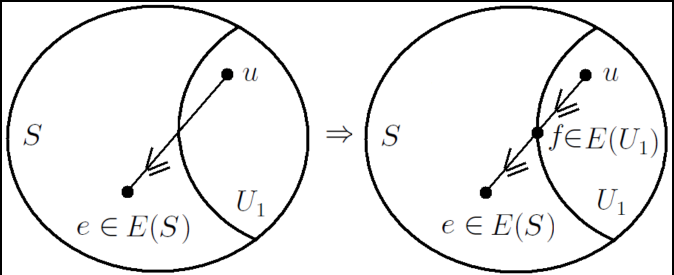


- U_1, U_2 inverse monoids, S inverse semigroup.
- $\phi : U_1 \to U_2$ isomorphism, $e_i =$ identity of U_i, $i = 1, 2$.
- Yamamura, 1997: $S \hookrightarrow S^* = [S; U_1, U_2; \phi]$.
- $tt^{-1} = e_1$, $t^{-1}t = e_2$, $t^{-1}ut = (u)\phi$, $u \in U_1$, in S^*.

Literature on $S^* = [S; U_1, U_2; \phi]$.

- Yamamura, 1997-2006.
- Jajcayová, 1997.
- Cherubini and Rodaro, 2008–2011.
- Ayyash, 2014–2019.

Definition: U_1 lower bounded in S



U_2 lower bounded in S, similar.

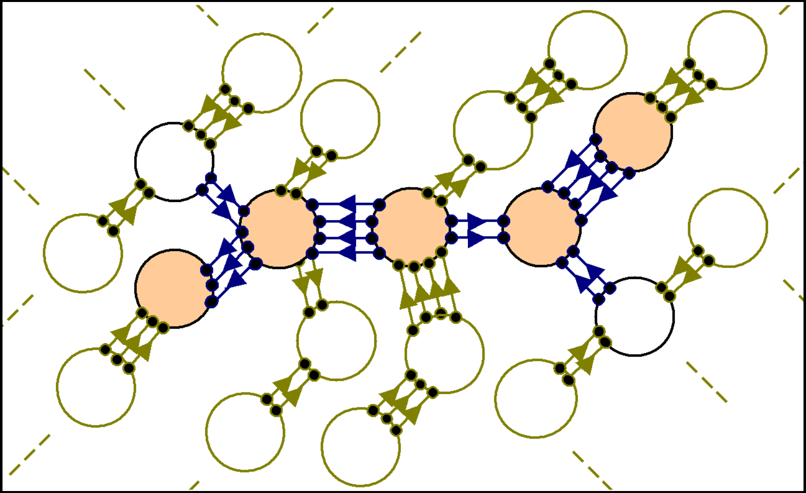
Lower bounded case

Theorems (B. and Jajcayová, 2020)
If U_1 are U_2 are lower bounded in S then, for S^*, we have:

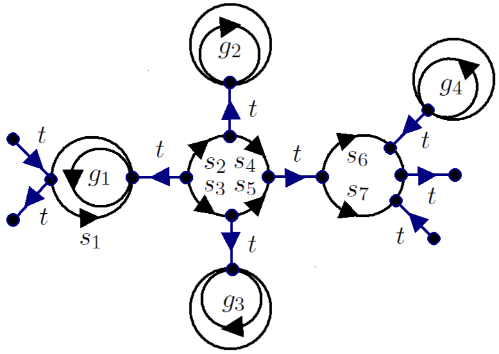
- Schützenberger automata descriptions.
- Structure of maximal subgroups (Bass-Serre theory).
- Preservational properties (e.g. completely semisimple).
- Conditions for decidable word problem (e.g. finite U).

Opuntia 'Pricky Pear' Cacti

Schützenberger 'Opuntoid' graphs of S^*

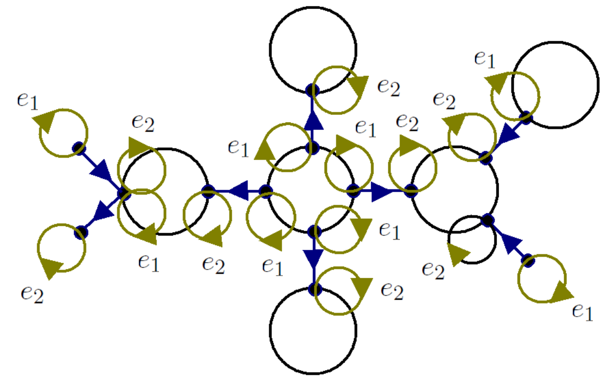


Schützenberger Automata Construction



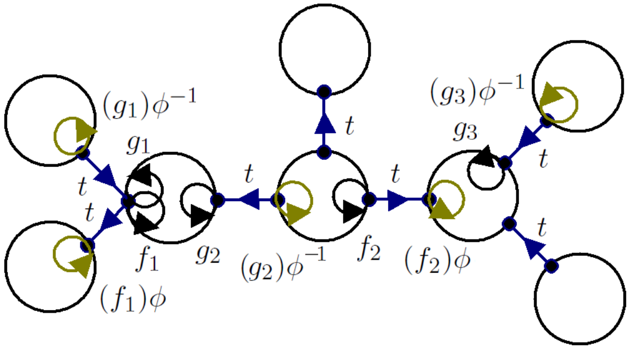
- Given word w over $\{t\}$ and the generators of S.
- Close relative $S * FIM(t)$, using Jones et al. (1994).
- Circles represent Schützenberger graphs of S.

Step 1: Sew e_1 and e_2 loops (green)



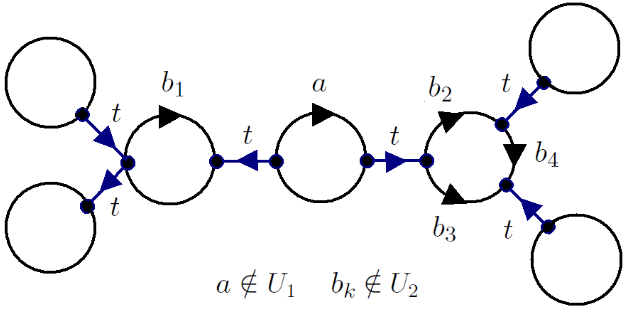
- Sew e_1-loop, using $tt^{-1} = e_1$ relation.
- Sew e_2-loop, using $t^{-1}t = e_2$ relation.
- Close relative $S * FIM(t)$, using Jones et al. (1994).

Step 2: sew $E(U_1)$ and $E(U_2)$ loops (green)



- Sew $(f)\phi$-loop, using $t^{-1}ft = (f)\phi$ relation, $f \in E(U_1)$.
- Sew $(g)\phi^{-1}$-loop, using $t(g)\phi^{-1}t^{-1} = g$ relation, $g \in E(U_2)$.
- Close relative $S * FIM(t)$.

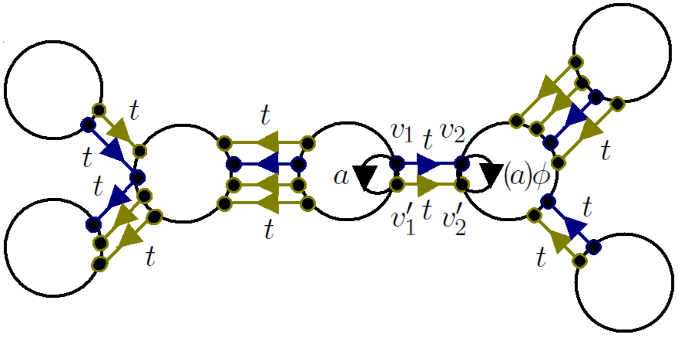
Take Direct Limit of Step 2



Use refinements:

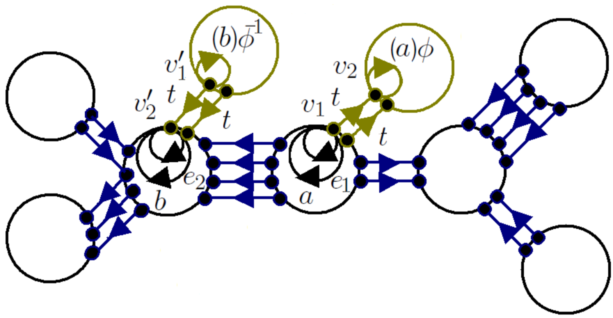
- Initial vertices of two t-edges not connected by U_1-paths.
- Terminal vertices of two t-edges not connected by U_2-paths.

Step 3: sew parallel t-edges



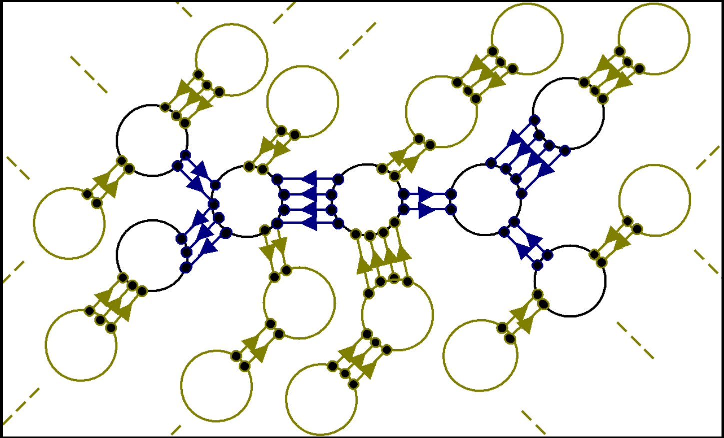
- Sew $v_1' \to^t v_2'$, given $v_1 \to^t v_2$, $v_1 \to^a v_1'$, for some $a \in U_1$, where v_2' is such that we have a path $v_2 \to^{(a)\phi} v_2'$.

Step 4: sew on new circles and t-edges (green)



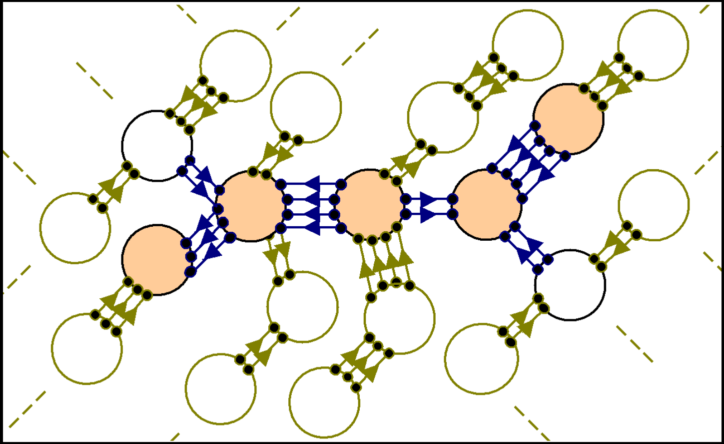
- Sew $v_1 \to^t v_2$ if we have $v_1 \to^{e_1} v_1$.
- Then sew $v_2 \to^{(a)\phi} v_2$, for all $v_1 \to^a v_1$ where $a \in U_1$.
- Sew $v_1' \to^t v_2'$ if we have $v_2' \to^{e_2} v_2'$.
- Then sew $v_1' \to^{(b)\phi^{-1}} v_1'$, for all $v_2' \to^b v_2'$ where $b \in U_2$.

Take Direct Limit of Step 4



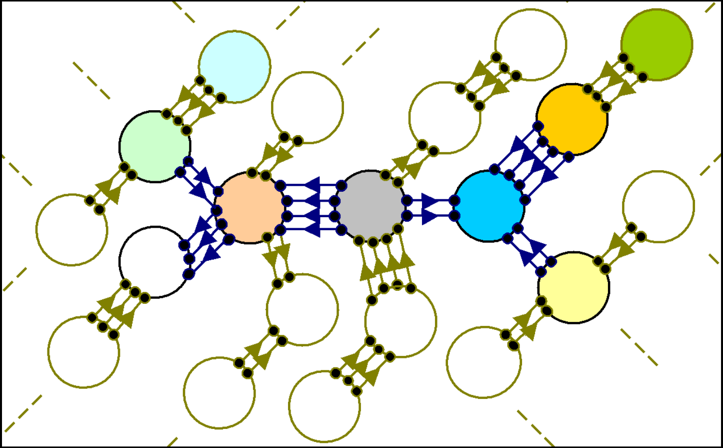
- Step 4 embeds each automaton in the directed system.
- Direct Limit is the Schützenberger automaton of w in S^*.

The Host(s)



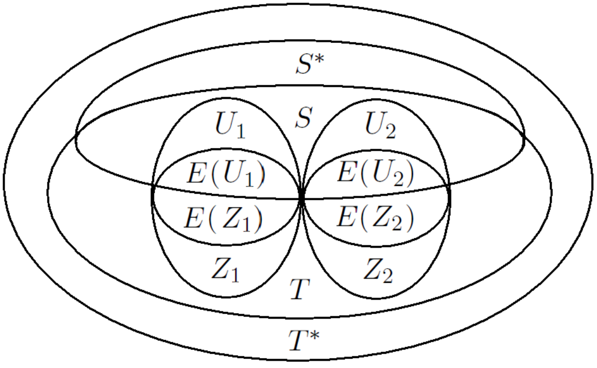
- Everything else feeds off the host(s).
- If multiple hosts then each host is a single circle.

Maximal Subgroups of S^*



- The Automorphism Group is that of the subgraph of all hosts.
- For multiple hosts, we have a graph of groups structure.

General Case: a new approach



- Construct a new HNN $T^* = [T; Z_1, Z_2; \pi]$.
- Show Z_1 and Z_2 lower bounded in T.
- Show $S^* \hookrightarrow T^*$.

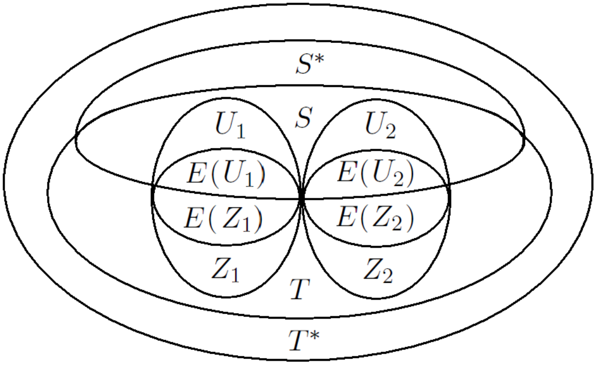
New HNN extension $T^* = [T; Z_1, Z_2; \pi]$.

- $U =$ inverse subsemigroup of S generated by $U_1 \cup U_2$.
- $M(U) =$ semilattice of closed inverse submonoids of U.
- $M_1 \cdot M_2 =$ inverse semigroup closure of $M_1 \cup M_2$ in U.
- $\langle u \rangle =$ closed inverse submonoid of U generated by $u \in U$.
- Construct $S *_{E(U)} M(U)$.
- μ_U is the least congruence on $S *_{E(U)} M(U)$ with:

$$g\mu_U \leq u\mu_U \Leftrightarrow g\mu_U \leq \langle u \rangle \mu_U$$

  $\forall u \in U, g \in E(S *_{E(U)} M(U))$.
- $T = (S *_{E(U)} M(U))/\mu_U$.
- $Z_i = (U_i *_{E(U_i)} M(U_i))/\mu_{U_i}$, $i = 1, 2$, similarly.
- $\pi : Z_1 \to Z_2$ isomorphism.

Theorem (B., 2020)



- $Z_1 \hookrightarrow T$, $Z_2 \hookrightarrow T$.
- Z_1 and Z_2 lower bounded in T.
- $S^* \hookrightarrow T^*$.

Generalisation 1

Theorem (Cherubini and Rodaro, 2008)
If S is finite then S^* has decidable word problem.

Theorem (B., 2020)
Suppose $U = \langle U_1 \cup U_2 \rangle$ is finite and S has:

- a finite presentation with decidable word problem,
- finite descending chains of idempotents of calculable length,
- finite subgroups of calculable order generated by $\mathcal{H}$-related partial conjugates of U.

Then $S^* = [S; U_1, U_2; \phi]$ has decidable word problem.

Generalisation 2

Theorem (Ayyash, 2014)
If S is finite then the maximal subgroup of S^* containing an idempotent of S has a Bass-Serre description.

Theorem (B., 2020)
The above result extends to when $U = \langle U_1 \cup U_2 \rangle$ is finite.

Theorem (B., 2020)
Suppose, in addition, S has:

- finite descending chains of idempotents,
- finite subgroups generated by $\mathcal{H}$-rel. partial conjugates of U.

Then any other subgroup of S^* is a homomorphic image of a subgroup of S.

Generalisation 3

- Define $f \prec_S g \Leftrightarrow f\mathcal{D}h \leq g$ in S, for some $h \in E(S)$, for all $f, g \in E(U_1) \cup E(U_2)$.
- Define $\prec$ as the transitive closure of $\prec_S$ and $\{(f, (f)\phi), ((f)\phi, f) : f \in E(U_1)\}$.

Theorem (Ayyash, 2014)
If S is finite then S^* is completely semisimple if and only if $\prec \cap \succ_S \subseteq \prec_S$.

Theorem (B., 2020)
The above result extends to when $U = \langle U_1 \cup U_2 \rangle$ is finite and:

- S is completely semisimple,
- S have finite descending chains of idempotents,
- S has finite $\mathcal{H}$-classes.

Analogue 1

Theorem (Higman, Neumann and Neumann, 1949)
For any HNN $S^* = [S; U_1, U_2; \phi]$ of groups, there is an amalgam of groups $[S_1, S_2; V]$ and $t \in S_1 *_V S_2$ with:

- $t^{-1}ut = (u)\phi$, for $u \in U_1$.
- $S^* \hookrightarrow S_1 *_V S_2$.

Theorem (B., 2020).
For any HNN $S^* = [S; U_1, U_2; \phi]$ of inverse semigroups, there is an amalgam of inverse semigroups $[S_1, S_2; V]$ and $t \in S_1 *_V S_2$ with:

- $t^{-1}ut = (u)\phi$, for $u \in U_1$.
- $S^* \hookrightarrow S_1 *_V S_2$.

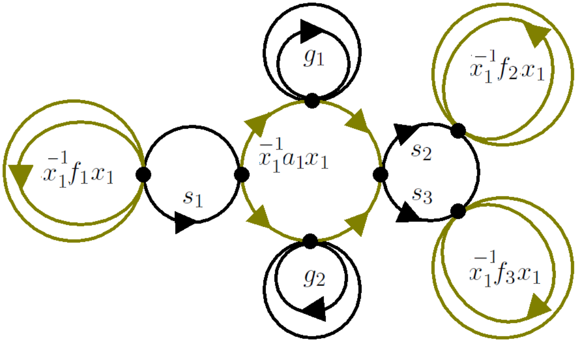
HNN Theorem (B., 2020)

- $S_1 = S *_{\{e_1\}} FIM(x_1)$.
- $V_1 =$ inverse subsemigroup generated by $S \cup x_1^{-1} U_1 x_1$.

- $S_2 = S *_{\{e_2\}} FIM(x_2)$.
- $V_2 =$ inverse subsemigroup generated by $S \cup x_2 U_2 x_2^{-1}$.

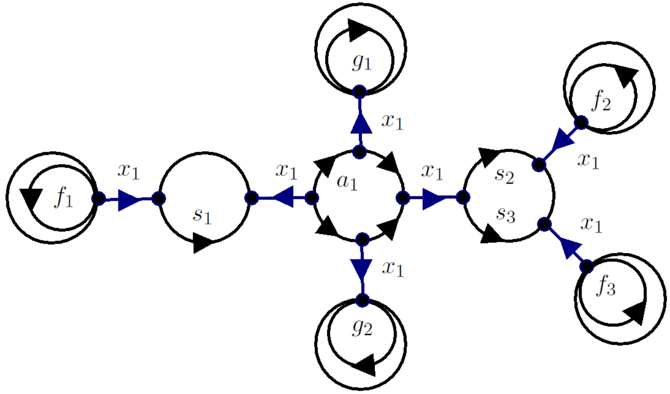
- Prove $V_1 \cong S * x_1^{-1} U_1 x_1 \cong S * x_2 U_2 x_2^{-1} \cong V_2$.
- The result follows, using $t = x_1 x_2$.

One-one map



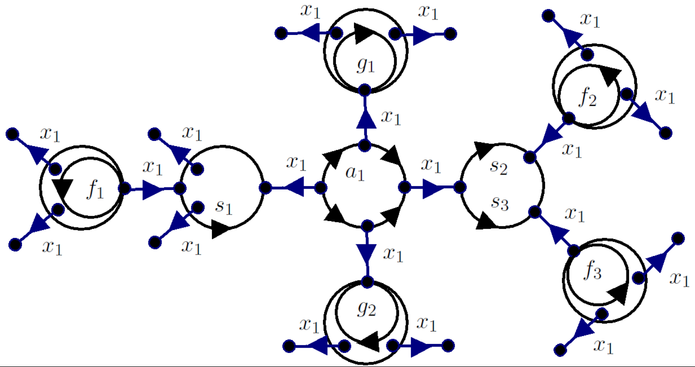
- From the Schützenberger automata of $S * x_1^{-1} U_1 x_1$
- To the Schützenberger automata of $S_1 = S *_{\{e_1\}} FIM(x_1)$.

One-one map



- Replace Schützenberger graphs of $x^{-1}U_1x_1$
- By Schützenberger graphs of $S * FIM(x_1)$.

One-one map



- Sew x_1-edges, using relation $e_1 = x_1 x_1^{-1}$.
- We obtain a Schützenberger graph of $S_1 = S *_{\{e_1\}} FIM(x_1)$.
- This proves $V_1 \cong S * x_1^{-1} U_1 x_1$.

Conclusions

Lower bounded case:

- Schützenberger graphs descriptions.
- Structural and preservational results.
- Conditions for decidable word problem.

General case:

- Construct containing amalgam (HNN), lower bounded case.
- Thus we can study the general case.
- Generalize the literature.
- Analogues of group theory results.