Cosetal extensions of monoids

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York 2021

Part 1: Group Extension Theory

Split extensions of groups

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a split extension of groups.

There is a bijection $f \colon N \times H \to G$ given by $f(n,h) = k(n)s(h)$.

We may equip $N \times H$ with a group multiplication making f as iso.

Let $k\varphi(h,n) = s(h)k(n)s(h)^{-1}$ and notice that
$k\varphi(h,n)s(h) = s(h)k(n)$.

The multiplication is given by $(n_1,h_1)(n_2,h_2) = (n_1\varphi(h_1,n_2), h_1h_2)$.

$$\begin{aligned} f((n_1,h_1)(n_2,h_2)) &= f(n_1\varphi(h_1,n_2), h_1h_2) \\ &= k(n_1)k\varphi(h_1,n_2)s(h_1)s(h_2) \\ &= k(n_1)s(h_1)k(n_1)s(h_2) \\ &= f(n_1,h_1)f(n_2,h_2). \end{aligned}$$

Semidirect products of groups

The map φ is a group action of H on N.

Given any action φ we may construct the semidirect product $N \rtimes_\varphi H$ with multiplication as discussed.

Semidirect products naturally give split extensions $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ with $k(n) = (n, 1)$, $e(n, h) = h$ and $s(h) = (1, h)$.

They provide a full characterization of split extensions in this way.

Group extensions with abelian kernel

Suppose we have a group extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$, where N is abelian.

The map e is guaranteed to be a surjection and so we may consider a set theoretic splitting s (that preserves unit).

Again each element g may be written uniquely as $g = k(n)se(g)$.

Now if $e(g_1) = e(g_2)$ then $g_1 = k(n_1)se(g_1)$ and $g_2 = k(n_2)se(g_2) = k(n_2)se(g_1)$.

Thus we have the $k(n_1)k(n_2)^{-1}g_2 = k(n_1)se(g_1) = g_1$.

Hence if $e(g_1) = e(g_2)$ there exists a unique $n \in N$ such that $g_1 = k(n)g_2$.

Factor sets

We would like to carry out the semidirect product construction in this new setting. However, we did use that $s(h_1h_2) = s(h_1)s(h_2)$.

Notice that $e(s(h_1h_2)) = e(s(h_1)s(h_2))$ and so there exists a unique element $\chi(h_1, h_2)$ such that $k\chi(h_1, h_2)s(h_1h_2) = s(h_1)s(h_2)$.

We may again define $\varphi(h, n) = s(h)k(n)s(h)^{-1}$. Surprisingly this is again an action. (The proof makes use of the abelian kernel).

We may then define the crossed product $N \rtimes_{\varphi}^{\chi} H$ with underlying set $N \times H$ and multiplication
$(n_1, h_1)(n_2, h_2) = (n_1\varphi(h_1, n_2)\chi(h_1, h_2), h_1h_2)$.

$f((n_1, h_1)(n_2, h_2)) = k(n_1)k\varphi(h_1, n_2)k\chi(h_1, h_2)s(h_1h_2) =$
$k(n_1)k\varphi(h_1, n_2)s(h_1)s(h_2) = k(n_1)s(h_1)k(n_2)s(h_2) =$
$f(n_1, h_1)f(n_2, h_2)$.

Second cohomology group

Factor sets $\chi\colon H \times H \to N$ may be defined generally relative to an action φ.

1. $\chi(1, h) = 1 = \chi(h, 1)$,
2. $\chi(x, y)\chi(xy, z) = \varphi(x, \chi(y, z))\chi(x, yz)$

The associated crossed product forms an extension
$N \xrightarrow{k} N \rtimes_{\varphi}^{\chi} H \xrightarrow{e} H$ where $k(n) = (n, 1)$ and $e(n, h) = h$.

These do not form a full characterisation as χ depends on the choice of splitting.

They have a natural abelian group structure. Quotienting by the subgroup of inner factor sets yields a full characterization.

This induces a Baer sum on the set of extensions.

Part 1: Monoid Extensions

Schreier split extensions

Given a split extension of monoids $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ can we extract an action?

In general no. In the group setting me made use of conjugation.

Schreier split extensions $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ satisfy that each g may be written uniquely as $k(n)se(g)$.

Thus there exists a unqiue element $\varphi(h,n) \in N$ such that $s(h)k(n) = k\varphi(h,n)se(s(h)k(n)) = k\varphi(h,n)s(h)$.

These give an action φ and in the group setting we only ever used that $\varphi(h,n)s(h) = s(h)k(n)$.

The entire argument carries through.

Special Schreier extensions

A monoid extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is special Schreier if whenever $e(g_1) = e(g_2)$ there exists a unique n such that $g_1 = k(n)g_2$.

If s is a set theoretic splitting of e, then there exists a $\chi(h_1, h_2)$ such that $s(h_1)s(h_2) = k\chi(h_1, h_2)s(h_1h_2)$.

When N is an abelian group we can extract an action φ of H on N.

The argument now completely carries through.

Okay, but what about other extensions?

λ-semidirect products

Given two inverse semigroups H and N and an action of H on N we may form the λ-semidirect product $N \rtimes_\varphi H$.

It has underlying set $\{(n,h) \in N \times H : \varphi(hh^{-1}, n) = n\}$ and multiplication
$(n_1, h_1)(n_2, h_2) = (\varphi(h_1h_2(h_1h_2)^{-1}, n_1)\varphi(h_1, n_2), h_1h_2)$

These form a split extension $N \overset{k}{\rightarrowtail} N \rtimes_\varphi H \underset{s}{\overset{e}{\rightleftarrows}} H$ where $k(n) = (n, 1)$, $e(n, h) = h$ and $s(h) = (1, h)$*

Now suppose $(n, h) \in N \rtimes_\varphi H$, we have
$k(n)s(h) = (n, 1)(1, h) = (\varphi(hh^{-1}, n), h) = (n, h)$.

However any n' satisfying that $\varphi(hh^{-1}, n') = n$ would also give $k(n')s(h) = (n, h)$.

Leech's normal extensions

Leech consider extensions $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ in which $gN = Ng$ for all g and where H is the monoid of cosets.

These are not in general special Schreier. Consider $\mathbb{Z} \overset{k}{\rightarrowtail} \mathbb{Z} \cup \{\infty\} \overset{e}{\twoheadrightarrow} 2$ where $k(n) = n$, $e(n) = \top$ and $e(\infty) = \bot$.

Since everything is commutative it is clearly Leech normal.

But $e(\infty) = e(\infty)$ and yet $k(n) + \infty = \infty = k(n') + \infty$.

Again we have a failure of uniqueness.

Weakly Schreier split extensions

Let $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ be a split extension of monoids. When we require that for each g there exists a (not necessarily unique) n such that $g = k(n) \cdot se(g)$ we call the extension weakly Schreier.

Weakly Schreier extensions may be characterized by a generalization of a semidirect product.

The map $t \colon N \times H \to G$ is now only a surjection.

We may thus quotient $N \times H$ by the equivalence relation $(n, h) \sim (n', h') \iff k(n)s(h) = k(n')s(h')$.

This induces a bijection $\bar{t} \colon N \times H/{\sim} \to G$ and the quotient then inherits a multiplication from G.

We call the combination of this equivalence relation and data specifying the multiplication a relaxed action.

Admissible equivalence relations

The equivalence relation E satisfies the following properties.

0. $(n_1, h_1) \sim (n_2, h_2)$ implies $h_1 = h_2$,
1. $(n_1, 1) \sim (n_2, 1)$ implies $n_1 = n_2$,
2. $(n_1, h) \sim (n_2, h)$ implies $(nn_1, h) \sim (nn_2, h)$,
3. $(n_1, h) \sim (n_2, h)$ implies $(n_1, hh') \sim (n_2, hh')$.

By condition 0 we may view E as an H-indexed equivalence relation.

If H has the divisibility order then the the map from H into equivalence relations

1. Preserves bottom,
2. Selects right conguences,
3. Preserves order.

Any such equivalence relation we call an H-relaxtion of N.

Compatible actions

We know that there exist $\varphi(h, n) \in N$ such that $k\varphi(h, n)s(h) = s(h)k(n)$.

The function φ may be characterised as follows.

Let $\varphi\colon H \times N \to N$ be a function.

1. $\varphi(h, nn') \sim^h \varphi(h, n) \cdot \varphi(h, n')$,
2. $\varphi(hh', n) \sim^{hh'} \varphi(h, \varphi(h', n))$,
3. $\varphi(h, 1) \sim^h 1$,
4. $\varphi(1, n) \sim^1 n$,
5. $n_1 \sim^h n_2$ implies $n_1\varphi(h, n) \sim^h n_2\varphi(h, n)$,
6. $n \sim^{h'} n'$ implies $\varphi(h, n) \sim^{hh'} \varphi(h, n')$,

Some of these actions give the same multiplication so we quotient them by $\varphi_1 \sim \varphi_2 \iff \varphi_1(h, n) \sim^h \varphi_2(h, n)$ for all $h \in H$ and $n \in N$.

Relaxed actions

We call an H-relaxation E and a compatible action φ, a relaxed action action $(E, [\varphi])$.

The idea is that in the group setting we were able to verify a number of identities involving the action by right multiplying equations by $s(h)$ and then cancelling later.

We cannot do this for monoids and so the H-relaxation remembers the $s(h)$.

This is sufficient to generalise all of the previous cases.

Characterizing weakly Schreier extensions

Let $(E, [\varphi])$ be a relaxed action of H on N.

Theorem

The set $N \times H/E$ equipped with multiplication

$$[n_1, h_1] \cdot [n_2, h_2] = [n_1 \varphi(h_1, n_2), h_1 h_2],$$

is a monoid.

Theorem

The diagram

$$N \overset{k}{\rightarrowtail} N \times H/E \underset{s}{\overset{e}{\rightleftarrows}} H$$

where $k(n) = [n, 1]$, $e([n, h]) = h$ and $s(h) = [1, h]$, is a weakly Schreier extension.

Cosetal extensions

We can construct a theory associated to relaxed actions in the obvious way.

A (right) cosetal extension $N \overset{k}{\rightarrowtail} G \overset{e}{\twoheadrightarrow} H$ is an extension in which if $e(g) = e(g')$ then there exists an n such that $k(n)g = g'$.

These are precisely the monoid extensions in which H is the monoid of right cosets of N.

These generalise special Schreier extensions, Leech's extensions of groups by monoids, Fulp and Steppe's central monoid extensions.

Each cosetal extension has an associated relaxed action.

Extracting the equivalence relation

Let $N \overset{k}{\rightarrowtriangle} G \overset{e}{\twoheadrightarrow} H$ be a cosetal extension and s a set theoretic splitting of e.

We may define an H-indexed equivalence relation where $n \sim^h n' \iff k(n)s(h) = k(n')s(h)$. This is an H-relaxation of N.

If s' is another splitting then $e(s(h)) = e(s'(h))$ and so there exists an a such that $k(a)s(h) = s'(h)$.

Now if $k(n)s(h) = k(n')s(h)$ then consider the following calculation.

$$\begin{aligned} k(n)s'(h) &= k(n)k(a)s(h) \\ &= k(a)k(n)s(h) \\ &= k(a)k(n')s(h) \\ &= k(n')s'(h). \end{aligned}$$

Extracting the action

For the action note that $e(s(h)) = h = e(s(h)k(n))$ thus there exist $\varphi(h,n)$ such that $k\varphi(h,n)s(h) = s(h)k(n)$.

For any choice of these $\varphi(h,n)$ they form a compatible action relative to the equivalence relation discussed.

To see that $\varphi(h,nn') \sim^h \varphi(h,n)\varphi(h,n')$ observe

$$\begin{aligned}
\varphi(h,nn')s(h) &= s(h)k(nn') \\
&= s(h)k(n)k(n') \\
&= \varphi(h,n)s(h)k(n') \\
&= \varphi(h,n)\varphi(h,n')s(h).
\end{aligned}$$

All choices of $\varphi(h,n)$ give compatible actions which are equivalent.

Relaxed factor sets

Cosetal extensions may then be characterised by a relaxed action and a class of relaxed factor sets.

These are function $g\colon H \times H \to N$ satisfying that

$$g(x, y)g(xy, z) \sim^{xyz} \varphi(x, g(y, z))g(x, yz)$$

The set of these relaxed factor sets form a group.

Quotienting by an appropriate notion of relaxed inner factor set gives the second cohomology group $\mathcal{H}_2(N, H, E, \varphi)$.

Its elements correspond to cosetal extensions with associated relaxed action (E, φ).