Strongly T-proper left Ehresmann monoids

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joint work with Victoria Gould

York Semigroup

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Proper Inverse Monoids

An inverse semigroup is called *proper* if $\mathcal{R} \cap \sigma = \iota$.

Let (X, f) be a presentation of a fixed group G. We construct a category $\mathbf{E}(X, f)$ as follows:

- The objects of the category $\mathbf{E}(X, f)$ are the pairs (g, M), where M is a proper inverse monoid and $g : X \to M$ such that $\langle Xg \rangle = M$ and

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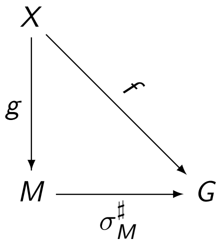

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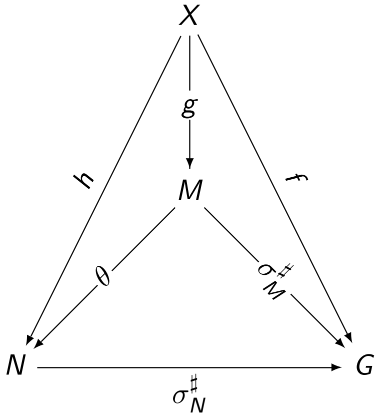

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The category $\mathbf{E}(X, f)$ is called the *category of X-generated proper inverse monoids with maximum group image G.*

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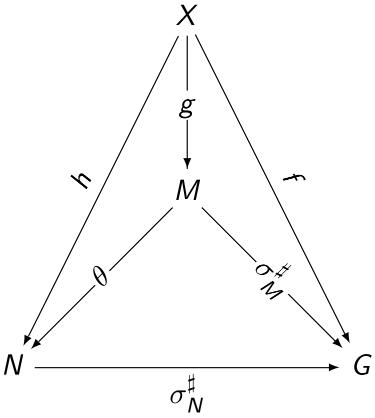

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Fact1: Let (X, f) be a presentation of a group G. Then $FIM(X)/\tau^*$ is the initial object in $\mathbf{E}(X, f)$, where

$$\tau = \{u^2 = u, \text{ whenever } u \in FIM(X) \text{ and } uf = 1 \text{ in } G\}$$

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Fact2: The monoids $FIM(X)/\tau^*$ and the graph expansion $M(X, f)$ of G are isomorphic.

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Graph Expansions of Groups

Let $\Gamma = \Gamma(X, f)$ be the Cayley graph of $G = gp\langle X; f\rangle$.

- $V(\Gamma)$: $=$ the set of vertices of Γ ($V(\Gamma) = G$)
- $E(\Gamma)$: $=$ the set of edges of Γ

an edge: $(g, x, g(xf))$ $\qquad g \xrightarrow{\;x\;} g(xf)$

We define an action of the group G on Γ by

$$t \cdot v = tv \text{ for } t \in G \text{ and } v \in V(\Gamma)$$

and

$$t \cdot (g, x, g(xf)) = (tg, x, tg(xf)), \text{ i.e.}$$

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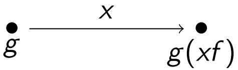

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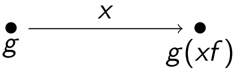

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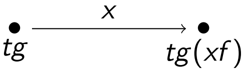

Graph Expansions of Groups

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$$M(X,f) = \{(\Delta, g) : \Delta \text{ is a finite connected subgraph of } \Gamma \text{ containing } 1 \text{ and } g \text{ as vertices}\}$$

with binary operation

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Historical Works

- V. Gould, Graph expansions of right cancellative monoids (1996)
- G.M.S. Gomes and V. Gould, Graph expansions of unipotent monoids (2000)
- V. Gould, Right cancellative and left ample monoids: Quasivarieties and proper covers (2000)
- C. Cornock, Proper left restriction monoids (2011)

The Free Left Ehresmann Monoid

Gomes and Gould:

X:= a non-empty set
E_X:= a semilattice constructed from X
X^* acts on E_X via order-preserving maps
$X^* * E_X$: = the free semigroup product of X^* and E_X
$\mathcal{P}(X^* * E_X)/\sim$:= the free left Ehresmann monoid $FLE(X)$

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Strongly T-proper Left Ehresmann Monoids

Let M be a left Ehresmann monoid with $M = \langle T \cup E \rangle_{(2,1,0)}$ and T a submonoid. We say that M is *strongly T-proper* if for any $u, v \in T$,

$$u \ \sigma_E \ v \ \Rightarrow u^\dagger v = v^\dagger u.$$

Let (X, f) be a presentation of a fixed monoid S. Then we form a category **SPLE**(X, f, S). The objects of **SPLE**(X, f, S) are pairs (g, M) such that $\langle Xg \rangle = M$ and

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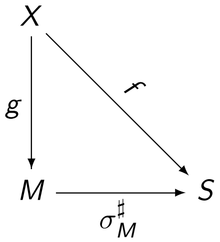

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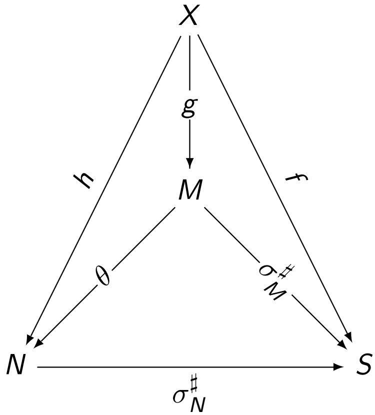

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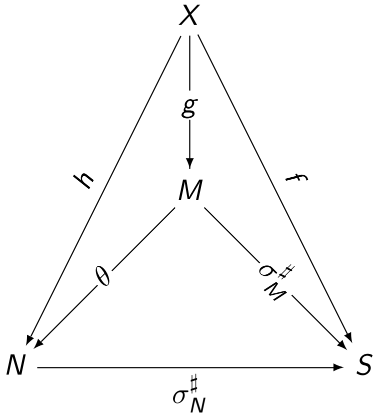

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Edge-labelled Directed Trees

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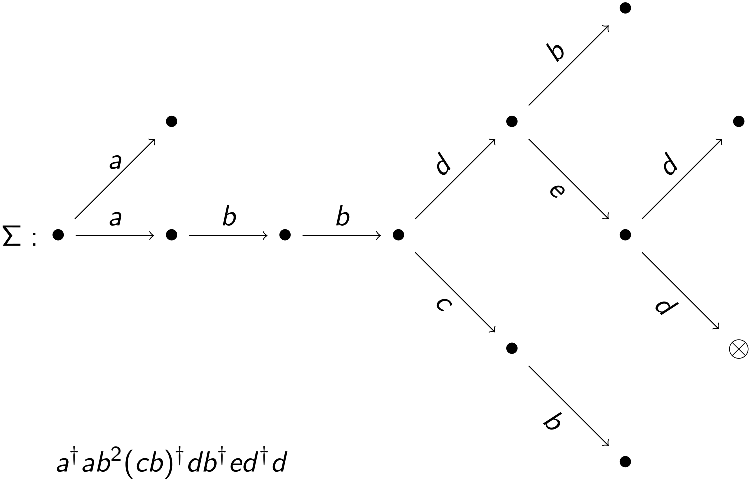

$a^{\dagger}ab^{2}(cb)^{\dagger}db^{\dagger}ed^{\dagger}d$

1-rooted c-graphs

S^1:= a monoid such that $ab \neq 1_S$ for all $a, b \in S$
$f : X \to S$ with $S = \langle Xf \rangle$ X, C, V:= non-empty sets

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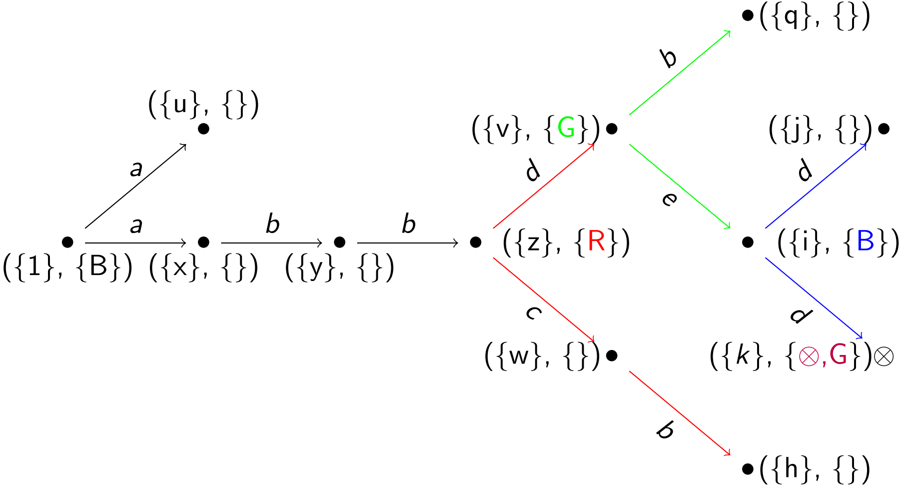

Pinching Methods

Rule 1: a linear path starting with a branching vertex

Suppose that $(a)f = (ab)f$.

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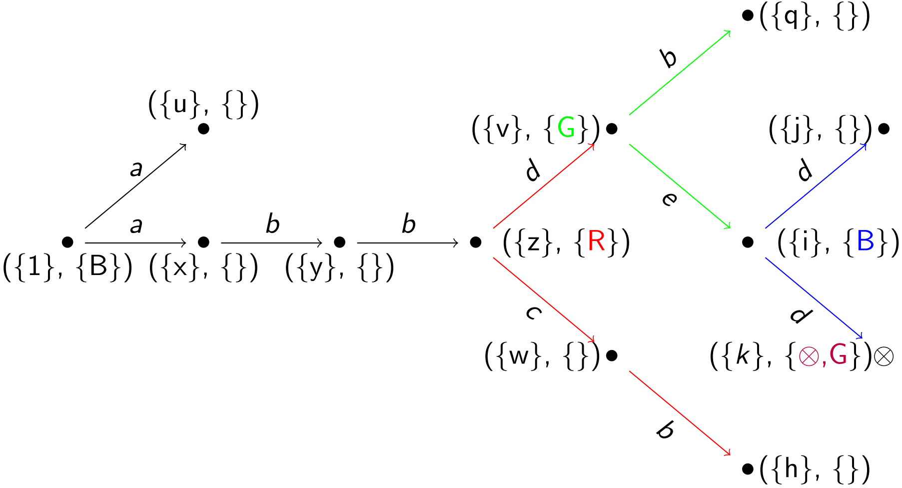

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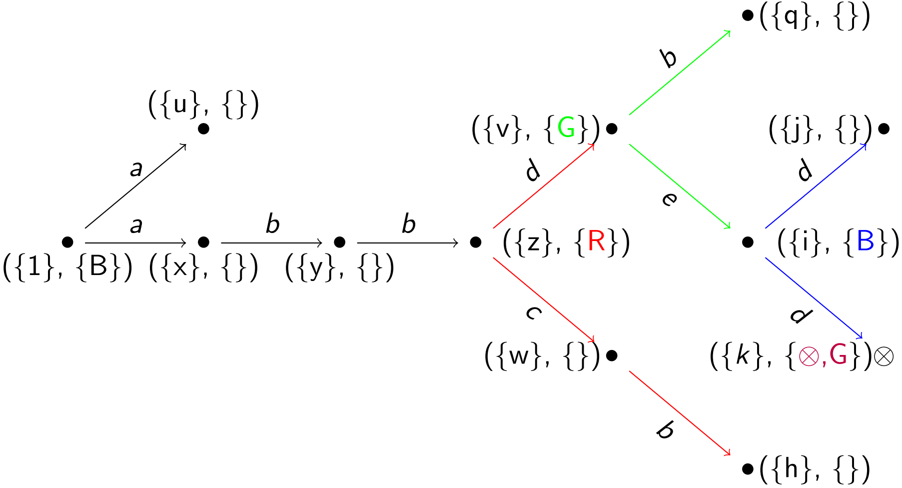

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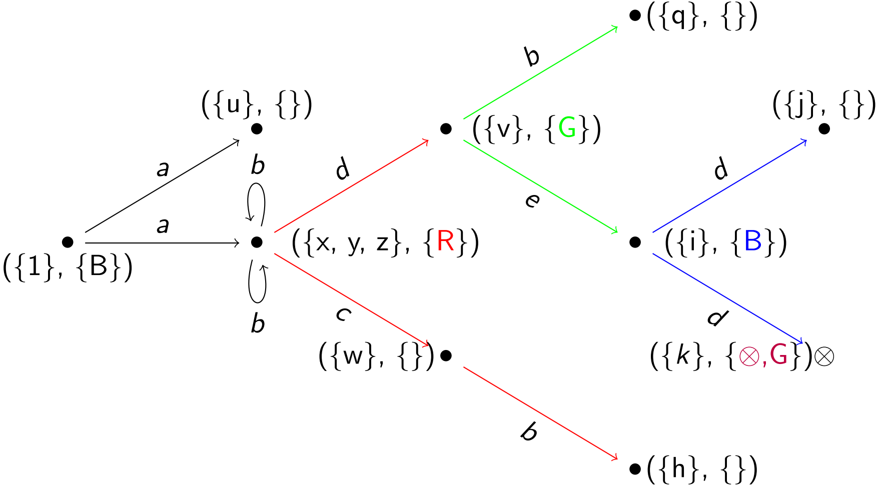

Pinching Methods

Rule 2: two linear paths starting with the same branching vertex

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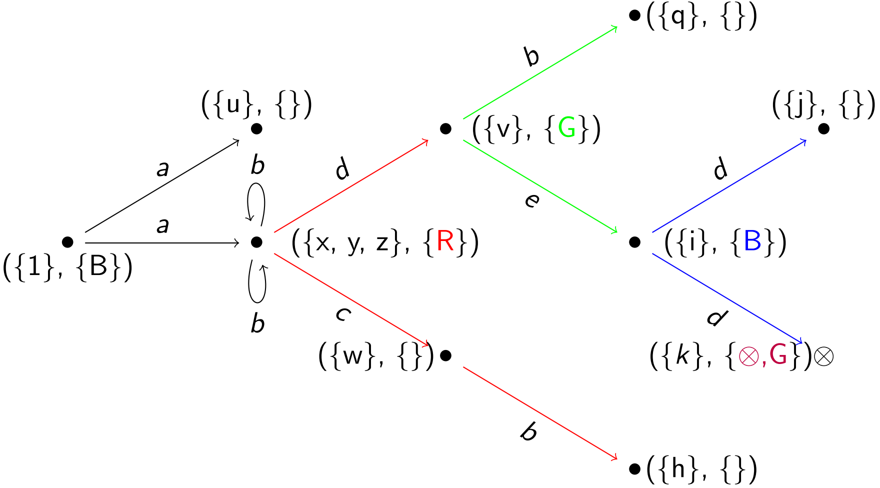

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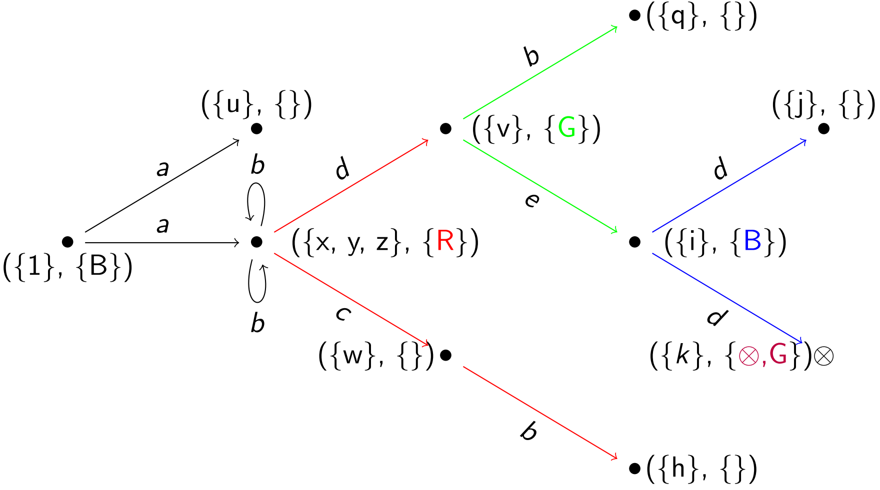

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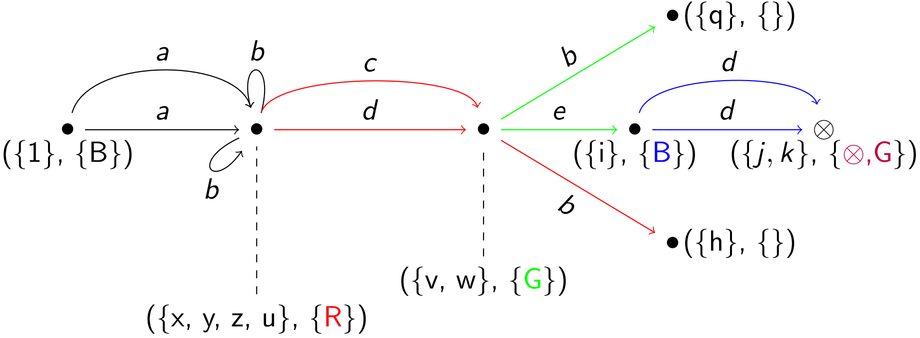
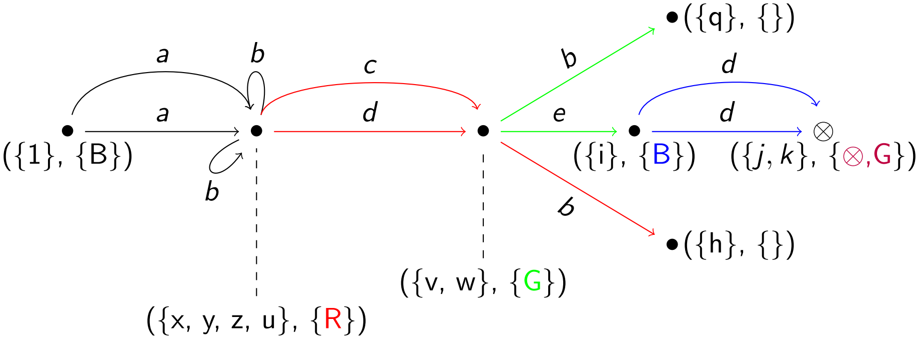


Reducing a Pinched Graph

Examples of allowed retractions:



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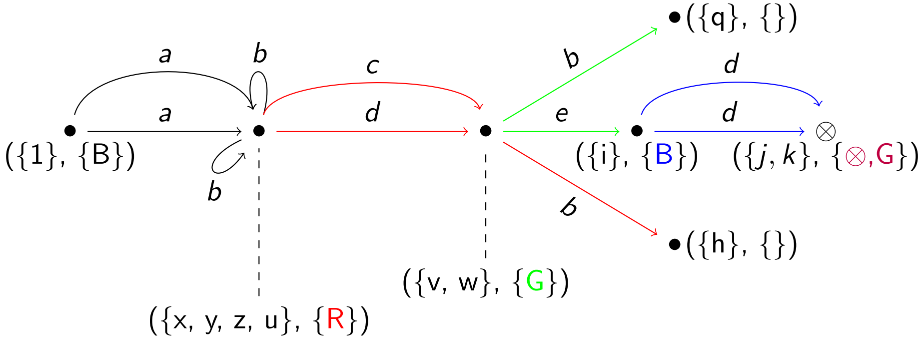

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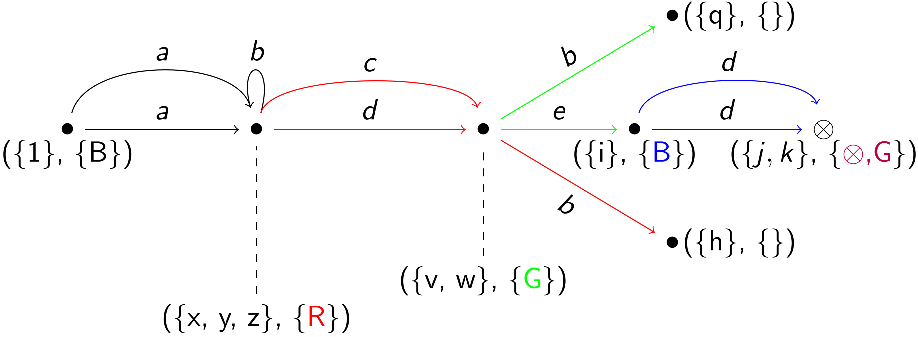

Case II: Parallel paths I (no tails)

Reducing a Pinched Graph: Step 1-Retraction

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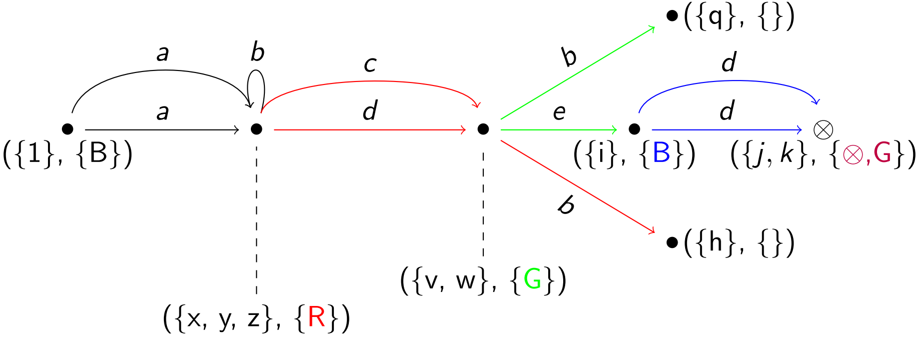

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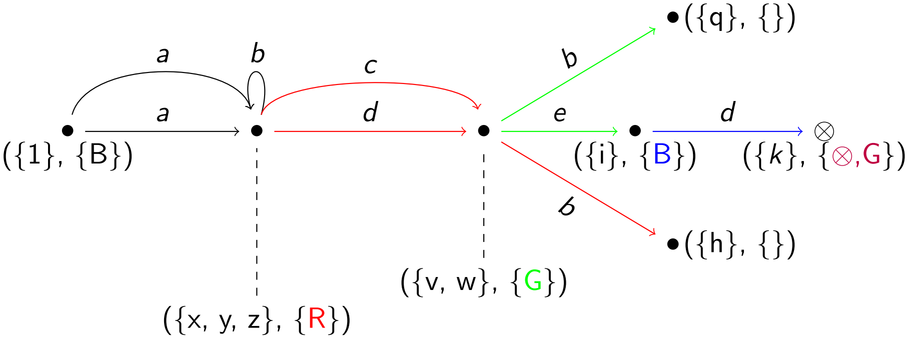

Case III: Parallel paths II (containing tails)

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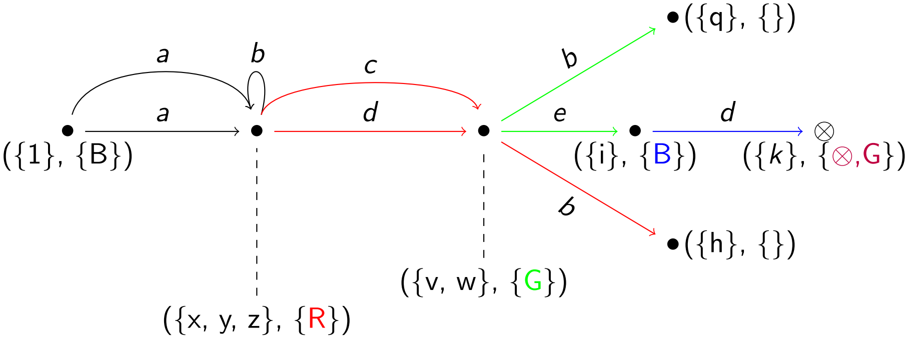

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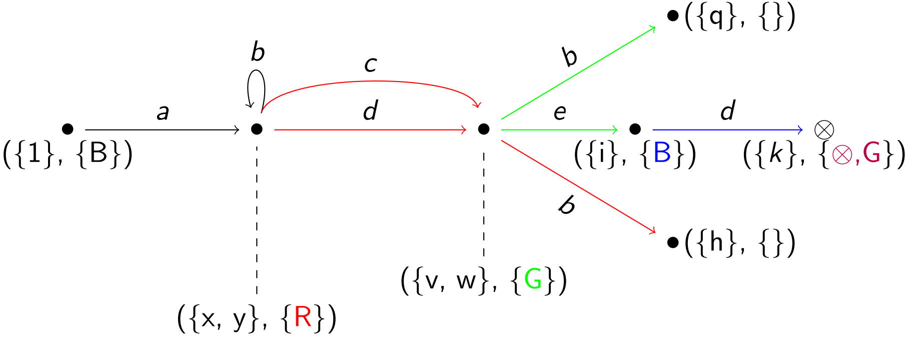

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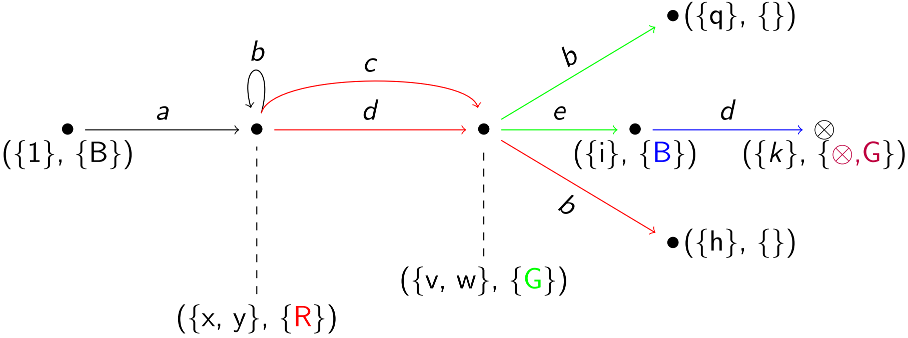

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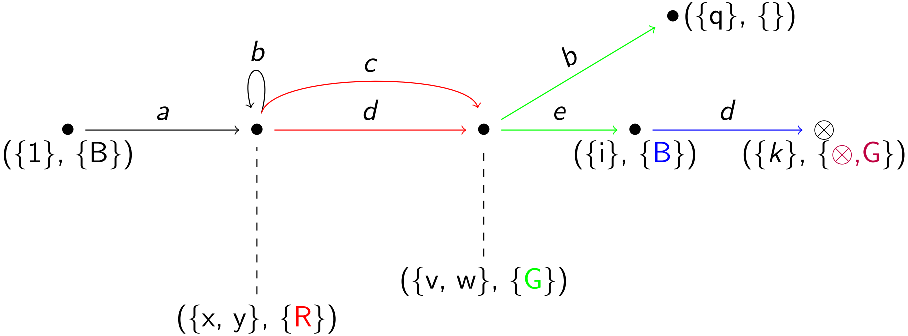

Let Σ be a pinched 1-rooted c-graph.

If there exists an A-subgraph of Σ, which is a linear path and a tail of a B-subgraph of Σ, then we color the A-subgraph in B and denote it by $\mathbf{d}(\Sigma)$.

Reducing a Pinched Graph: Step 2-coloring

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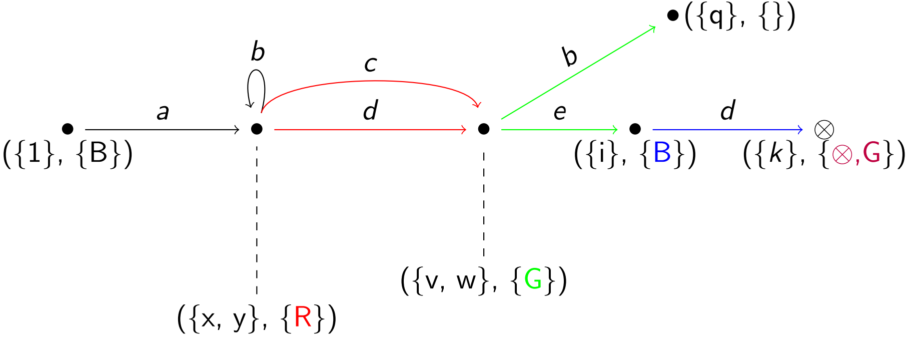

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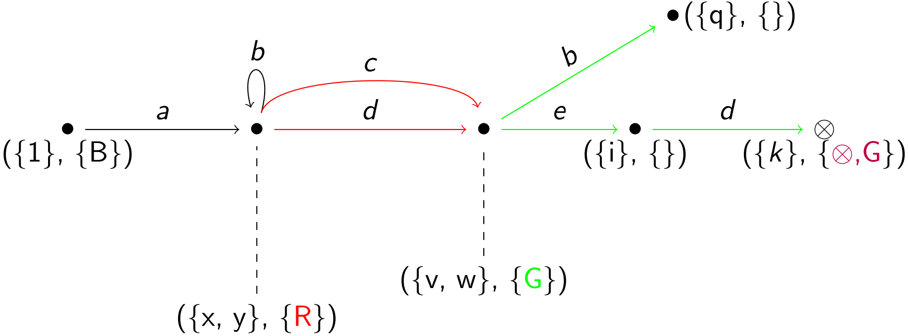
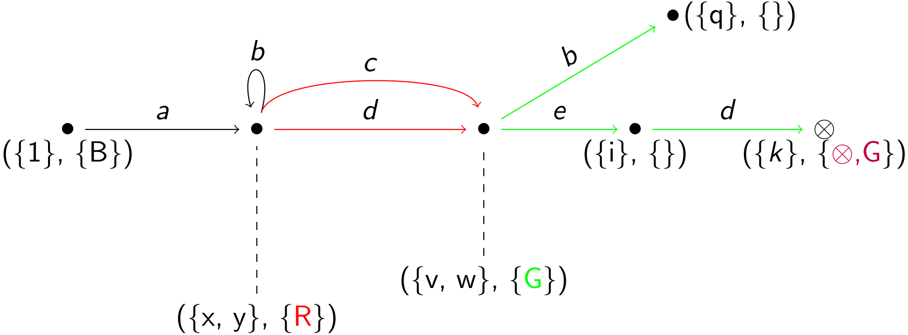


Reducing a Pinched Graph: Step 3-pinching

$\mathbf{pd}(\Sigma) :=$ the pinched graph of $\mathbf{d}(\Sigma)$.



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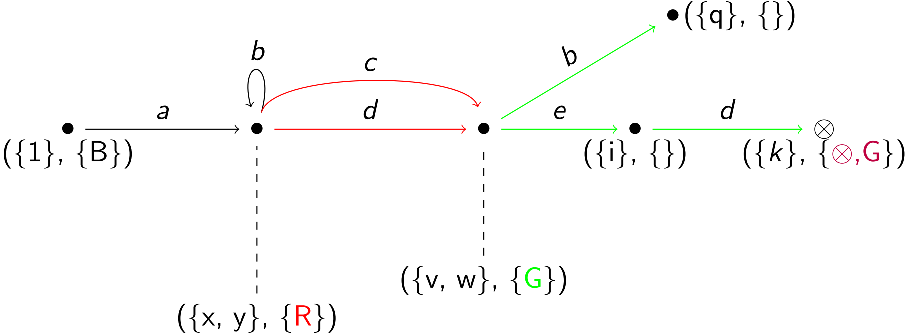

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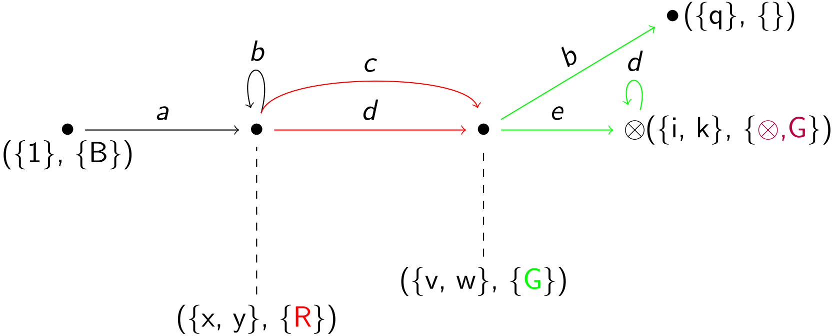

A pinched 1-rooted c-graph Σ is reduced if there is no any non-trivial retract of Σ and $\mathbf{pd}(\Sigma) = \Sigma$.

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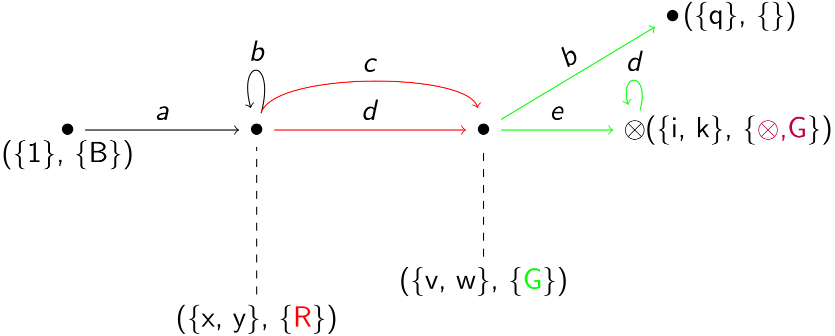

A pinched 1-rooted c-graph Σ is reduced if there is no any non-trivial retract of Σ and $\mathbf{pd}(\Sigma) = \Sigma$.

Lemma 1 Let P, Q be 1-rooted c-graphs and U, V be linear 1-rooted c-graphs satisfying that $Uf = Vf$. If $\Sigma = P \times U^\dagger \times V \times Q$ and $\Gamma = P \times V^\dagger \times U \times Q$, then $\mathbf{rp}(\Sigma) = \mathbf{rp}(\Gamma)$.

Lemma 2 If Σ is an 1-rooted c-graph and Γ is a retract of Σ, then $\mathbf{rp}(\Sigma) = \mathbf{rp}(\Gamma)$.

Theorem 1 If Σ and Γ are 1-rooted c-graphs with $\Sigma \ \rho^* \ \Gamma$, then $\mathbf{rp}(\Sigma) = \mathbf{rp}(\Gamma)$.

Remark: $\Sigma \ \rho^* \ \Gamma$ if and only if there exists a sequence

$$\Sigma = P_1 U_1^\dagger V_1 Q_1 \quad P_2 V_2^\dagger U_2 Q_2 = \cdots = P_n V^\dagger U_n Q_n$$

$$P_1 V_1^\dagger U_1 Q_1 = P_2 U_2^\dagger V_2 Q_2 \quad \cdots \quad P_n U_n^\dagger V_n Q_n = \Gamma$$

where P_i, Q_i are 1-rooted c-graphs and U_i, V_i are linear 1-rooted c-graphs satisfying that $U_i f = V_i f$.

A Unique Reduced Pinched Graph in Each ρ^*-class

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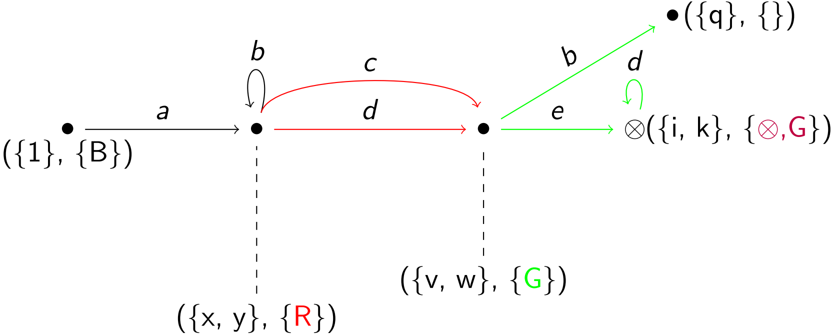
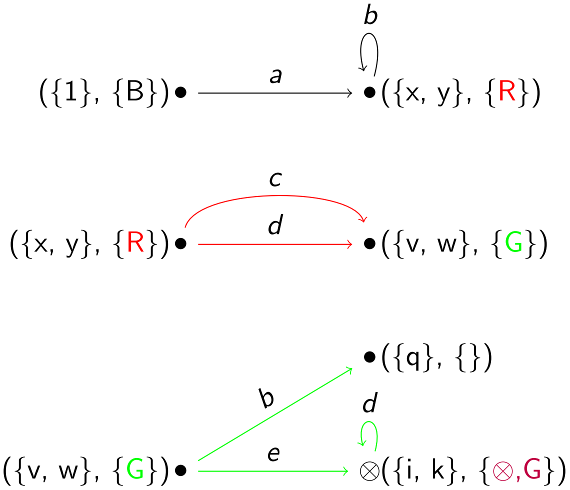
Unpinching methods

Examples:



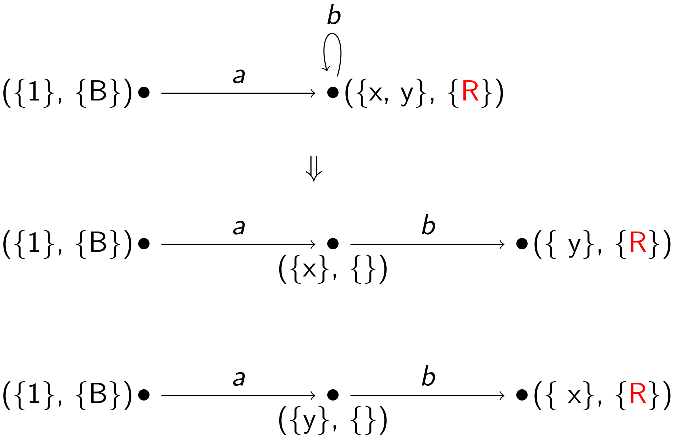
Unpinching methods

Step 1:



Unpinching methods

Step 2:



Unpinching methods

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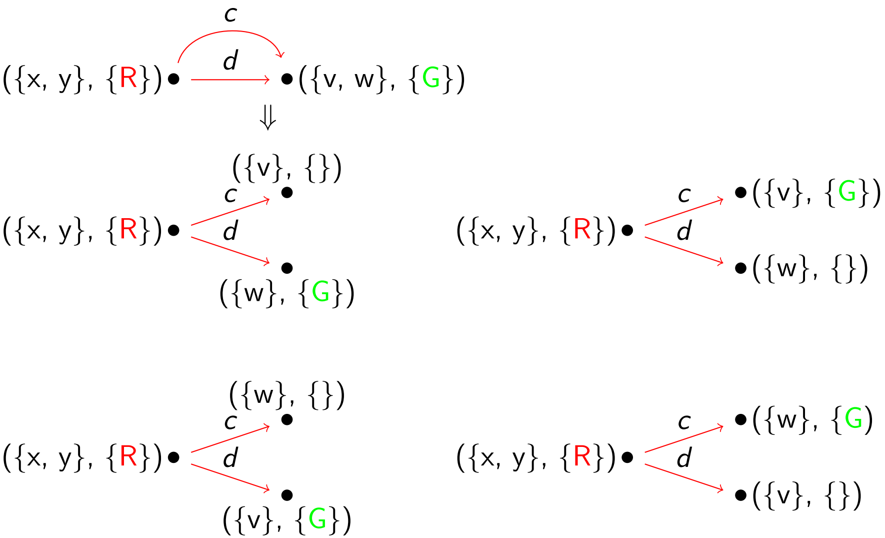

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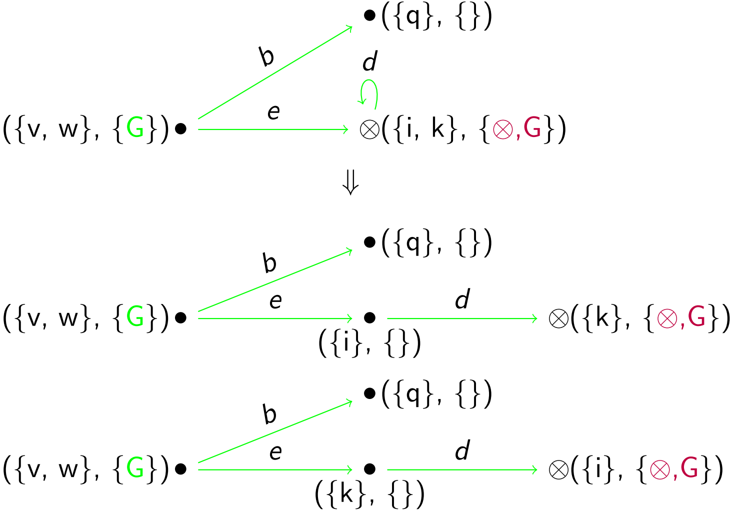
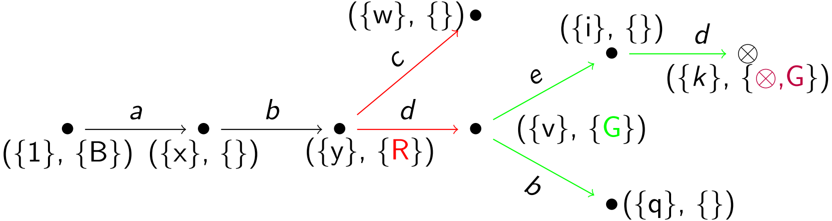
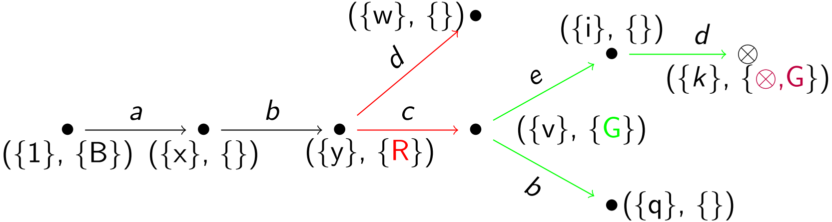


Unpinching Methods

Step 3:





Unpinching Graphs Being ρ^*-related

Theorem 2 Let Γ be a pinched graph, and let Γ_1 and Γ_2 be unpinching graphs of Γ. Then $\Gamma_1 \ \rho^* \ \Gamma_2$.

outline of the proof

- For each c-subgraph K of Γ, any two unpinching graphs of K are ρ^*-related.
- For any connected c-subgraphs K_1 and K_2 of Γ, if W_1 and W_2 are unpinching graphs of K_1, and U_1 and U_2 are unpinching graphs of K_2, then $W_1 U_1 \ \rho^* \ W_2 U_2$.

Proposition 1 Let Σ be a 1-rooted c-graph. Then $\Sigma \ \rho^* \ \mathbf{unp}(\Sigma)$.

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Word Problems

Theorem 3 If Σ and Γ are 1-rooted c-graphs with $\mathbf{rp}(\Sigma) = \mathbf{rp}(\Gamma)$, then $\Sigma\ \rho^*\ \Gamma$.

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Theorem 4 If Σ and Γ are 1-rooted c-graphs, then $\Sigma\ \rho^*\ \Gamma$ if and only if $\mathbf{rp}(\Sigma) = \mathbf{rp}(\Gamma)$.

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