Solving equation systems in ω-categorical algebras

Thomas Quinn-Gregson
TU Dresden
Joint work with Manuel Bodirsky

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Our problem

Given an algebra A, we are interested in the problem of deciding whether a given system of term equalities and inequalities has a solution in A.

Example

Let L be a left zero semigroup. Does the following system have a solution over $(L; \cdot)$?

$$
\begin{aligned}
x_1 \cdot x_2 &= x_3 \cdot x_4, \\
x_3 \cdot x_4 \cdot x_5 &= x_2, \\
x_2 \cdot x_5 &\neq x_1 \cdot x_3.
\end{aligned}
$$

Equivalent: does $x_1 = x_3, x_3 = x_2, x_2 \neq x_1$ have a solution in $(|L|; \neq)$?

- Given an algebra A, can we create a "fast" algorithm which solves any given system over A?
- **Spoilers:** The problem for a left zero semigroup is solvable in polynomial time when $|L| = 1, 2$ or infinite.

Part 1: CSPs

A rough definition

A **constraint satisfaction problem** consists of:

1. a finite list of variables V,
2. a domain of possible values A,
3. a set of constraints on those variables $\mathcal{C}$.

Problem: Can we assign values to all the variables so that all the constraints are satisfied?

Example (Graph 3-colouring)

Let G be a finite graph. Each vertex can be coloured either red, green or blue. Problem: can we colour G such that no two adjacent variables have the same colour?
$V =$ vertices of G.
$A = \{\text{Red, Green, Blue}\}$.
$\mathcal{C} =$ "no two adjacent vertices have the same colour".

Constraint language

Much attention has been paid to the case where the constraints arise from finitely many relations and functions on a fixed domain.

Definition

Given a (first-order) structure $(A; \Gamma)$ where Γ is finite, we define $\mathrm{CSP}(A; \Gamma)$, or simply $\mathrm{CSP}(\Gamma)$, to be the CSP with:

- **Instance:** $I = (V, A, \mathcal{C})$ in which each constraint is simply a relation from Γ.
- **Question:** Does I have a solution?

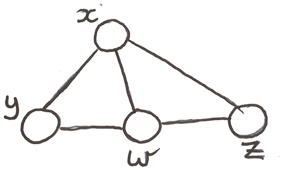
Example

Graph 3-colouring can be considered as $\mathrm{CSP}(A; \neq)$ where $A = \{R, B, G\}$ i.e. $\mathrm{CSP}(K_3)$, where K_3 is the complete graph on 3 vertices.

Graph 3-colouring

Instance: $x \neq y$, $x \neq z$, $x \neq w$,
$y \neq w$, $z \neq w$.

Graph:

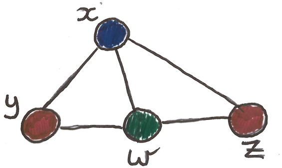


Q: Can we colour the vertices Red, Blue or Green such that no adjacent vertices are the same colour?

3-Graph colouring

Instance: $x \neq y$, $x \neq z$, $x \neq w$, $y \neq w$, $z \neq w$.

Graph:



Q^{n}: Can we colour the vertices Red, Blue or Green such that no adjacent vertices are the same colour?

Computational Complexity

Key question: How does the structure $\mathcal{A}$ affect the computational complexity of CSP($\mathcal{A}$)?

Definition

1. *P*: the class of all problems solved in polynomial time. Its members are called **tractable**.
2. *NP*: the class of problems solvable in nondeterministic polynomial time.
3. *NP*-hard: the class of problems which at least as hard as the hardest problems in *NP*.
4. *NP*-complete: the class of problems which are *NP* and *NP*-hard (the "hardest problems in *NP*").

Theorem (Ladner, 1975)

If $P \neq NP$ then there are problems in $NP \setminus P$ that are not NP-complete.

Dichotomy Theorem for finite structures

Example

Graph n-colouring is NP-complete if $n > 2$, and tractable otherwise. Equivalently, $\mathrm{CSP}(\mathbf{n};\neq) = \mathrm{CSP}(K_n)$ is NP-complete when $n > 2$, and tractable otherwise.

Example (Hell and Nešetři, 90')

Let G be a finite undirected graph. Then $\mathrm{CSP}(G)$ is either tractable (if bipartite) or NP-complete.

Theorem (Dichotomy Theorem (Bulatov, Zhuk 17'))

Let $\mathcal{A}$ be a finite structure. Then CSP($\mathcal{A}$) is either tractable or is NP-complete.

Part 2: CSPs arising from algebras

System of equations satisfactibility

Definition (The *system of equations satisfactibility* problem)

Given a finite algebra $\mathcal{A} = (A; F)$, the problem $\text{EQN}^*_{\mathcal{A}}$ is:
Instance: a system of equations $\mathcal{E}$ over $\mathcal{A}$ (constants and variables).
Question: does $\mathcal{E}$ have a solution?

Example

Consider the abelian group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. An instance of $\text{EQN}^*_{\mathbb{Z}_5}$ could be

$$\begin{aligned} x + y &= 1, \\ z + u + 2 &= x + v, \\ u &= v + 1. \end{aligned}$$

Solve by Gaussian Elimination e.g. $x = v = 0, y = u = 1, z = 2$.

System of equations satisfactibility

The problem $\mathrm{EQN}^*_{\mathcal{A}}$ is equivalent to $\mathrm{CSP}(\mathcal{A}, c_1, \ldots, c_n)$ where $A = \{c_1, \ldots, c_n\}$.

Theorem (Goldmann, Russell 2002)

Let G be a finite group. If G is abelian then EQN^*_G is tractable, and is NP-complete otherwise.

Theorem (Klíma, Tesson, Thérien 2007)

Every CSP over a finite domain is polynomial-time equivalent to EQN^*_S for some finite semigroup S.

To infinity...

We are interested in building non-trivial CSPs from an infinite algebra $\mathcal{A} = (A; F)$. Possibilities include:

1. $\mathrm{EQN}^*_{\mathcal{A}}$

Pro: Natural problem.

Cons: $\mathrm{CSP}(\mathcal{A}, a : a \in A)$ has an infinite language.

2. Get rid of constants i.e. $\mathrm{CSP}(\mathcal{A})$.

Pros: Natural problem, finite language.

Con: Often a trivial problem e.g. if A is a group, then every equation can be solved by substituting in the identity element.

3. Replace constants by disequality i.e. $\mathrm{CSP}(\mathcal{A}, \neq)$

Pros: natural problem, finite language, non-trivial, core,...

Con: rather boring for finite algebras - NP-complete if $2 < |A| < \omega$.

Our problem

We study CSP$(A, \neq)$ for algebras A. Motivation include:

- **A natural problem:** CSP$(A, \neq)$ is polynomial time equivalent to the problem of deciding whether a given set of term equalities and inequalities has a solution in A.
- **A non-trivial problem:** As we will see, even in our very restrictive setting we obtain both tractability and hardness.
- **Constraint entailment:** Testing if a list of equations $\mathcal{E}$ implies an equation $u = v$ is equivalent of testing if $\mathcal{E} \cup \{u \neq v\}$ is satisfiable.
- **The Identity Checking Problem ICP(A)**: CSP$(A, \neq) \in P \Rightarrow$ ICP$(A) \in P$.
- **Sporadically studied problem:** CSP$(A, \neq)$ for a number of well-known algebras have served as key examples:
  - the lattice reduct of the atomless Boolean algebra $(\mathbb{A}; \cup, \cap)$ (NP-hard; Bodirsky, Hils, Krimkevitch, 2011)
  - the infinite-dimensional vector space over the finite field $\mathbb{F}_q$ (tractable; Bodirsky, Chen, Kára, von Oertzen, 2007).

ω-categoricity

Much progress has been made in understanding the CSPs of infinite structures: often in the (highly symmetric) ω-categorical setting.
e.g. If M and N are ω-categorical then CSP(M)=CSP(N) if and only if $M \to N$ and $N \to M$.

Definition

A structure M is ω-**categorical** if Th(M) has one countable model, up to isomorphism. Equivalently, if Aut(M) has only finitely many orbits on its action on M^n for each $n \geq 1$.

Example

A right zero semigroup S has $\text{Aut}(S) = \mathcal{S}_{|S|}$ and is ω-categorical:

- $\forall x, y, z \ [(xy)z = x(yz)]$
- $\forall x, y \ [xy = y]$
- 'correct cardinality'

ω-categoricity

Example

An abelian group is ω-categorical if and only if it has finite exponent i.e. $\exists n \in \mathbb{N}$ with $g^n = 1$ for all $g \in G$.

Example

CSP($\mathbb{Q}; <$) and CSP($\mathbb{N}; \neq$) are tractable (!).

Well studied ω-categorical algebras also include:

- Groups (Rosenstein, Felgner, Apps,...),
- Rings (Baldwin, Rose,...),
- Semigroups (my PhD,...),
- Boolean algebras (classified - finitely many atoms),
- Fields (must be finite).

Part 3: The power of polymorphisms

Polymorphisms

- The hardness of a problem often comes from a lack of symmetry.
- Our usual objects that capture symmetry (automorphism group or endomorphism monoid) are not sufficient.
- We require a more general symmetry - polymorphisms!

Definition

A **polymorphism** of a structure M is an n-ary homomorphism $f : M^n \to M$. The set of all polymorphisms of M is denoted $\mathrm{Pol}(M)$.

For any structure M, the set $\mathrm{Pol}(M)$ forms a *clone* i.e. is closed under composition and contains the projections.

Polymorphisms of $(\mathcal{A}; \neq)$

Lemma

Let $\mathcal{A} = (A; F)$ be an algebra. Then $f : A^n \to A$ is a polymorphism of $(\mathcal{A}, \neq)$ if and only if f is an algebra homomorphism and

$$x_1 \neq y_1, \ldots, x_n \neq y_n \Rightarrow f(x_1, \ldots, x_n) \neq f(y_1, \ldots, y_n)$$

or, equivalently, if

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n) \Rightarrow x_i = y_i \text{ for some } 1 \leq i \leq n.$$

In particular, every endomorphism of $(\mathcal{A}, \neq)$ is an embedding.
i.e. $\mathcal{A}$ is a core.

Hence if $\mathcal{A}$ and $\mathcal{B}$ are ω-categorical then CSP$(\mathcal{A}, \neq)$=CSP$(\mathcal{B}, \neq)$ if and only if A and B are *bi-embeddable* i.e. $A \hookrightarrow B$ and $B \hookrightarrow A$.

We can thus work up to bi-embeddablity!

Siggers vs pseudo-Siggers

Definition

A 6-ary operation $f \in \mathrm{Pol}(\mathcal{A})$ is called a **Siggers** polymorphism if

$$f(x, y, x, z, y, z) \approx f(y, x, z, x, z, y).$$

For finite CSPs, the existence of a *Siggers* polymorphism is necessary and sufficient for tractability (Bulatov, Zhuk 2017).
For infinite structures this is no longer true...We need greater generality!

Definition

A 6-ary operation $f \in \mathrm{Pol}(\mathcal{A})$ is called a **pseudo-Siggers** polymorphism if

$$\alpha f(x, y, x, z, y, z) \approx \beta f(y, x, z, x, z, y)$$

for some unary operations $\alpha, \beta \in \mathrm{Pol}(\mathcal{A})$.

Model-complete

We call a structure M **model-complete** if every first-order sentence is equivalent to an existential sentence over M.

Theorem (Bodirsky, 07')

Every ω-categorical structure is homomorphically equivalent to a model-complete core, which is unique and ω-categorical.

Corollary

Let $\mathcal{A}$ be an ω-categorical algebra. Then there exists a unique ω-categorical algebra $\mathcal{B}$ which is bi-embeddable with $\mathcal{A}$ and with $(\mathcal{B}, \neq)$ model-complete.

Example

The abelian groups $\mathbb{Z}_2 \oplus \mathbb{Z}_4^{(\omega)}$ and $\mathbb{Z}_4^{(\omega)}$ are bi-embeddable, and $(\mathbb{Z}_4^{(\omega)}, \neq)$ is model-complete.

The pseudo-Siggers theorem

Let $\mathscr{P}$ denote the clone of projections on a two-element set.

Theorem (Barto, Pinsker 06')

Let M be an ω-categorical structure which is a model-complete core. Then at least one of the following holds.

- *M has a pseudo-Siggers polymorphism.*
- *M Pol(M) is "small" (has a uniformly continuous minor-preserving map to $\mathscr{P}$); in this case, CSP(M) is NP-hard.*

However, the two possibilities in the theorem are not mutually exclusive.

If $\mathbb{A}$ is the atomless Boolean algebra then $(\mathbb{A}; \neq)$ has a pseudo-Siggers polymorphism, but $\text{Pol}(\mathbb{A}, \neq)$ has a u.c. minor-preserving map to $\mathscr{P}$.

Aim

Show that a dichotomy exists for both abelian groups and semilattices; Either $(A; \neq)$ has a pseudo-Siggers polymorphism, or Pol($A; \neq$) has a u.c. minor-preserving map to $\mathscr{P}$.

Part 4: Groups

Groups

- Given an ω-categorical algebra $\mathcal{A}$, if f is a pseudo-Siggers operation of $(\mathcal{A}, \neq)$ then for all $x, y, z, u, v, w \in A$

$$
\begin{aligned}
& f(x,y,x,z,y,z) = f(u,v,u,w,v,w) \qquad \text{(Property 1)} \\
\Leftrightarrow \quad & f(y,x,z,x,z,y) = f(v,u,w,u,w,v).
\end{aligned}
$$

- Let G be a group with identity 1 and $f \in Pol(G; \neq)$. Then

$$f(x_1, \ldots, x_n) = f(x_1, 1, 1, \ldots, 1) f(1, x_2, 1, \ldots, 1) \cdots f(1, 1, \ldots, 1, x_n).$$

- This, together with Property (1) shows that if $(G; \neq)$ has a pseudo-Siggers polymorphism then it is 'close' to being bi-embeddable with $G \times G$.

Groups

Proposition (Bodirsky, TQG)

Let G be an ω-categorical group such that $(G, \neq)$ has a pseudo-Siggers polymorphism. Then at least one of the following holds.

- *G is bi-embeddable with $G \times G$.*
- *G is bi-embeddable with $G \times (G/\langle x \rangle)$ for some $x \in G$ of order 2.*

Rough proof.

One of the maps $x \mapsto f(x, 1, x, 1, 1, 1)$ and $x \mapsto f(1, x, 1, x, x, x)$ is injective as f preserves $\neq$:

$$f(1, x, 1, x, x, x) = 1 = f(y, x, y, x, x, x) \Rightarrow f(y, x, y, x, x, x) = 1.$$

Similarly, their images are disjoint. For $y \neq 1$, use Property (1):

$$f(1, y, 1, y, y, y) = 1 = f(1, 1, 1, 1, 1, 1) \Rightarrow f(y, 1, y, 1, y, y) = 1.$$

Hence $f(y, y, y, y, y^2, y^2) = 1 = f(1, 1, 1, 1, 1, 1)$, so $y^2 = 1$ etc... □

Abelian groups

Theorem

Every abelian group of finite exponent is a direct sum of cyclic groups $\mathbb{Z}_n$.

It is then a relatively simple exercise to find those which satisfy the necessary condition to having a pseudo-Siggers:

Proposition (Bodirsky, TQG)

Let G be an abelian group of finite exponent. Then $(G, \neq)$ has a pseudo-Siggers polymorphism if and only if G is bi-embeddable with $\mathbb{Z}_m^{(\omega)}$ or with $\mathbb{Z}_m^{(\omega)} \oplus \mathbb{Z}_{2m}$ for some $m \geq 1$.

Abelian groups

Theorem (Bodirsky, TQG)

Let G be an ω-categorical abelian group. Then the following are equivalent:

(i) *$Pol(G, \neq)$ has no u.c. minor-preserving map to $\mathscr{P}$,*

(ii) *$(G, \neq)$ has a pseudo-Siggers polymorphism,*

(iii) *G is bi-embeddable with $\mathbb{Z}_m^{(\omega)}$ or with $\mathbb{Z}_m^{(\omega)} \oplus \mathbb{Z}_{2m}$ for some $m \geq 1$.*

Moreover, in this case $CSP(G, \neq)$ is in P, and is NP-hard otherwise.

Key: If M is an ω-categorical structure with both a pseudo-Siggers polymorphism and with $\mathrm{Pol}(M)$ having a u.c. minor-preserving map to $\mathscr{P}$, then M is not ω-stable.

General groups

The non-abelian case remains open.

In particular, we have no example of an ω-categorical non-abelian group G with $\mathrm{CSP}(G; \neq)$ in P.

Theorem (Sarcino, Wood 1982)

There are 2^ω distinct (up to isomorphism) ω-categorical groups which are pairwise non bi-embeddable.

$\Rightarrow \exists$ ω-categorical groups G such that $\mathrm{CSP}(G; \neq)$ is undecidable.

Part 5: Semilattices

Semilattices

- A **semilattice** is an algebra $(Y; \wedge)$ where $\wedge$ is an associative, commutative, and idempotent binary operation.
- There exists a unique ω-categorical semilattice which embeds all finite semilattices and is **homogeneous**. We call this the *universal semilattice*, denoted $\mathbb{U}$.
- $\mathbb{U}$ is bi-embeddable with the meet-reduct of the atomless boolean algebra $(\mathbb{A}; \wedge, \vee, \neg, 0, 1)$.

Lemma (Bodirsky, TQG)

CSP$(\mathbb{U}; \neq)$ is tractable.

While semilattices do not necessarily possess an identity, property (1) still proves to be useful for proving hardness of CSP$(Y; \neq)$.

$$\begin{aligned} & f(x,y,x,z,y,z) = f(u,v,u,w,v,w) \qquad \text{(Property 1)} \\ \Leftrightarrow \quad & f(y,x,z,x,z,y) = f(v,u,w,u,w,v). \end{aligned}$$

Semilattices

Theorem (Bodirsky, TQG)

Let Y be a non-trivial ω-categorical semilattice. Then CSP$(Y; \neq)$ is tractable if Y is bi-embeddable with $\mathbb{U}$, and is NP-hard otherwise.

Proof idea:

- Y is bi-embeddable with $\mathbb{U}$ if and only if it embeds all finite Boolean algebras $(\mathbb{P}_n; \wedge)$.
- Show that every $\mathbb{P}_n$ embeds into Y if CSP$(Y; \neq)$ is not NP-hard.
- Use induction: true for $n = 2$ (since CSP$(Y, \neq)$ is NP-hard for $Y = (\mathbb{Q}; \min)$ - Bodirsky '09).
- Induction step: Use the existence of a pseudo-Siggers polymorphism.

□

Dichotomy

Theorem (Bodirsky, TQG)

Let Y be a countable ω-categorical semilattice. Then either

(i) *there is a u.c. minor-preserving map from Pol(Y; $\neq$) to $\mathscr{P}$, in which case CSP($Y, \neq$) is NP-hard, or*

(ii) *the model-complete core of $(Y, \neq)$ is isomorphic to $(\mathbb{U}, \neq)$, in which case CSP($Y, \neq$) is in P.*

Proof.

The following (height one) identities, discovered by Jakub Rydval, are preserved by all minor-preserving maps and are not satisfied by $\mathscr{P}$:
There are $f, g_1, \ldots, g_4 \in \mathrm{Pol}(\mathbb{U}, \neq)$ such that for all $x, y \in \mathbb{U}$

$$\begin{array}{ll} g_1(y,x,x) = f(x,y,x,x), & g_2(y,x,x) = f(y,x,x,x), \\ g_1(x,y,x) = f(x,x,y,x), & g_2(x,y,x) = f(x,x,y,x), \ \textit{etc} \\ g_1(x,x,y) = f(x,x,x,y), & g_2(x,x,y) = f(x,x,x,y), \end{array}$$

□

Lattices

Similar occurrences holds for lattices:

- An ω-categorical lattice L in which $(L; \neq)$ has a pseudo-Siggers polymorphism is bi-embeddable with $L \times L$.
- If L is distributive then $\mathrm{CSP}(L; \neq)$ is NP-hard.
- However: the universal lattice (which embeds all finite lattices) is not ω-categorical.

Open: Let L be a non-distributive ω-categorical lattice which is bi-embeddable with $L \times L$. What is the computational complexity of $\mathrm{CSP}(L; \neq)$?

Key: Can we classify the ω-categorical (model-complete) lattices L such that L is bi-embeddable with $L \times L$?