A tetrachotomy for positive first-order logic without equality

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Model Checking problem

We are interested in the parameterisation of the model checking problem by the model. Fix a logic $\mathscr{L}$ and fix $\mathcal{D}$.

The problem "$\mathscr{L}(\mathcal{D})$" has

- Input: a sentence φ of $\mathscr{L}$.
- Question: does $\mathcal{D} \models \varphi$?

We consider syntactic fragments $\mathscr{L}$ of FO and structures $\mathcal{D}$ that are relational and finite.

Complexity of Model Checking

Fragment	Dual	Classification?
$\{\exists,\vee\}$ $\{\exists,\vee,=\}$ $\{\exists,\vee,\neq\}$	$\{\forall,\wedge\}$ $\{\forall,\wedge,\neq\}$ $\{\forall,\wedge,=\}$	Logspace
$\{\exists,\wedge,\vee\}$ $\{\exists,\wedge,\vee,=\}$ $\{\exists,\wedge,\vee,\neq\}$	$\{\forall,\wedge,\vee\}$ $\{\forall,\wedge,\vee,\neq\}$ $\{\forall,\wedge,\vee,=\}$	Logspace if there is some element a s.t. all relations are a-valid, and NP-complete otherwise
$\{\exists,\wedge\}$ $\{\exists,\wedge,=\}$	$\{\forall,\vee\}$ $\{\forall,\vee,\neq\}$	CSP dichotomy conjecture: P or NP-complete
$\{\exists,\wedge,\neq\}$	$\{\forall,\vee,=\}$	NP-complete for $\lvert\mathcal{D}\rvert \geq 3$, reduces to Schaefer classes otherwise.
$\{\exists,\forall,\wedge\}$ $\{\exists,\forall,\wedge,=\}$	$\{\exists,\forall,\vee\}$ $\{\exists,\forall,\vee,\neq\}$	QCSP polychotomy: P, NP-complete, or Pspace-complete ?
$\{\exists,\forall,\wedge,\neq\}$	$\{\exists,\forall,\vee,=\}$	Pspace-complete for $\lvert\mathcal{D}\rvert \geq 3$, reduces to Schaefer classes for Quantified Sat otherwise.
$\{\forall,\exists,\wedge,\vee\}$		**Positive equality free: the rest of this talk**
$\{\forall,\exists,\wedge,\vee,=\}$ $\{\neg,\exists,\forall,\wedge,\vee,=\}$	$\{\forall,\exists,\wedge,\vee,\neq\}$	Logspace when $\lvert\mathcal{D}\rvert \leq 1$, Pspace-complete otherwise
$\{\neg,\exists,\forall,\wedge,\vee\}$		Logspace when $\mathcal{D}$ contains only empty or full relations, Pspace-complete otherwise

Tetrachotomy for $\{\exists, \forall, \wedge, \vee\}$-FO

Ingredients

- Galois Connection
- “Tractability” via relativisation of quantifiers
- Complexity classifications for suitable templates

Ferdinand Börner's tips for Galois Connections

relation closed under	preserved by "operation"
absence of $\exists$	partial
presence of $\forall$	surjective
presence of $\vee$	unary
presence of $=$ absence of $=$	functions hyperfunctions
presence of $\neq$	injective
presence of atomic $\neg$	full

For $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}$, we will need to consider the surjective hyper endomorphisms of the structure $\mathcal{D}$.

Surjective hyper endomorphisms

A *surjective hyper-operation* (shop) on a set D is a function

$$f : D \to \mathcal{P}(D)$$

that satisfies

- for all $x \in D$, $f(x) \neq \emptyset$ (totality).
- for all $y \in D$, there exists $x \in D$ s.t. $y \in f(x)$ (surjectivity).

A *surjective hyper-endomorphism* (she) of $\mathcal{D}$ is a surjective hyper-operation f on D that preserves all extensional relations R of $\mathcal{D}$,

- if $R(x_1, \ldots, x_i) \in \mathcal{D}$ then, for all $y_1 \in f(x_1), \ldots, y_i \in f(x_i)$, $R(y_1, \ldots, y_i) \in \mathcal{D}$.

Example

preserves

0	$\{0\}$
1	$\{1\}$
2	$\{1,2\}$

does not preserve

0	$\{0\}$
1	$\{1,2\}$
2	$\{1,2\}$

Example

preserves

0	$\{0\}$
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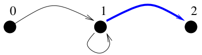
2	$\{1,2\}$



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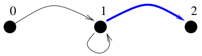

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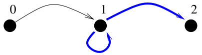

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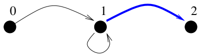

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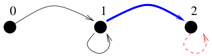
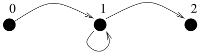


does not preserve

0	$\{0\}$
1	$\{1, 2\}$
2	$\{1, 2\}$

Monoid



has the following set of surjective hyper endomorphisms:

$$\left\{ \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\}$$

which forms in fact a monoid:

$$\left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\rangle.$$

Monoid

$\mathcal{D}$

has the following set of surjective hyper endomorphisms:

$$\mathrm{shE}(\mathcal{D}) = \left\{ \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 2 \end{array}, \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\}$$

which forms in fact a monoid:

$$\mathrm{shE}(\mathcal{D}) = \left\langle \begin{array}{c|c} 0 & 01 \\ \hline 1 & 1 \\ \hline 2 & 12 \end{array} \right\rangle.$$

Down Shop Monoid

- A set of shops on D is a down-shop-monoid, if it contains id_D, and is closed under composition and sub-shops.

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The *identity* shop id_S is defined by $x \mapsto \{x\}$.

Down Shop Monoid

- A set of shops on D is a down-shop-monoid, if it contains id_D, and is closed under composition and sub-shops.

Given shops f and g, define the *composition* $g \circ f$ by

$$x \mapsto \{z : \exists y \; z \in g(y) \wedge y \in f(x)\}.$$

Down Shop Monoid

- A set of shops on D is a down-shop-monoid, if it contains id_D, and is closed under composition and sub-shops.

A shop f is a *sub-shop* of g if $f(x) \subseteq g(x)$, for all x.

Down Shop Monoid

- A set of shops on D is a down-shop-monoid, if it contains id_D, and is closed under composition and sub-shops.
- We write $\langle F \rangle$ for the down-shop-monoid generated by a set of surjective hyper-operations F.

A suitable Galois Connection

Theorem (Madelaine-M.'09/ M.'10)

For a finite structure $\mathcal{D}$ and a set of shops F, the following holds,

- $\langle \mathcal{D} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-FO}} = \mathsf{Inv}(\mathsf{shE}(\mathcal{D}))$; *and,*
- $\langle F \rangle = \mathsf{shE}(\mathsf{Inv}(F))$.

A suitable Galois Connection

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For a finite structure $\mathcal{D}$ and a set of shops F, the following holds,

- $\langle \mathcal{D} \rangle_{\{\exists,\forall,\wedge,\vee\}\text{-FO}} = \mathsf{Inv}(\mathsf{shE}(\mathcal{D}))$; *and,*
- $\langle F \rangle = \mathsf{shE}(\mathsf{Inv}(F))$.

Theorem

For finite $\mathcal{D}$ and $\mathcal{D}'$ (s.t. $D = D'$),

$\mathsf{shE}(\mathcal{D}) \subseteq \mathsf{shE}(\mathcal{D}') \Rightarrow \{\exists,\forall,\wedge,\vee\}\text{-FO}(\mathcal{D}') \leq \{\exists,\forall,\wedge,\vee\}\text{-FO}(\mathcal{D})$.

Motto. surjective hyper-endomorphisms control expressive power and complexity.

Surjective hyper operations of special interest

Let D be a finite set with elements c, d. We define the following types of surjective hyper operations.

$$A_c(x) := \begin{cases} D & \text{if } x = c \\ \{?\} & \text{otherwise.} \end{cases}$$

e.g.

0	0
1	3
2	0123
3	12

i.e. $A_c(c) = D$

$$E_c(x) := \{?, c\}$$

e.g.

0	012
1	1
2	12
3	13

i.e. $E_c{}^{-1}(c) = D$

$$\forall\exists_{c,d}(x) := \begin{cases} D & \text{if } x = c \\ \{d\} & \text{otherwise.} \end{cases}$$

e.g.

0	0123
1	2
2	2
3	2

Quantifier Elimination

$$A_c(c) = D \qquad {E_d}^{-1}(d) = D$$

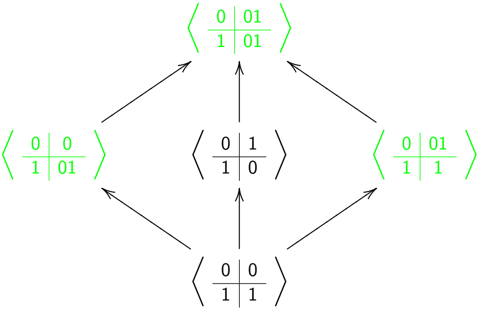
$$\forall\exists_{c,d}(c) = D \qquad {\forall\exists_{c,d}}^{-1}(d) = D$$

presence of	complexity drops to	"algorithm"
A_c	NP	evaluate all $\forall$ to c
E_d	co-NP	evaluate all $\exists$ to d
$\forall\exists_{c,d}$	Logspace	simultaneously do both

- We shall see that these special surjective hyper operations characterise fully the complexity.
- For example, if a relational structure $\mathcal{D}$ is preserved by an A-shop but no $\forall\exists$-shop, the model checking problem $\{\exists, \forall, \wedge, \vee\}\text{-FO}(\mathcal{D})$ is NP-complete

Warm-up: the boolean case

There are five monoids in this case.



Theorem (Madelaine-M. '09)

If shE($\mathcal{D}$) *is green above, then* $\{\exists, \forall, \wedge, \vee\}$-FO($\mathcal{D}$) *is in* Logspace*;*
otherwise it is Pspace*-complete.*

The three-element case

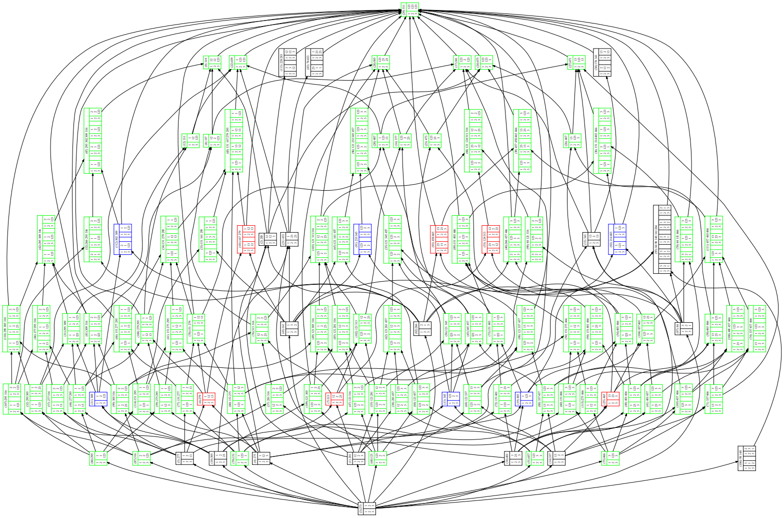
The lattice is considerably richer. The problem class $\{\exists, \forall, \wedge, \vee\}$-$\textsc{fo}(\mathcal{D})$ displays tetrachotomy, between

Logspace

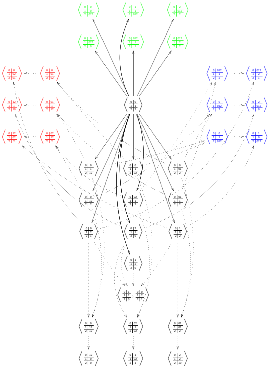
NP-complete co-NP-complete

Pspace-complete

lattice in the 3 element case



Most of these are green “L” cases. The bottom of the lattice is

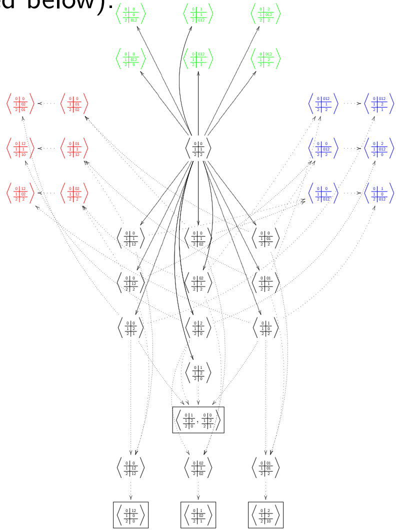


Theorem (Madelaine-M. ’09)

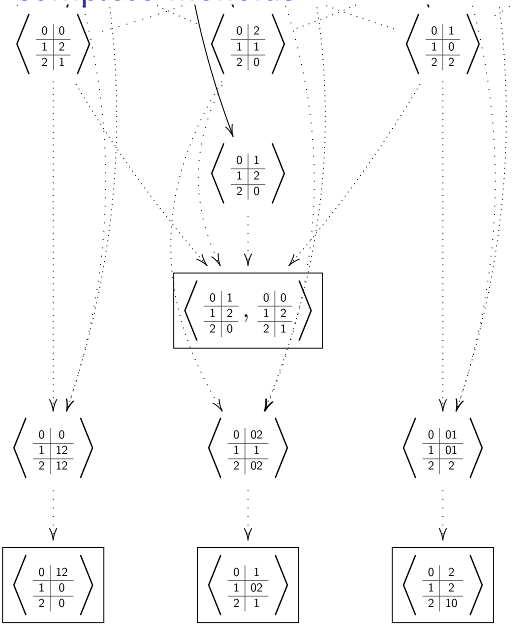
If shE($\mathcal{D}$) *is green, blue or red, above, then* $\{\exists, \forall, \wedge, \vee\}$-$\textrm{FO}(\mathcal{D})$ *is in* L, *is* NP-*complete or is* co-NP-*complete, respectively; otherwise it is* Pspace-*complete.*

Maximal Pspace-complete monoids

There are four maximal Pspace-complete monoids in the 3 element case (drawn boxed below).



Maximal Pspace-complete monoids



Maximal Pspace-complete monoids

There are 20 maximal Pspace-complete monoids in the 4 element case.

Class I

0	1	0
1	0	1
2	012	013
3	013	012

0	2	0
1	012	023
2	0	2
3	023	012

0	3	0
1	013	023
2	023	013
3	0	3

0	012	123
1	2	1
2	1	2
3	123	012

0	013	123
1	3	1
2	123	013
3	1	3

0	023	123
1	123	023
2	3	2
3	2	3

Class II

0	23
1	23
2	01
3	01

0	13
1	02
2	13
3	02

0	12
1	03
2	03
3	12

Class III

0	3
1	3
2	3
3	012

0	2
1	2
2	013
3	2

0	1
1	023
2	1
3	1

0	123
1	0
2	0
3	0

Class IV

0	2	01
1	2	01
2	01	3
3	3	2

0	1	02
1	02	3
2	1	02
3	3	1

0	1	03
1	03	2
2	2	1
3	1	03

0	12	3
1	0	12
2	0	12
3	3	0

0	13	2
1	0	13
2	2	0
3	0	13

0	23	1
1	1	0
2	0	23
3	0	23

Class V

0	1	0	0
1	0	2	1
2	2	1	3
3	3	3	2

The hard part is in proving there are no others.

Maximal Pspace-complete monoids

I think(!) there are 161 maximal Pspace-complete monoids in the 5 element case.

$S_2(1;4)$

0	4
1	4
2	4
3	4
4	0123

+ 4 others

S_2 (2;3)

0	34
1	34
2	34
3	012
4	012

+ 9 others

S_3 (1;1;3)

0	1
1	0
2	234
3	234
4	234

0	0
1	234
2	1
3	1
4	1

+ 9 others

S_3 (1;2;2)

0	1
1	0
2	0
3	34
4	34

0	0
1	34
2	34
3	12
4	12

+ 14 others

S_4 (1;1;1;2)

0	1
1	0
2	2
3	34
4	34

0	0
1	2
2	1
3	34
4	34

0	0
1	1
2	34
3	2
4	2

+ 9 others

Class Ia

0	234
1	234
2	2
3	3
4	0123

0	0123
1	0123
2	3
3	2
4	234

+ 29 others

Class Ib

0	1234
1	12
2	12
3	3
4	0123

0	0123
1	3
2	3
3	12
4	1234

+ 29 others

Class A

0	1234
1	1
2	2
3	3
4	0123

0	0123
1	2
2	1
3	3
4	1234

0	0123
1	1
2	3
3	2
4	1234

+ 9 others

Class B

0	4
1	014
2	024
3	034
4	0

0	0
1	024
2	014
3	034
4	4

0	0
1	014
2	034
3	024
4	4

+ 9 others

Class C

0	0
1	0234
2	024
3	0124
4	4

0	4
1	0124
2	024
3	0234
4	0

+ 29 others

Ia and Ib are inverse, A and B are inverse and C is self-inverse.

Limitation of the “classification by lattice” method

Domain	Classification	Method	Maximally hard monoids
2	done	by hand	1
3	done	by hand, (computer checked)	4
4	done	by computer	20
5	failed attempt	by computer	161

Stuck. We need to move away from the lattice.

Tetrachotomy for all finite domains

Theorem (Madelaine-M. '11)

Let $\mathcal{D}$ be any finite structure.

I. *If* shE($\mathcal{D}$) *contains both an* A*-shop and an* E*-shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* Logspace.

II. *If* shE($\mathcal{D}$) *contains an* A*-shop but no* E*-shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is* NP*-complete.*

III. *If* shE($\mathcal{D}$) *contains an* E*-shop but no* A*-shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is* co-NP*-complete.*

IV. *If* shE($\mathcal{D}$) *contains neither an* A*-shop nor an* E*-shop, then* $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\mathcal{D})$ *is in* Pspace*-complete.*

Proved for domain size 2,3,4 (using the lattice).

Settled for larger domains (without the lattice).

Ingredients of our approach

Previous ingredients:

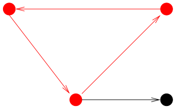
- Galois Connection
- "Tractability" via relativisation of quantifiers

New ingredients:

- A suitable notion of core for $\{\exists, \forall, \wedge, \vee\}$-FO
- Normal form for the monoid associated with the core of a structure $\mathcal{D}$
- Generic hardness proof

Core

For CSP there is the well-established notion of core. The core of a structure $\mathcal{D}$ is a minimal induced substructure $\mathcal{X} \subseteq \mathcal{D}$ all of whose endomorphisms are automorphisms.



It is well-known that $\mathcal{X}$ is unique and $\mathrm{CSP}(\mathcal{D}) = \mathrm{CSP}(\mathcal{X})$.

Core and relativisation

Another way to define the core is as a minimal subset $X \subseteq D$ such that for all positive conjunctive $\phi(\overline{x})$:

$$\mathcal{D} \models \exists \overline{x}\ \phi(\overline{x}) \ \text{ iff } \ \mathcal{D} \models \exists \overline{x} \in X\ \phi(\overline{x}).$$

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Does there exist a "core"-like notion for $\{\exists, \forall, \wedge, \vee\}$-FO?

Core and relativisation

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Does there exist a "core"-like notion for $\{\exists, \forall, \wedge, \vee\}$-FO?

Yes.
But we need 2 relativising sets U (universal) and X (existential).

U-X-core

Theorem (Madelaine-M. '11)

The following are equivalent

1. *There is* $f \in \mathrm{shE}(\mathcal{D})$ *s.t.* $f(U) = D$ *and* $f^{-1}(X) = D$
2. *for all positive equality-free* ϕ, $\mathcal{D} \models \phi \Leftrightarrow \mathcal{D} \models \phi_{[\forall/U,\exists/X]}$.

U-X-core

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The following are equivalent

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2. *for all positive equality-free* ϕ, $\mathcal{D} \models \phi \Leftrightarrow \mathcal{D} \models \phi_{[\forall/U, \exists/X]}$.

We may minimise X and U, then maximise their intersection to obtain a monoid we call reduced.

The substructure of $\mathcal{D}$ induced by $U \cup X$ satisfies the same sentences of $\{\exists, \forall, \wedge, \vee\}$-FO as $\mathcal{D}$. We call it the U-X-**core** (as it is unique up to isomorphism).

Example of a reduced monoid

Consider the domain 5 maximal monoid ⟨

0	0
1	0234
2	024
3	0124
4	4

,

0	4
1	0124
2	024
3	0234
4	0

⟩.

0	0
1	0234
2	024
3	0124
4	4

,

0	0
1	0234
2	024
3	0124
4	4

$U := \{1,3\}$ and $X := \{0,4\}$.

Thus we are equivalent to the reduced monoid

⟨

0	0
1	034
3	014
4	4

,

0	4
1	014
3	034
4	0

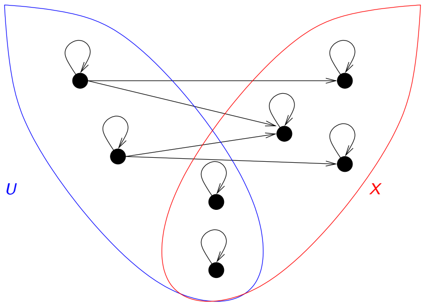
⟩.

Tractable cases

Case	Complexity	A-shop	E-shop	U-X-core	Relativises into	Dual
I	Logspace	yes	yes	$\|U\| = 1, \|X\| = 1$	$\{\wedge, \vee\}$-FO	I
II	NP-complete	yes	no	$\|U\| = 1, \|X\| \geq 2$	$\{\exists, \wedge, \vee\}$-FO	III
III	co-NP-complete	no	yes	$\|U\| \geq 2, \|X\| = 1$	$\{\forall, \vee, \wedge\}$-FO	II

Remaining case. when both $|U| \geq 2$ and $|X| \geq 2$.

Canonical shop and normal form of the reduced monoid



Canonical shop

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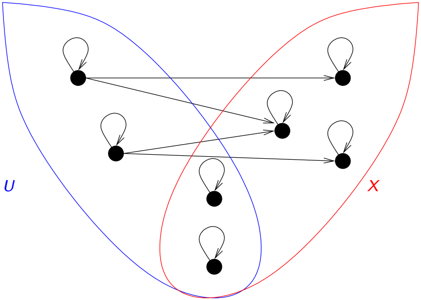
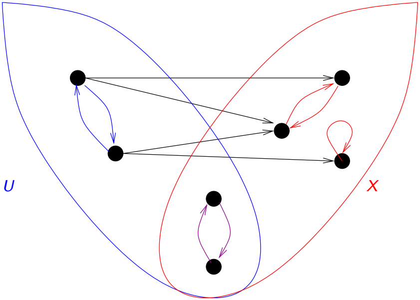


Canonical shop



3-permuted form

All shops in the reduced monoid are in a similar form up to permutation of $U \cap X$, $X \setminus U$ and $U \setminus X$, or sub-shops thereof.

Pspace-hardness

U and X have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.

Pspace-hardness

U and X have both size at least 2.

We consider three cases:

- $U = X$.
  - shops are necessarily "permutations".
  - We know from previous results that this case is Pspace-complete.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.

Pspace-hardness

U and X have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
  - one set can not be included in another.
  - We complete the monoid by adding more shops to blur $U \cap X$ to a single element and $U \Delta X$ to a single element.
  - This amounts to consider a Pspace-hard monoid from the 2-element case.
- $U \cap X = \emptyset$.

Pspace-hardness

U and X have both size at least 2.

We consider three cases:

- $U = X$.
- $U \neq X$ and $U \cap X \neq \emptyset$.
- $U \cap X = \emptyset$.
  - we are unable to exhibit such a simple proof.
  - we complete the monoid by adding all shops in the 3-permuted form.
  - thanks to the relative simplicity of this completed monoid, we can provide a generic hardness proof inspired from the 4 element case

Tetrachotomy for all domains

Tetrachotomy for $\{\exists, \forall, \wedge, \vee\}$-FO($\mathcal{D}$)						
Case	Complexity	A-shop	E-shop	U-X-core	Relativises into	Dual
I	Logspace	yes	yes	$\lvert U\rvert = 1$, $\lvert X\rvert = 1$	$\{\wedge, \vee\}$-FO	I
II	NP-complete	yes	no	$\lvert U\rvert = 1$, $\lvert X\rvert \geq 2$	$\{\exists, \wedge, \vee\}$-FO	III
III	co-NP-complete	no	yes	$\lvert U\rvert \geq 2$, $\lvert X\rvert = 1$	$\{\forall, \vee, \wedge\}$-FO	II
IV	Pspace-complete	no	no	$\lvert U\rvert \geq 2$, $\lvert X\rvert \geq 2$	$\{\exists, \forall, \vee, \wedge\}$-FO	IV

Bonus. A notion of core for quantified constraints.

The meta problem is NP-complete.

The $\{\exists, \forall, \wedge, \vee\}$-$\mathrm{FO}(\sigma)$ meta-problem takes as input a finite σ-structure $\mathcal{D}$ and answers L, NP-complete, co-NP-complete or Pspace-complete, according to the complexity of $\{\exists, \forall, \wedge, \vee\}$-$\textsc{fo}(\mathcal{D})$.
It is NP-hard even for some fixed and finite signature σ_0.

Conclusion

Fragment	Dual	Classification?
$\{\exists, \wedge\}$ $\{\exists, \wedge, =\}$	$\{\forall, \vee\}$ $\{\forall, \vee, \neq\}$	CSP Dichotomy conjecture (P or NP-complete). solved for (undirected) graphs (Hell & Nešetřil), in the boolean case (Schaefer), the 3 element case (Bulatov) and the conservative case (Bulatov, Barto).
$\{\exists, \forall, \wedge\}$ $\{\exists, \forall, \wedge, =\}$	$\{\exists, \forall, \vee\}$ $\{\exists, \forall, \vee, \neq\}$	P/Pspace-complete dichotomy in the boolean case (Schaefer). In general, no precise conjecture. Partial results exhibit P, NP-complete, and Pspace-complete complexities: via the algebraic approach by Chen *et. al.* or a combinatorial approach for graphs and digraphs (Madelaine & Martin). Even the case of (undirected) graphs remains open.
$\{\forall, \exists, \wedge, \vee\}$		Tetrachotomy