Proceeding of ICA 2010
pp. xx–xx.

RESTRICTION AND EHRESMANN SEMIGROUPS

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Abstract. Inverse semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, in this case $a \mapsto a^{-1}$. The theory of inverse semigroups is perhaps the best developed within semigroup theory, and relies on two factors: an inverse semigroup S is regular, and has semilattice of idempotents. Three major approaches to the structure of inverse semigroups have emerged. Effectively, they each succeed in classifying inverse semigroups via groups (or groupoids) and semilattices (or partially ordered sets). These are (a) the Ehresmann-Schein-Nambooripad characterisation of inverse semigroups in terms of inductive groupoids, (b) Munn's use of fundamental inverse semigroups and his construction of the semigroup T_E from a semilattice E, and (c) McAlister's results showing on the one hand that every inverse semigroup has a proper (E-unitary) cover, and on the other, determining the structure of proper inverse semigroups in terms of groups, semilattices and partially ordered sets.
The aim of this article is to explain how the above techniques, which were developed to study inverse semigroups, may be adapted for certain classes of bi-unary semigroups. The classes we consider are those of restriction and Ehresmann semigroups. The common feature is that the semigroups in each class possess a semilattice of idempotents; however, there is no assumption of regularity.

For Professor K.P. Shum on the occasion of his 70th birthday

1. **Introduction**

A semigroup S is *(von Neumann) regular* if for every $a \in S$ there exists $b \in S$ such that $a = aba$. Much of the early push in semigroup theory was to study and characterise regular semigroups (whether by means of a structure theory, or by representations) and in particular *inverse* semigroups, that is, regular semigroups

2000 Mathematics Subject Classification: 20 M 10
Received: dd-mm-yyyy, accepted: dd-mm-yyyy.*restriction semigroup, Ehresmann semigroup, proper cover*

with commuting idempotents. This direction was entirely natural from the point of view that every semigroup S embeds into a full transformation semigroup $\mathcal{T}_S$, which is regular. Moreover, the use of Green's relations yields, in many cases, vital information about the structure and properties of regular semigroups. But, just as group theory cannot be reduced to the theory of symmetric groups, neither can semigroup theory be boiled down to the study of full transformation (and hence regular) semigroups.

This article will survey three techniques for studying each of two classes of semigroups that may be regarded as non-regular analogues of inverse semigroups. For the first class, that of restriction semigroups, the connections with the inverse case may be clearly seen. For the second, that of Ehresmann semigroups, new insights are required. In fact, we have four classes of semigroups, since, unlike the case for inverse semigroups, which are defined in a left-right dual manner, restriction and Ehresmann semigroups come in left (and, dually, right) and two-sided versions. Left restriction and left Ehresmann semigroups are varieties of unary semigroups, whereas restriction and Ehresmann semigroups are varieties of bi-unary semigroups. Note that a semigroup in each of these classes contains a distinguished semilattice, which is the image of the unary operation(s). As classes of semigroups, all of these classes contain the class of inverse semigroups. The latter is itself a variety of unary semigroups, where the unary operation is $a \mapsto a^{-1}$, but we remark that the unary operations we employ are different.

The three approaches that we consider are inspired by those in the classical theory of inverse semigroups, where they each succeed in classifying inverse semigroups via groups (or groupoids) and semilattices (or partially ordered sets). They are (a) the Ehresmann-Schein-Nambooripad characterisation of inverse semigroups in terms of inductive groupoids (hereafter referred to as the *categorical* approach), (b) Munn's use of fundamental inverse semigroups and his construction of the semigroup T_E from a semilattice E (hereafter referred to as the *fundamental* approach), and (c) McAlister's results showing on the one hand that every inverse semigroup has a proper (E-unitary) cover, and on the other, determining the structure of proper inverse semigroups in terms of groups, semilattices and partially ordered sets (hereafter referred to as the *covering* approach).

In Section 2 we introduce the varieties of (bi)-unary semigroups under consideration. We present them as both varieties, and as determined by relations that may be thought of as analogues of Green's relations. The three subsequent sections take each of the above methods - categorical, fundamental and covering - and examine how they can be used to study restriction and Ehresmann semigroups. Our aim is to give as complete a picture as possible in the two-sided case, but, in order to do so, we must make some mention of the one-sided varieties.

2. Preliminaries

The classes of semigroups we consider may be arrived at in two ways: by using relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, which may be thought of as 'generalisations' of Green's relations $\mathcal{R}$ and $\mathcal{L}$, or as varieties of (bi)-unary semigroups. We give both descriptions, beginning with the former, since this was the original route taken by the 'York school', to which the author belongs. Further details may be found in the notes [21].

Let S be a semigroup and let $E \subseteq E(S)$, where $E(S)$ will always denote the set of idempotents of S. The relation $\widetilde{\mathcal{R}}_E$ is defined on S by the rule that for any $a, b \in S$, we have $a \,\widetilde{\mathcal{R}}_E\, b$ if

$$ea = a \Leftrightarrow eb = b, \text{ for all } e \in E.$$

Where $E = E(S)$ it is usual to drop the subscript and write $\widetilde{\mathcal{R}}_{E(S)}$ more simply as $\widetilde{\mathcal{R}}$. It is easy to see that $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E$, moreover if S is regular, then $\mathcal{R} = \widetilde{\mathcal{R}}$. For, in this case, if $a \,\widetilde{\mathcal{R}}\, b$, then choosing $x, y \in S$ with $a = axa$ and $b = byb$, we have that $ax, by \in E(S)$ and so $a = bya$ and $b = axb$. In general, for any S and any $e, f \in E \subseteq E(S)$, we see that $e \,\mathcal{R}\, f$ if and only if $e \,\widetilde{\mathcal{R}}_E\, f$.

We say that the semigroup S is *weakly left E-abundant* if every $\widetilde{\mathcal{R}}_E$-class contains an element of E (the reader should be aware that some authors also insist that $\widetilde{\mathcal{R}}_E$ be a left congruence). Notice that if $a \,\widetilde{\mathcal{R}}_E\, e$ where $e \in E$, then $ea = a$. If E is a semilattice in S (by which we mean, a commutative subsemigroup of idempotents of S), then it is clear that any $\widetilde{\mathcal{R}}_E$-class contains at most one element of E. In this case we denote the unique idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a, if it exists, by a^+.

Definition 2.1. A semigroup S is *left Ehresmann*, or *left E-Ehresmann*, if E is a semilattice in S, every $\widetilde{\mathcal{R}}_E$-class contains a (unique) element of E, and $\widetilde{\mathcal{R}}_E$ is a left congruence.

Some remarks on Definition 2.1 are appropriate. El Qallali [9] introduced the relation $\widetilde{\mathcal{R}}$ and later Lawson [29] defined $\widetilde{\mathcal{R}}_E$. In addition they both used the left/right duals $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_E$. El Qallali talks of a semigroup as being *semi-abundant* if every $\widetilde{\mathcal{R}}$- and $\widetilde{\mathcal{L}}$-class contains an idempotent, and Lawson calls a semigroup *E-semiabundant* if every $\widetilde{\mathcal{R}}_E$- and $\widetilde{\mathcal{L}}_E$-class contains an element of E. From now on we drop the hyphen that follows the prefix 'semi' in various articles. We recall that a semigroup S is *abundant* [12] if every $\mathcal{R}^*$-class and $\mathcal{L}^*$-class contains an idempotent, where $\mathcal{R}^*$ is defined by the rule that $a \,\mathcal{R}^*\, b$ if and only if for all $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb$$

and $\mathcal{L}^*$ is the dual. An abundant semigroup is *adequate* [10, 11] if $E(S)$ is a semilattice. It is clear that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, so that ($E$)-semiabundant semigroups were being thought of as a generalisation of abundant semigroups, which, in turn, are a generalisation of regular semigroups. Hence if E is a semilattice, then an E-semiabundant semigroup, which Lawson calls *E-semiadequate*, is a generalisation of an adequate semigroup, and again, all inverse semigroups are adequate. Note that in an inverse semigroup, $a^+ = aa^{-1}$. Whereas $\mathcal{R}$ and $\mathcal{R}^*$ are always left congruences, the same is not true of $\widetilde{\mathcal{R}}_E$, which is why the extra condition in Definition 2.1 appears.

Specialising to the case where E forms a semilattice, it was again Lawson [29] who coined the term 'Ehresmann semigroup', making the connection with the work of C. Ehresmann on small ordered categories [8], and that of Schein and Nambooripad on inductive groupoids, which we will come to later. Lawson only considered the two-sided version of Definition 2.1 and to 'keep track' of the idempotents under consideration, he talks of (S, E) as being Ehresmann; subsequent authors have said that S is 'E-Ehresmann' or just 'Ehresmann', where E is understood. Similar conventions apply in the one sided case.

From the comments above, it is clear that an inverse semigroup S is left $E(S)$-Ehresmann. Inverse semigroups form a variety, not of semigroups, but of unary semigroups, that is, of semigroups possessing a unary operation. In this case the operation is $a \mapsto a^{-1}$, where a^{-1} is the unique element such that

$$a = aa^{-1}a \text{ and } a^{-1} = a^{-1}aa^{-1}.$$

A set of three identities defining inverse semigroups is given in [1].

Let S be left E-Ehresmann. By definition, every $\widetilde{\mathcal{R}}_E$-class contains a (unique) element of E, giving a unary operation $a \mapsto a^+$ on S. We may thus regard S as a unary semigroup; the reader should bear in mind that we therefore have two different unary operations defined on an inverse semigroup.

Lemma 2.2. [21] *A semigroup S is left E-Ehresmann such that for any $a \in S$ we have that $a \,\widetilde{\mathcal{R}}_E\, a^+$ where $a^+ \in E$, if and only if it satisfies the identities*

$$x^+x = x,\ (x^+y^+)^+ = x^+y^+,\ x^+y^+ = y^+x^+,\ x^+(xy)^+ = (xy)^+,\ (xy)^+ = (xy^+)^+$$

and

$$E = \{a^+ : a \in S\}.$$

Proof. The result follows from [21], with the exception of showing that the given identities imply $(x^+)^+ = x^+$.

Suppose the identities hold. Then for any $a \in S$,

$$a^+ = (a^+a)^+ = (a^+a^+)^+ = a^+a^+,$$

using the first, fifth and second identities. Then

$$a^+ = a^+a^+ = (a^+a^+)^+ = (a^+)^+,$$

again from the second identity. □

From the above, left Ehresmann semigroups form a variety of algebras. We shall hereafter regard a left Ehresmann semigroup S as a unary semigroup and always use E (or, occasionally, E_S,) for the image of the unary operation. We refer to E as the *distinguished semilattice*. If the reader finds such semigroups a little esoteric, then we point out that the more familiar *left adequate* semigroups (where, following the standard pattern of terminology, a semigroup S is left adequate if $E(S)$ is a semilattice and every $\mathcal{R}^*$-class contains an idempotent) form a generating sub-quasivariety of the variety of left Ehresmann semigroups [18].

Right Ehresmann semigroups are defined dually, where now we use a^* to denote the idempotent in the $\widetilde{\mathcal{L}}_E$-class of a. A semigroup is *Ehresmann* if it is both left and right Ehresmann with respect to the same distinguished semilattices. Ehresmann semigroups form a variety of bi-unary semigroups (that is, of semigroups possessing two unary operations). The defining identities (other than associativity) are those of Lemma 2.2, their duals, together with

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

An inverse semigroup is therefore Ehresmann with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

Definition 2.3. A semigroup S is *left restriction* (or left E-restriction), if it is left Ehresmann and satisfies the *ample condition*

$$ae = (ae)^+a$$

for all $a \in S, e \in E$.

Lemma 2.4. [21] *A semigroup S is left restriction such that for any $a \in S$ we have that $a\,\widetilde{\mathcal{R}}_E\,a^+$ where $a^+ \in E$, if and only if it satisfies the identities*

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x$$

and

$$E = \{a^+ : a \in S\}.$$

The presence of the last identity is therefore very strong, and corresponds to the ample condition. It tells us that in any product, we can replace an idempotent of E on the right by one on the left. We remark that any inverse semigroup S is left restriction, since for any $a, b \in S$,

$$(ab)^+a = (ab)(ab)^{-1}a = a(bb^{-1})(a^{-1}a) = a(a^{-1}a)(bb^{-1}) = abb^{-1} = ab^+.$$

A left restriction semigroup is *left ample* (formerly, *left type A*) if $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$. In this case, we are forced to have $E = E(S)$, since for any $e \in E(S)$,

$$ee^+ = (ee)^+e = e^+e = e$$

so that as $e = e^+e = ee$ and $e\,\mathcal{R}^*\,e^+$,

$$e^+ = e^+e^+ = ee^+ = e.$$

Left ample semigroups have been widely studied, beginning with [10]. They form a generating sub-quasivariety of the variety of left restriction semigroups (see the description of free objects in [16]).

The following result is essentially folklore. The first time it appears in print is in the work of Trokhimenko [45]. To set the scene, we recall that the *symmetric inverse semigroup* $\mathcal{I}_X$ on a set X is the inverse semigroup of partial one-one maps of X, under left-to-right composition of partial maps. By the Vagner-Preston Representation Theorem [23], every inverse semigroup embeds in some $\mathcal{I}_X$. Indeed every left ample semigroup embeds into some $\mathcal{I}_X$ as a unary semigroup [11]. Of course, $\mathcal{I}_X$ is contained in the larger semigroup $\mathcal{PT}_X$ of partial maps of X.

Proposition 2.5. [45, 21] *A unary semigroup S is left restriction if and only if it is isomorphic to a subalgebra of $\mathcal{PT}_X$, where the unary operation on $\mathcal{PT}_X$ is given by $\alpha \mapsto \alpha^+ = I_{\operatorname{dom}\alpha}$.*

It is in essence the above result that determines the importance of left restriction semigroups. Owing to their intimate connections with functions, they have arisen in a number of contexts, and under a number of names, in the last half century. We cite here [5, 15, 24, 43, 31]. Further background and references are given in [21].

The term 'restriction semigroup' is taken from the work of the category theorists, Cockett and Lack [5], where in fact they are using the term for what we call *right restriction* semigroups, where such unary semigroups are defined in a dual manner to left restriction semigroups. We say that a semigroup is *restriction* if it is left and right restriction with respect to the same set of idempotents. Hence restriction semigroups form a variety of bi-unary semigroups.

Clearly inverse semigroups are restriction. We will see that the key to developing analogues of results and techniques for inverse semigroups to the class of (left) restriction semigroups is the existence of the ample condition(s). In the more general case of (left) Ehresmann semigroups, new techniques, or at least new insights, are often required. The next sections will provide ample evidence for this claim.

As mentioned earlier, our thrust here is to consider restriction and Ehresmann semigroups, that is, the two-sided versions. But, there are situations in which these semigroups are harder to deal with than their left-handed cousins. For example, an ample semigroup S (that is, S is both left, and dually, right, ample) must embed into inverse semigroups I and J in a way that preserves $^+$ and *, respectively. However, it is undecidable whether a finite ample semigroup embeds as a bi-unary semigroup into an inverse semigroup [19]. In the final section we find it useful to outline the approach in the one-sided case to inform our discussion in the two-sided.

3. The categorical approach

We begin by outlining the connection between inverse semigroups and inductive groupoids. Of all the approaches to inverse semigroups, this one makes most explicit use of the fact that every inverse semigroup possesses a natural partial order $\leq$, that is, a partial order on S that is compatible with multiplication, and restricts to the usual ordering on the semilattice $E(S)$. We recall that $\leq$ is defined on an inverse semigroup S by the rule that

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S)$$

and it is easy to see that

$$a \leq b \text{ if and only if } a = aa^{-1}b.$$

Further, this relation may also be defined by its dual, that is,

$$a \leq b \text{ if and only if } a = bf \text{ for some } f \in E(S) \text{ if and only if } a = ba^{-1}a.$$

By Vagner-Preston, S embeds into some $\mathcal{I}_X$, and for $\alpha, \beta \in \mathcal{I}_X$, we have $\alpha \leq \beta$ if and only if α is a restriction of β.

All of the categories that we consider will be small (that is, the collection of morphisms is a *set*) unless otherwise stated. We will denote the domain and range of a morphism p in a category $\mathbf{C}$ (that is, $p \in \operatorname{Mor}\mathbf{C}$) by $\mathbf{d}(p)$ and $\mathbf{r}(p)$, respectively. Furthermore, we identify an object α of $\mathbf{C}$ (that is, $\alpha \in \operatorname{Ob}\mathbf{C}$) with the identity map I_α at that object. We think of a category $\mathbf{C}$ as a pair $\mathbf{C} = (C, \cdot)$, where C corresponds to $\operatorname{Mor}\mathbf{C}$ and $\cdot$ is the partial binary operation of 'composition' of morphisms. The existence of domains and ranges tells us when the product $p \cdot q$ exists - it exists if and only if $\mathbf{r}(p) = \mathbf{d}(q)$. For a category $\mathbf{C}$ we denote by $E_{\mathbf{C}}$ the set of identity maps (or objects) of $\mathbf{C}$.

Definition 3.1. A *groupoid* is a category $\mathbf{G} = (G, \cdot)$ in which for every $p \in G$ we have $p^{-1} \in G$ with

$$p \cdot p^{-1} = I_{\mathbf{d}(p)} \text{ and } p^{-1} \cdot p = I_{\mathbf{r}(p)}.$$

A one-object groupoid is precisely a group.

Let S be an inverse semigroup. It is easy to construct a groupoid $\mathcal{G}(S) = (G, \cdot)$ from S as follows:

$$G = \operatorname{Mor}\mathcal{G}(S) = S, \quad \operatorname{Ob}\mathcal{G}(S) = E(S), \; \mathbf{d}(a) = aa^{-1}, \mathbf{r}(a) = a^{-1}a$$

and when $\mathbf{r}(a) = \mathbf{d}(b)$,

$$a \cdot b = ab.$$

On the other hand, let $\mathbf{G} = (G, \cdot)$ be a small groupoid. Let S be the semigroup obtained from $\mathbf{G}$ by declaring all undefined products to be 0. Then $S = G \cup \{0\}$ is an inverse semigroup with 0 with $E(S) = E_{\mathbf{G}} \cup \{0\}$. The problem is, semigroups obtained in this manner are always primitive, that is, non-zero idempotents are incomparable under the natural partial order. For this reason, we need to consider *inductive groupoids*.

We begin with the more general notion of an ordered category, since it will be this we will require subsequently.

Definition 3.2. Let $\mathbf{C} = (C, \cdot)$ be a category. Suppose that $\leq$ is a partial order on C such that:

(OC1) $x \leq y$ implies that $\mathbf{d}(x) \leq \mathbf{d}(y)$ and $\mathbf{r}(x) \leq \mathbf{r}(y)$;
(OC2) $x \leq y, u \leq v, \exists x \cdot u, \exists y \cdot v$ implies that $x \cdot u \leq y \cdot v$;

(OC3) if $a \in C$ and $e \in E_{\mathbf{C}}$ with $e \leq \mathbf{d}(a)$, then there exists a unique ***restriction*** $(e|a) \in C$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;

(OC4) if $a \in G$ and $e \in E_{\mathbf{C}}$ with $e \leq \mathbf{r}(a)$, then there exists a unique ***co-restriction*** $(a|e) \in C$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$.

Then $\mathbf{C} = (C, \cdot, \leq)$ is called an *ordered category.* If in addition

(I) $E_{\mathbf{C}}$ is a semilattice

then $\mathbf{C} = (C, \cdot, \leq)$ is called an *inductive category.*

Definition 3.3. Let $\mathbf{G} = (G, \cdot)$ be a groupoid. Suppose that $\leq$ is a partial order on G such that $\mathbf{G} = (G, \cdot, \leq)$ is an ordered (inductive) category. If in addition

(OG1) $x \leq y$ implies that $x^{-1} \leq y^{-1}$,

then $\mathbf{G} = (G, \cdot, \leq)$ is called an *ordered (inductive) groupoid.*

We note that (OG1) and (OC2) together imply (OC1), so that, if we only wished to define ordered *groupoids*, this could have been done without mention of (OC1).

Let S be an inverse semigroup. Let $\mathcal{G}(S) = (S, \cdot)$ be defined as above. Then $\mathcal{G}(S) = (S, \cdot, \leq)$ (where $\leq$ is the partial order in S) is an inductive groupoid with

$$(e|a) = ea \text{ and } (a|e) = ae.$$

Conversely, let $\mathbf{G} = (G, \cdot, \leq)$ be an inductive groupoid. The *pseudo-product* $\otimes$ is defined on G by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Then $\mathcal{S}(\mathbf{G}) = (G, \otimes)$ is an inverse semigroup (having the same partial order as G) such that the inverse of a in $\mathcal{S}(\mathbf{G})$ coincides with the inverse of a in $\mathbf{G}$. We remark that establishing associativity is the tricky part in proving this assertion.

We have outlined the object correspondence in the theorem below, which has become known as the Ehresmann-Schein-Nambooripad or ESN Theorem, due to its varied authorship. Further details may be found in [30]. The categories in the result below are, of course, general categories. An inductive functor between two inductive groupoids is a functor preserving the partial ordering and meets of identities.

Theorem 3.4. *The category* $\mathbf{I}$ *of inverse of semigroups and morphisms is isomorphic to the category* $\mathbf{G}$ *of inductive groupoids and inductive functors.*

As Lawson points out in his book [30], Theorem 3.4 is not the end of the story. If we want to work with an inverse semigroup S, we can associate to it an inductive groupoid $\mathcal{G}(S)$. We can then work in the larger category of ordered groupoids, and, having obtained a result for $\mathcal{G}(S)$, translate it back to S. Indeed, Theorem 5.3 can be proven in this manner.

A few words of background. In the 1950s Ehresmann developed what are called here inductive groupoids as a way of providing an abstract framework for pseudogroups, where a *pseudogroup* is an inverse semigroup of partial homeomorphims between open sets of a topological space. Indeed he developed a much wider categorical framework. Pseudogroups had been introduced as the suitable vehicle for characterising differentiable manifolds, in the same way that Klein's Erlanger Programme attempted to characterise geometries by their groups of symmetries. Ehresmann's work and its relation to Semigroup Theory has been championed by Lawson, as witnessed by [29, 30]. In the mid 1960s Schein [42] realised that, dropping the differential geometry framework, Ehresmann's ideas could be used to show the connection between inverse semigroups and inductive groupoids. Western semigroup theorists probably first became aware of this work when a translation appeared in 1979 [44]. Nambooripad shortly thereafter [40] extended Schein's results to regular semigroups (using ordered categories with bi-ordered sets of idempotents); he was the first to state the correspondence results at the category level. Again, further details and references may be found in [30], from which this historical sketch is taken.

We now show how this theory adapts to the wider classes of restriction and Ehresmann semigroups.

3.1. **Restriction semigroups.** For restriction semigroups and the sub-quasivariety of ample semigroups, there are smooth and natural analogues of the ESN Theorem.

We first observe that, as for inverse semigroups, a left restriction semigroup S is naturally partially ordered by $\leq$ where here

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E \text{ if and only if } a = a^+b$$

and this partial order is 'inherited' from the embedding of S into $\mathcal{PT}_X$. Moreover, if S is restriction, then we have the equivalent characterisation that

$$a \leq b \text{ if and only if } a = bf \text{ for some } f \in E \text{ if and only if } a = ba^*.$$

Let S be a restriction semigroup. We can construct a category $\mathcal{C}(S) = (C, \cdot)$ from S as follows:

$$C = \mathrm{Mor}\,\mathcal{C}(S) = S, \quad \mathrm{Ob}\,\mathcal{C}(S) = E(S),\ \mathbf{d}(a) = a^+,\ \mathbf{r}(a) = a^*$$

and when $\mathbf{r}(a) = \mathbf{d}(b)$,

$$a \cdot b = ab$$

and then an inductive category $\mathcal{C}(S) = (C, \cdot, \leq)$ where $\leq$ is the natural partial order inherited from S, with

$$(e|a) = ea \text{ and } (a|e) = ae.$$

Conversely, let $\mathbf{C} = (C, \cdot, \leq)$ be an inductive category. The *pseudo-product* $\otimes$ is defined on C in exactly the same way as for inductive groupoids. Then $\mathcal{S}(\mathbf{C}) = (C, \otimes)$ is a restriction semigroup having the same partial order as C, with $a^+ = \mathbf{d}(a)$ and $a^* = \mathbf{r}(a)$.

Again, the 'big' categories in the result below are general categories; inductive functors are defined as between inductive groupoids.

Theorem 3.5. [29] *The category of restriction semigroups and morphisms is isomorphic to the category of inductive categories and inductive functors.*

The forerunner to Theorem 3.5 appears in the ample case in [2], albeit with different terminology. We say that a category $\mathbf{C} = (C, \cdot)$ is *cancellative* if for all $a, b, c \in C$,

$$\exists a \cdot b, \exists a \cdot c \text{ and } a \cdot b = a \cdot c \text{ implies that } b = c$$

and

$$\exists b \cdot a, \exists c \cdot a \text{ and } b \cdot a = c \cdot a \text{ implies that } b = c.$$

Corollary 3.6. [2] *The category of ample semigroups and morphisms is isomorphic to the category of inductive cancellative categories and inductive functors.*

3.2. **Ehresmann semigroups.** Let S be an Ehresmann semigroup. Again, we may define relations $\leq_r$ and $\leq_\ell$ on S by the rule that

$$a \leq_r b \text{ if and only if } a = eb \text{ for some } e \in E \text{ if and only if } a = a^+ b$$

and dually

$$a \leq_\ell b \text{ if and only if } a = bf \text{ for some } f \in E \text{ if and only if } a = ba^*.$$

It is easy to see that $\leq_r$ ($\leq_\ell$) are partial orders, that restrict to the usual partial order on E and which are right (left) compatible with multiplication. However, as explained in [29], unlike the case for restriction semigroups, we need not have $\leq_r = \leq_\ell$ and hence we need not have that either relation is compatible with multiplication *on both sides*. For this reason, we cannot describe an Ehresmann semigroup by merely using an inductive category.

Instead, we consider what Lawson calls *Ehresmann categories*. Our account below is entirely taken from [29], but we use different terminology in parts, since an 'ordered category' in [29] has a weaker definition than that we have used so far in this survey, and we are composing morphisms from left to right.

Definition 3.7. A *category with order* $\mathbf{C} = (C, \cdot, \leq)$ is a category $\mathbf{C} = (C, \cdot)$ partially ordered by $\leq$ such that (OC1), (OC2) and (R) hold:

(R) if $x \leq y$, $\mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$, then $x = y$.

Let S be an Ehresmann semigroup. As in the restriction case, we may define a category $\mathcal{C}(S) = (C, \cdot)$ as follows:

$$C = \operatorname{Mor} \mathcal{C}(S) = S, \quad \operatorname{Ob} \mathcal{C}(S) = E(S), \; \mathbf{d}(a) = a^+, \; \mathbf{r}(a) = a^*$$

and when $\mathbf{r}(a) = \mathbf{d}(b)$,

$$a \cdot b = ab.$$

This category is partially ordered by both $\leq_r$ and $\leq_\ell$. Moreover, we obtain categories with order $\mathcal{C}(S)_r = (C, \cdot, \leq_r)$ and $\mathcal{C}(S)_\ell = (C, \cdot, \leq_\ell)$. This is in spite of the fact that $\leq_r$ and $\leq_\ell$ may not be compatible with multiplication everywhere *on S*.

But by restricting the products under consideration in this way we find that (OC2) holds. For example, in $\mathcal{C}(S)_r$, if

$$x \leq_r y,\ u \leq_r v,\ \exists x \cdot u \text{ and } \exists y \cdot v$$

then

$$x = x^+y,\ u = u^+v,\ x^* = u^+ \text{ and } y^* = v^+.$$

Calculating,

$$xu = x(u^+v) = x(x^*v) = (xx^*)v = xv = x^+yv$$

so that

$$x \cdot u = xu \leq_r yv = y \cdot v,$$

as required.

However, $\mathcal{C}(S)_r$ and $\mathcal{C}(S)_\ell$ may not have both restrictions and co-restrictions. This leads to the following definition.

Definition 3.8. An *Ehresmann category* $\mathbf{C} = (C, \cdot, \leq_r, \leq_\ell)$ is a category $\mathbf{C} = (C, \cdot)$ equipped with two relations $\leq_r$ and $\leq_\ell$ satisfying the following axioms:

(EC1) $(C, \leq_r)$ is a category with order satisfying (OC3);
(EC21) $(C, \leq_\ell)$ is a category with order satisfying (OC4);
(EC3) if $e, f \in E_{\mathbf{C}}$, then $e \leq_r f$ if and only if $e \leq_\ell f$;
(I) $E_{\mathbf{C}}$ is a semilattice under $\leq_r$ (equivalently, $\leq_\ell$);
(EC4) $\leq_r \circ \leq_\ell = \leq_\ell \circ \leq_r$;
(EC5) if $x \leq_r y$ and $f \in E_{\mathbf{C}}$, then $(x|\mathbf{r}(x) \wedge f) \leq_r (y|\mathbf{r}(y) \wedge f)$;
(EC6) if $x \leq_\ell y$ and $f \in E_{\mathbf{C}}$, then $(\mathbf{d}(x) \wedge f|x) \leq_\ell (\mathbf{d}(y) \wedge f|y)$.

Let S be an Ehresmann semigroup and let $\mathcal{C}(S)$ be defined as above. Then $\mathcal{C}(S) = (S, \cdot, \leq_r, \leq_\ell)$ is an Ehresmann category.

Conversely, if $\mathbf{C} = (C, \cdot, \leq_r, \leq_\ell)$ is an Ehresmann category, we may define the pseudo-product in the usual way. As proven in [29], $\mathcal{S}(\mathbf{C}) = (C, \otimes)$ is an Ehresmann semigroup having the same partial orders as $\mathbf{C}$, such that $a^+ = \mathbf{d}(a)$ and $a^* = \mathbf{r}(a)$ for any $a \in C$.

We say that a functor between two Ehresmann categories is *inductive* if it preserves *both* partial orders, and meets of identities.

Theorem 3.9. [29] *The category of Ehresmann semigroups and morphisms is isomorphic to the category of Ehresmann categories and inductive functors.*

Theorem 3.9 is specialised in [29] to the case of adequate semigroups and to restriction semigroups, obtaining Theorem 3.5.

3.3. **Left restriction semigroups.** If S is a left restriction semigroup, then S possesses a natural partial order, but we cannot make a category from S in the above manner, since, although we can define a domain a^+ for $a \in S$, we cannot define a range. With this in mind, the author and her former student Hollings introduced the notion of a *left constellation*.

Definition 3.10. Let P be a set, let $\cdot$ be a partial binary operation and let $^+$ be a unary operation on P with image E, such that E consists of idempotents. We call $(P, \cdot, ^+)$ a *left constellation* if the following axioms hold:

(C1) $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$, in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C2) $\exists x \cdot (y \cdot z) \Leftrightarrow \exists x \cdot y$ and $\exists y \cdot z$;
(C3) for each $x \in P$, x^+ is the unique left identity of x in E;
(C4) $a \in P$, $g \in E$, $\exists a \cdot g \Rightarrow a \cdot g = a$.

It has been suggested by Cockett that left constellations should be called left categories, and we are inclined to agree. Hollings and the author defined the notion of an *inductive left constellation*, and proved an analogue of Theorem 3.4 for left restriction semigroups and inductive left constellations [20].

To the author's knowledge, thus far there has been no association of a left Ehresmann semigroup to a left constellation.

4. The fundamental approach

As in Section 3, we begin by outlining the techniques used originally to study inverse semigroups.

Let S be an inverse semigroup and let us denote the largest congruence contained in $\mathcal{H}$ by μ; equivalently, μ is the greatest idempotent separating congruence on S [35]. By [22], μ is given by the formula

$$a \,\mu\, b \text{ if and only if } a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S).$$

We will say that a regular semigroup is *fundamental* if μ is trivial.

The seminal paper of Munn [36] shows how to construct a fundamental inverse semigroup having a given semilattice of idempotents, as follows.

Let E be a semilattice and let T_E be the subset of $\mathcal{I}_E$ consisting of all isomorphisms between principal ideals of E. It is not hard to check that T_E is a subsemigroup of $\mathcal{I}_E$, that is clearly inverse. Moreover, T_E is fundamental and has semilattice of idempotents isomorphic to E. Unsurprisingly, semigroups of the form T_E have come to be called *Munn semigroups*.

Conversely, any fundamental inverse semigroup with semilattice of idempotents E embeds in T_E, as we now explain.

Let S be an inverse semigroup with $E(S) = E$. For any $a \in S$ we define a partial map α_a of E by:

$$\operatorname{dom} \alpha_a = Eaa^{-1} \text{ and } e\alpha_a = a^{-1}ea \text{ for all } e \in Eaa^{-1}.$$

Then $\alpha_a : Eaa^{-1} \to Ea^{-1}a$ is an isomorphism between principal ideals of E, that is, it lies in T_E.

For any set X we denote by I_X the identity map of X.

Theorem 4.1. [36] *Let E be a semilattice. Then T_E is a fundamental inverse semigroup with semilattice of idempotents*

$$\overline{E} = \{I_{Ee} : e \in E\}$$

isomorphic to E.

Conversely, let S be an inverse semigroup with $E(S) = E$. Let $\theta : S \to T_E$ be given by

$$a\theta = \alpha_a.$$

Then θ is morphism with kernel μ such that

$$\theta|_E : E \to \overline{E}$$

is an isomorphism.

It follows from the above that fundamental inverse semigroups are precisely full subsemigroups of Munn semigroups, and one strategy for describing *any* inverse semigroup S with semilattice E is to solve an extension problem for a full subsemigroup of T_E by μ. Munn [37] also explains how, for example, his results can be used to obtain arbitrary 0-bisimple inverse semigroups in terms of semilattices and groups.

4.1. **Restriction semigroups.** When moving from inverse semigroups to restriction and then Ehresmann semigroups, the step to restriction semigroups is, as in Section 3, straightforward.

Let S be an Ehresmann semigroup with distinguished semilattice E. We denote by μ_E the greatest congruence contained in $\widetilde{\mathcal{H}}_E$, where $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{R}}_E \cap \widetilde{\mathcal{L}}_E$, and say that S is *fundamental* (or *E-fundamental*) if μ_E is trivial. This extends the definition given above, as in an inverse semigroup, $\mathcal{H} = \widetilde{\mathcal{H}}$. However, it is also not unusual to say that an arbitrary semigroup is fundamental if the largest congruence contained in $\mathcal{H}$ is trivial, or if the largest idempotent separating congruence is trivial - neither of which concepts is quite right for us here. We remark that for an Ehresmann semigroup, or indeed an ample semigroup, there may be no greatest idempotent separating congruence [11]. However, it is clear that any congruence contained in $\widetilde{\mathcal{H}}_E$ separates the *idempotents of E.*

Recall that in an inverse semigroup we have that $a^+ = aa^{-1}, a^* = a^{-1}a$ and $a^{-1}ea = (ea)^{-1}ea = (ea)^*$, for any idempotent e. If S is a restriction semigroup we may happily adapt the definition of α_a ($a \in S$) by putting

$$\alpha_a : Ea^+ \to Ea^*,\ e\alpha_a = (ea)^* \text{ for all } e \in Ea^+,$$

retaining the property that α_a is an isomorphism.

The following is taken from [13], where restriction semigroups are called *weakly E-ample.* It extends the corresponding result for ample semigroups, which may be found in [11].

Theorem 4.2. *Let S be a restriction semigroup with distinguished semilattice E. Then $\theta : S \to T_E$ given by $a\theta = \alpha_a$, is a morphism with kernel μ_E such that*

$$\theta|_E : E \to \overline{E}$$

is an isomorphism.

We remark that Theorem 4.2 characterises the relation μ_E on a restriction semigroup S with distinguished semilattice E by the rule that

$$a \,\mu_E\, b \Leftrightarrow a^+ = b^+ \text{ and } (ea)^* = (eb)^* \text{ for all } e \in E;$$

by duality,

$$a \,\mu_E\, b \Leftrightarrow a^* = b^* \text{ and } (ae)^+ = (be)^+ \text{ for all } e \in E.$$

4.2. **Ehresmann semigroups.** To further generalise the above to the Ehresmann case, we must analyse the method a little.

If S is an Ehresmann semigroup with distinguished semilattice E, then we may define α_a as before, and dually, β_a. For restriction semigroups, these two maps were mutually inverse, but this will not be true in general. Moreover, for a purpose that will become clear, we need to extend their domains to E^1, that is, E with an identity adjoined if necessary. To avoid confusion we call the new maps α_a^1 and β_a^1. Specifically,

$$\alpha_a^1 : E^1 \to E \text{ and } \beta_a^1 : E^1 \to E$$

are given by

$$e\alpha_a^1 = (ea)^* \text{ and } e\beta_a^1 = (ae)^+.$$

For a restriction semigroup, the maps α_a and β_a were morphisms. In general, α_a^1 and β_a^1 (and indeed α_a and β_a) need not be, but they will be order preserving. We define $\mathcal{O}_1(E^1)$ to be the semigroup of those order preserving maps of the semilattice E^1 having image contained in E, and we let $\mathcal{O}_1^*(E^1)$ be its dual (in which maps are composed right to left). It transpires that the analogue of $\mathcal{I}_E$ useful for our purposes here is $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$. If S is Ehresmann then it is not hard to define a morphism from S to $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ having kernel μ_E, as we indicate below. The real difficulty is in picking out an Ehresmann subsemigroup of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ that contains all the images of the relevant morphisms from Ehresmann semigroups having semilattice E, in other words, a maximal E-fundamental Ehresmann semigroup.

The semigroups $\mathcal{O}_1(E^1)$ and $\mathcal{O}_1^*(E^1)$ are partially ordered by $\leq$ where

$$\alpha \leq \beta \text{ if and only if } x\alpha \leq x\beta \text{ for all } x \in E^1.$$

It is easy to see that $\leq$ is compatible with multiplication. The subset C_E of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ is then defined by

$$C_E = \{(\alpha, \beta) \in \mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1) : \forall x \in E^1, \rho_{x\alpha} \leq \beta\rho_x\alpha \text{ and } \rho_{x\beta} \leq \alpha\rho_x\beta\}.$$

We note that for any $e \in E$, the pair $\overline{e} = (\rho_e, \rho_e) \in C_E$, where ρ_e is the order preserving map of E^1 given by multiplication with e.

Theorem 4.3. [17] *The set C_E is a fundamental Ehresmann subsemigroup of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ having distinguished semilattice*

$$\overline{E} = \{\overline{e} : e \in E\}$$

isomorphic to E and such that for any $(\alpha, \beta) \in C_E$,

$$(\alpha, \beta)^* = (\rho_{1\alpha}, \rho_{1\alpha}) \text{ and } (\alpha, \beta)^+ = (\rho_{1\beta}, \rho_{1\beta}).$$

Conversely, if S is an Ehresmann semigroup with distinguished semilattice E, then $\phi : S \to C_E$ given by

$$a\phi = (\alpha_a^1, \beta_a^1)$$

is a morphism with kernel μ_E.

As in Subsection 4.1, Theorem 4.3 gives us a closed form for determining μ_E on an Ehresmann semigroup.

4.3. **Left restriction and left Ehresmann semigroups.** If S is left restriction or even left Ehresmann, then $\beta_a^1 : E^1 \to E$ is order preserving, and as $\beta_a \beta_b = \beta_{ba}$, we can certainly represent S as a subsemigroup of $\mathcal{O}_1^*(E^1)$. Moreover the kernel of the representative morphism will be μ_E^R, the largest congruence contained in $\widetilde{\mathcal{R}}_E$ (Cf [11]). Indeed, calling a left Ehresmann semigroup *left fundamental* if μ_E^R is trivial, then $\mathcal{O}_1^*(E^1)$ is itself left fundamental [25], a more straightforward situation than the two-sided case. This approach is currently being considered by Jones [25], as part of a fresh view of the varieties of semigroups related to those presented in this article.

5. **The covering approach**

Whereas Munn's theory, outlined at the beginning of Section 4, constructs an *image* $S\theta$ of an inverse semigroup S, with $E(S) \cong E(S\theta)$, McAlister's seminal papers [32, 33] construct a *preimage* $\widehat{S}$ of S such that $E(\widehat{S}) \cong E(S)$. Just as we know how to construct $T_{E(S)}$ (of which $S\theta$ is a full subsemigroup), so we can determine the structure of $\widehat{S}$ - it is 'almost' a semidirect product of $E(S)$ by the maximal group image of S.

Let S be a semigroup and let $E \subseteq E(S)$. The relation σ_E on S is the least congruence identifying all the elements of E. It is well known that if S is inverse and $\sigma = \sigma_{E(S)}$, then $a \, \sigma \, b$ if and only if $ea = eb$ for some $e \in E(S)$ [34], and further, σ is the least congruence such that S/σ is a group.

An inverse semigroup is *proper* if $\mathcal{R} \cap \sigma = \iota$, where ι denotes the trivial congruence. This definition is only apparently one sided, for it is easily seen to be equivalent to $\mathcal{L} \cap \sigma = \iota$. Moreover, an inverse semigroup is proper if and only if it is *E-unitary*, that is, $E(S)$ forms a σ-class.

The first of McAlister's results tell us that every inverse semigroup is closely related to a proper one, and is knows as *McAlister's Covering Theorem.*

Theorem 5.1. [32] *Let S be an inverse semigroup. Then there exists a proper inverse semigroup $\widehat{S}$ and an idempotent separating morphism $\theta : \widehat{S} \twoheadrightarrow S$ from $\widehat{S}$ onto S.*

The semigroup $\widehat{S}$ appearing in Theorem 5.1 is a *proper cover* of S.

The second of McAlister's results determines the structure of proper inverse semigroups, as follows. We first recall the notion of a monoid acting on a set.

Definition 5.2. Let T be a monoid and let X be a set. Then T *acts* on X (on the left) if there is a map $T \times X \to X$, $(t, x) \mapsto t \cdot x$, such that for all $x \in X$ and $s, t \in T$ we have

$$1 \cdot x = x \text{ and } st \cdot x = s \cdot (t \cdot x).$$

Let G be a group acting on the left of a partially ordered set $\mathcal{X}$ by order automorphisms, which contains $\mathcal{Y}$ as an ideal and subsemilattice, such that

$$(a)\ G \cdot \mathcal{Y} = \mathcal{X} \text{ and } (b) \text{ for all } g \in G,\ g \cdot \mathcal{Y} \cap \mathcal{Y} \neq \emptyset.$$

Then $(G, \mathcal{X}, \mathcal{Y})$ is a **McAlister triple**.

If $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple, we put

$$\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y}) = \{(e, g) \in \mathcal{Y} \times G : g^{-1} \cdot e \in \mathcal{Y}\},$$

and define a binary operation on $\mathcal{P}$ by

$$(e, g)(f, h) = (e \wedge g \cdot f, gh).$$

Of course, if $\mathcal{X} = \mathcal{Y}$, then (a) and (b) would be redundant and $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ would be a semidirect product. In general, however, we cannot dispense with the $\mathcal{X}$ and remark that although the conditions of a McAlister triple ensure that $e \wedge g \cdot f \in \mathcal{Y}$ in the definition of the binary operation, we need not have that $g \cdot f \in \mathcal{Y}$.

Theorem 5.3. [33] *An inverse semigroup is proper if and only if it is isomorphic to a semigroup of the form $\mathcal{P} = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$.*

Semigroups of the form $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ are known as *P-semigroups* and Theorem 5.3 as the *McAlister P-Theorem.* Given a proper inverse semigroup S, the group and semilattice that appear in the corresponding McAlister triple are S/σ and $E(S)$, respectively. Finding the $\mathcal{X}$ is the major difficulty. Proper inverse semigroups form an important class in their own right - for example, the free inverse semigroup is proper [38]. Having a good structure theory for such semigroups enables the investigation of classes of proper inverse semigroups, such as 1-dimensional tiling semigroups. Further, it guides the way for the study of the corresponding classes of inverse semigroups with zero, which include tiling semigroups [30].

5.1. **Left restriction semigroups.** Unlike the case for the 'categorical' and 'fundamental' approaches of the previous two sections, for the 'covering' method it helps to deal first with the one-sided situation.

Let S be left restriction. It is clear that the least *semigroup* congruence identifying all the elements of E is also the least unary semigroup congruence to do so. Thus there is no ambiguity in talking of σ_E as the least congruence identifying all the elements of E.

We will say that a (left) restriction semigroup S is *reduced* if its distinguished semilattice is trivial. A reduced (left) restriction semigroup is simply a monoid, in augmented signature. Clearly, if S is (left) restriction, then S/σ_E is reduced.

Lemma 5.4. [21] *Let S be a left restriction semigroup. Then for any $a, b \in S$, we have that $a\,\sigma_E\,b$ if and only if $ea = eb$ for some $e \in E$.*

A left restriction semigroup S is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$. A *proper cover* of S is a proper left restriction semigroup $\widehat{S}$ such that there exists an epimorphism $\theta : \widehat{S} \twoheadrightarrow S$ that separates distinguished idempotents. It follows that $E_{\widehat{S}} \cong E_S$.

Theorem 5.5. [4] *Let S be left restriction. Then there is a proper cover $\widehat{S}$ of S.*

We now give a recipe for constructing proper left restriction semigroups, inspired by the P-theorem. Naturally, the group in a McAlister triple will be replaced by a monoid.

Let T be a monoid acting on the left of a semilattice $\mathcal{X}$ via morphisms. Suppose that $\mathcal{X}$ has subsemilattice $\mathcal{Y}$ with upper bound ε such that

(a) for all $t \in T$ there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$;

(b) if $e \leq t \cdot \varepsilon$ then for all $f \in \mathcal{Y}$, $e \wedge t \cdot f \in \mathcal{Y}$.

Then $(T, \mathcal{X}, \mathcal{Y})$ is a **strong left M-triple**.

We remark that we have 'gained' over the formulation of McAlister triples in that we may take $\mathcal{X}$ to be a semilattice, but 'lost' in that we no longer have $T \cdot \mathcal{Y} = \mathcal{X}$.

For a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, t) \in \mathcal{Y} \times T : e \leq t \cdot \varepsilon\}$$

and define

$$(e, s)(f, t) = (e \wedge s \cdot f, st), \ (e, s)^+ = (e, 1).$$

Theorem 5.6. [4] *A left restriction semigroup is proper if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$.*

Whilst properly stated for left restriction semigroups in [4], Theorems 5.5 and 5.6 may essentially be found in [15], where they are stated for 'weakly left ample' semigroups, which are left restriction semigroups S with $E = E(S)$. If S is proper left restriction, then the T and $\mathcal{Y}$ that appear in Theorem 5.6 are S/σ_E and E, respectively. In the weakly left ample case, we must insist that T be unipotent, that is, $|E(T)| = 1$.

By replacing T with a right cancellative monoid, we can specialise to the left ample case. The first result is not immediate, but easy to obtain following a similar method to that given in [14, Theorem 7.1]. It was first proven, via a different technique, in [10].

Corollary 5.7. [10] *Let S be left ample. Then there is a proper left ample cover $\widehat{S}$ of S.*

Corollary 5.8. *A left ample semigroup is proper if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ where T is right cancellative.*

Corollary 5.8 also appears under a different formulation in [10, 28].

5.2. **Restriction semigroups.** It follows by duality that if S is right restriction, then $a\,\sigma_E\,b$ if and only if $af = bf$ for some $f \in E$, so that if S is restriction, then either characterisation of σ_E will suffice.

A restriction semigroup S is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota = \widetilde{\mathcal{L}}_E \cap \sigma_E$, that is, if S is proper as a left and as a right restriction semigroup. A *proper cover* of S is a proper restriction semigroup $\widehat{S}$ such that there exists an epimorphism $\theta : \widehat{S} \twoheadrightarrow S$ that separates distinguished idempotents. The next result is stated in [14] for *monoids*, but a remark at the end of that article explains how to deduce the corresponding result for *semigroups*.

Theorem 5.9. [14] *Let S be a restriction semigroup. Then S has a proper ample cover.*

Of course the question then is, can we find a structure theorem for proper restriction semigroups? Of course, a proper ample semigroup S is proper left ample, so that there is a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ (with T right cancellative) such that $S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$. In [28] conditions are put on a strong left M-triple (which has a slightly different formulation in [28]) such that the semigroup built from it be proper ample. This is extended to the restriction case in [6]. However, there is no left-right symmetry about these descriptions.

Again, if S is proper restriction, then as it is proper left restriction, there is a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ such that $S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$, and as S is proper right restriction, there is a strong right M-triple $(T, \mathcal{X}', \mathcal{Y})$ (where a strong right M-triple is defined in a dual manner to a strong left M-triple) such that $S \cong \mathcal{M}'(T, \mathcal{X}, \mathcal{Y})$ (where $\mathcal{M}'(T, \mathcal{X}', \mathcal{Y})$ is also defined in a dual manner). In both cases, we can take $T = S/\sigma_E$ and $\mathcal{Y} = E$. For the strong left M-triple, T acts on the left of $\mathcal{X}$, and for the strong right M-triple, T acts on the right of $\mathcal{X}'$. Now, $\mathcal{X}$ and $\mathcal{X}'$ both contain $\mathcal{Y} = E$ as a subsemilattice, and clearly these actions must be linked in some way. In [14] the author and colleagues developed the notion of *double action* of a monoid acting on the left and the right of a semilattice, satisfying what were called 'compatibility conditions'. This inspired the following definitions.

Let $(T, \mathcal{X}, \mathcal{Y})$ and $(T, \mathcal{X}', \mathcal{Y})$ be strong left and right M-triples respectively. Denote the left action of T on $\mathcal{X}$ by $\cdot$ and the right action of T on $\mathcal{X}'$ by $\circ$. Suppose that for all $t \in T$ and $e \in \mathcal{Y}$, the following and the dual holds:

$$e \leq t \cdot \varepsilon \Rightarrow e \circ t \in \mathcal{Y} \text{ and } t \cdot (e \circ t) = e.$$

Then $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is a *strong M-quadruple.*

Proposition 5.10. [7] *For a strong M-quadruple* $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$*,*

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) \cong \mathcal{M}'(T, \mathcal{X}', \mathcal{Y})$$

via an isomorphism preserving the distinguished semilattices.

Consequently, if $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is a *strong M-quadruple*, then putting

$$\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}),$$

we have that $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is proper restriction.

From results of [14], every restriction monoid has a proper cover of the form $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$, where, in fact, we can take $\mathcal{X} = \mathcal{X}' = \mathcal{Y}$ to be a semilattice with identity, and, moreover, the *free* restriction monoid has this form. However, it is shown in [6] that if $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is *finite*, then it must be inverse. As there are certainly finite proper ample semigroups that are *not* inverse, it follows that not all proper ample (and hence certainly not all proper restriction) semigroups are isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. In [7] a condition is given on a proper restriction semigroup S such that $S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ for some strong M-quadruple $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$.

To obtain a truly two-sided structure theorem for proper restriction semigroups, we are forced to use *partial actions.*

Definition 5.11. Let T be a monoid and let X be a set. Then T *acts partially* on X (on the left) if there is a partial map $T \times X \to X, (t, x) \mapsto t \cdot x$, such that for all $s, t \in T$ and $x \in X$,

$$\exists 1 \cdot x \text{ and } 1 \cdot x = x$$

and

$$\text{if } \exists t \cdot x \text{ and } \exists\, s \cdot (t \cdot x) \text{ then } \exists\, st \cdot x \text{ and } s \cdot (t \cdot x) = st \cdot x,$$

where we write $\exists\, u \cdot y$ to indicate that $u \cdot y$ is defined.

Of course, a partial left action of T on X with domain of the action $T \times X$ is an action. Dually, we may define the (partial) right action of T on X, using the symbol '$\circ$' to replace '$\cdot$'.

Let T be a monoid, acting partially on the left and right of a semilattice $\mathcal{Y}$, via $\cdot$ and $\circ$ respectively. Suppose that both actions preserve the partial order and the domains of each $t \in T$ are order ideals. Suppose in addition that for $e \in \mathcal{Y}$ and $t \in T$, the following and their duals hold:

(a) if $\exists e \circ t$, then $\exists t \cdot (e \circ t)$ and $t \cdot (e \circ t) = e$;

(b) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $\exists e \circ t$.

Then $(T, \mathcal{Y})$ is a **strong M-pair**.

For a strong M-pair $(T, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : \exists e \circ s\}$$

and define operations by

$$(e, s)(f, t) = (s \cdot (e \circ s \wedge f), st),\ (e, s)^+ = (e, 1) \text{ and } (e, s)^* = (e \circ s, 1).$$

Theorem 5.12. [7]*A semigroup is proper restriction if and only if it is isomorphic to some* $\mathcal{M}(T, \mathcal{Y})$.

Our approach to the above result is very direct, inspired by the philosophy of [39]. If S is proper restriction, then the T and $\mathcal{Y}$ that we take for Theorem 5.12 are, naturally, S/σ_E and E. The following corollaries are then almost immediate.

Corollary 5.13. [28] *A semigroup is proper ample if and only if it is isomorphic to* $\mathcal{M}(C, \mathcal{Y})$ *for a cancellative monoid* C.

Corollary 5.14. [41] *A semigroup is proper inverse if and only if it is isomorphic to* $\mathcal{M}(G, \mathcal{Y})$ *for a group* G.

Corollories 5.13 and 5.14 are presented in a slightly different form in [28] and [41], respectively. In those articles more explicit mention is made of the fact that, effectively, our compatibility conditions are insisting that the left and right actions of an element of T be mutually inverse on certain domains.

The reader might wonder why we claim that Theorem 5.12 is truly left/right dual. As we explain in [7], if $(T, \mathcal{Y})$ is a strong M-pair, then

$$\mathcal{M}'(T, \mathcal{Y}) = \{(s, e) \in T \times \mathcal{Y} : \exists s \cdot e\}$$

with operations given by

$$(s, e)(t, f) = (st, (e \wedge t \cdot f) \circ t), (s, e)^* = (1, e) \text{ and } (s, e)^+ = (1, s \cdot e)$$

is a restriction semigroup isomorphic to $\mathcal{M}(T, \mathcal{Y})$.

5.3. **Left Ehresmann semigroups.** We recall that for left Ehresmann semigroups, we do *not* have the identity $xy^+ = (xy)^+x$. This identity ensures that for a left restriction semigroup S, if we have a product

$$s = t_0 e_1 t_1 \dots t_{n-1} e_n t_n$$

where $t_0, \dots, t_n \in S$ and $e_1, \dots, e_n \in E$, then we can re-write this as

$$s = f t_0 t_1 \dots t_n$$

for some $f \in E$. Moreover, S acts by morphisms on E on the left by

$$s \cdot e = (se)^{+}.$$

These two facts underly the description of a proper left restriction semigroup as a semigroup of the form $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ for some strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$. Moreover, building on the first observation, if $S = \langle E \cup T \rangle$ for a *subsemigroup* T of S, then any element of S can be written as

$$t \text{ or } et$$

where $e \in E, t \in T$ and $e < t^{+}$. If this expression is unique, then it is easy to see that S is proper. Consider now a strong left M-triple $(T, \mathcal{Y}, \mathcal{Y})$ and put

$$T' = \{(t \cdot 1, t) : t \in T\}.$$

We know that

$$E = \{(e, 1) : e \in \mathcal{Y}\}$$

and it is easy to see that T' is a subsemigroup isomorphic to T, such that $\mathcal{M} = \mathcal{M}(T, \mathcal{Y}, \mathcal{Y}) = \langle E \cup T' \rangle$ and every element of $\mathcal{M}$ can be written uniquely as

$$(t \cdot 1, t) \text{ or as } (e, 1)(t \cdot 1, t) \text{ for some } (e, 1)^{+} < (t \cdot 1, t)^{+},$$

the latter condition being equivalent to $e < t \cdot 1$.

Having lost our identity, we must think again. We cannot move idempotents to the front in products, and hence we cannot find expressions of the form t/et where $E \cup T$ is a set of generators for a left Ehremann semigroup S, $t \in T$ and $e \in E$. Moreover, the action of S on E as above will be by order order preserving maps, but not usually by morphisms.

We now briefly describe work taken from recent preprints [3],[18]. These articles consider the case for monoids, which for convenience we focus on here.

Let M be a left Ehresmann monoid and let T be a submonoid of M such that $E \cup T$ generates M as a semigroup (equivalently, as a unary monoid). Then any $m \in M$ can be written as

$$m = t_0 e_1 t_1 \dots e_n t_n$$

for some $t_0, t_n \in T$, $t_i \in T \setminus \{1\}$ (for $1 \le i \le n-1$) and $e_j \in E \setminus \{1\}$ with $e_i < (t_i \dots e_n t_n)^{+}$ (for $1 \le j \le n$).

Such an expression is a *T-normal form.* If this expression for each m is unique, then M *has uniqueness of T-normal forms.* We remark that if If $M = \langle X \rangle$ and T is the *submonoid* generated by X, then $M = \langle E \cup T \rangle$.

From an action of T on a semilattice $\mathcal{Y}$ by order preserving maps, we can construct a semigroup $\mathcal{P}(T, \mathcal{Y})$ that has uniqueness of T-normal forms. We refer to [18] for the details.

Theorem 5.15. [18] *A left Ehresmann monoid S with set of generators $E \cup T$, for some submonoid T, has uniqueness of T-normal forms if and only if it is isomorphic to some $\mathcal{P}(T, E)$.*

The notion of *cover* for a left Ehresmann monoid is defined in the obvious way.

Theorem 5.16. [18] *Every left Ehresmann monoid has a cover of the form $\mathcal{P}(X^*, \mathcal{Y})$. Moreover, $\mathcal{P}(X^*, \mathcal{Y})$ is left adequate.*

Since left Ehresmann monoids form a variety, the free object on any set X exists, and as it is left adequate, it is also the free left adequate monoid on X.

Theorem 5.17. [18] *The free left Ehresmann monoid on X is isomorphic to some $\mathcal{P}(X^*, \mathcal{Y})$.*

Kambites [26] has also determined the structure of the free left Ehresmann monoid, using a completely different approach. He also determines the structure of the free Ehresmann monoid [27]. In both cases he uses labelled trees, in what can be seen as an analogue of Munn's approach [38] to the structure of free inverse semigroups.

A strong M-triple with $\mathcal{X} = \mathcal{Y}$ gives rise to a proper left restriction semigroup with the normal form property. We therefore anticipate that our class of monoids of the form $\mathcal{P}(T, \mathcal{Y})$ are a subclass of a class of monoids of the form $\mathcal{P}(T, \mathcal{X}, \mathcal{Y})$, where $\mathcal{P}(T, \mathcal{X}, \mathcal{Y})$ is constructed from a monoid T acting by order preserving maps on a partially ordered set $\mathcal{X}$ containing $\mathcal{Y}$ as semilattice. We also anticipate that that this class can be abstractly characterised by some notion of 'proper', which will of necessity be a little more technical than that for left restriction semigroups.

A suggested notion, that of 'T-proper' appears in [3]. The author and her colleagues and students are working on semigroups of the form $\mathcal{P}(T, \mathcal{X}, \mathcal{Y})$, and, indeed, the corresponding approaches in the two-sided case.

Acknowledgement The author would like to thank the organisers of ICA 2010, and in particular, Professor K.P. Shum, for the opportunity to participate in this meeting.

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