The Number of Countable Subdirect Powers of Finite Unary Algebras

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Joint work with Nik Ruškuc

2021

Introductory Definition: Unary Algebras

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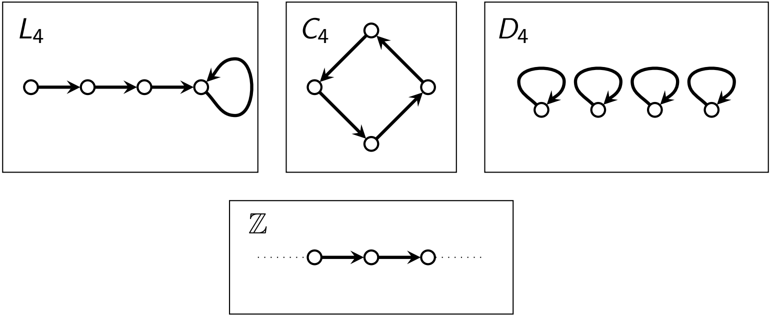

Useful ideas

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The format of a is an equivalence relation on X with (x, y) is in the format iff $a_x = a_y$.

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Projection Maps:

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Can be extended to an arbitrary number of factors.

Universality

Theorem

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An algebra is subdirectly irreducible if whenever it is expressed as a subdirect product of $\Pi_{i \in I} A_i$, then some projection π_i is an isomorphism.

Fiber Products

For algebras A, B, Q and surjective homomorphisms $\phi : A \to Q$ and $\psi : B \to Q$,

$$\{(a, b) \in A \times B : \phi(a) = \psi(b)\}$$

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Theorem

(Fleischer's Lemma) Every subdirect product of two algebras in a congruence permutable variety is a fiber product.

Boolean Powers

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Theorem

(Ruškuc, de Witt) A finite unary algebra $(A, \mathcal{F})$ has countably many non-isomorphic countable subdirect iff each $f \in \mathcal{F}$ is either a bijection or a constant map.

Monounary case

Lemma

Let (A, f) be a finite monounary algebra, and let f be a bijection. Then A has countably many non-isomorphic subdirect powers.

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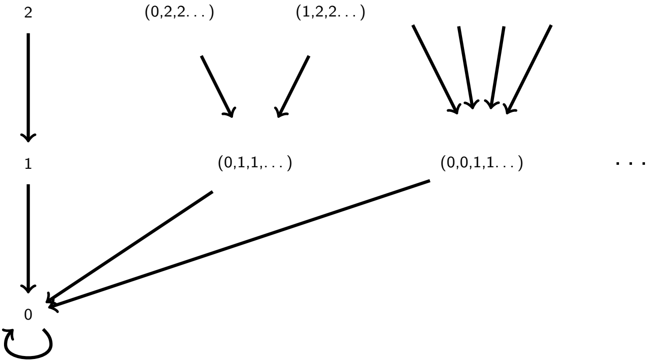

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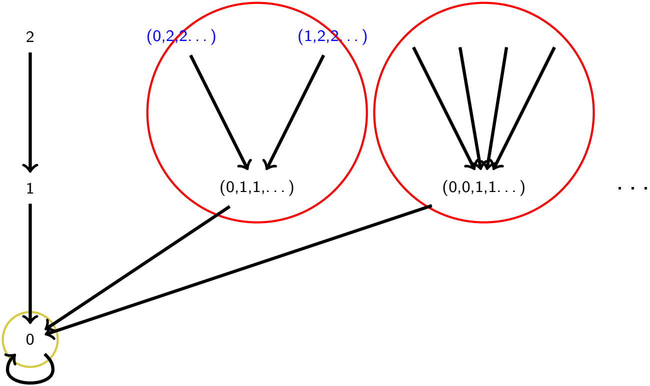

Tools for Unary

Definition

Let $(A, \mathcal{F})$ be a unary algebra. Then we define the following:

1. $B \subseteq A$ is a bottom level component if it is strongly connected and for all $a \in B$ and $f \in \mathcal{F}$, we have $f(a) \in B$.
2. for a bottom level component B, an outer section of A with respect to B is a connected component of the graph $A \setminus B$.
3. $T \subseteq A$ is a top level component if it is strongly connected and there does not exist $a \in A \setminus T$ and $f \in \mathcal{F}$ such that $f(a) \in T$.

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Lemma

The above are preserved under isomorphism.

Tools for Uncountable Type

Lemma

Let $(A, \mathcal{F})$ be a finite unary algebra, and $\mathrm{a} \in A^{\mathbb{N}}$ be a tuple with $\mathrm{cont}(\mathrm{a}) = A$. *Then the set* $\{f_1 \circ \cdots \circ f_n(\mathrm{a}) : f_1, \ldots, f_n$ *are bijections in* $\mathcal{F}, n \in \mathbb{N}\}$ *is a top level component of* $A^{\mathbb{N}}$.

Proof outline

Let $\text{Mon}(A)$ be the monoid of functions on A generated by $\mathcal{F}$,and pick an $g \in \text{Mon}(A)$ such that $|g(A)| > 1$ is minimal.

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Take arbitrary unions of the S_n, and add in the diagonal to ensure subdirectness, giving uncountably many subdirect powers.

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T_2 has uncountably many countable subdirect powers.

$(\mathbb{N}, +1)$ has countably many subdirect powers.

Related Questions

Question

Does an algebra have countably many countable subdirect powers if and only if it is abelian?

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Question

Is being boolean separating algebras equivalent to having uncountably many countable subdirect powers?

Related Questions

For finite groups we know the answer:

Countably many subdirect powers $\Rightarrow$ Non-Boolean Separating

Countably many subdirect powers $\not\Leftarrow$ Non-Boolean Separating

Abelian $\Rightarrow$ Countably many subdirect powers

Countably many subdirect powers $\Rightarrow$ Abelian

Abelian $\Rightarrow$ Non-Boolean Separating

Non-Boolean Separating $\not\Rightarrow$ Abelian

Lawrence,1981

Abelian

Related Questions

Using our results we have the following for the general case:

Countably many subdirect powers $\Rightarrow$ Non-Boolean Separating

Countably many subdirect powers $\not\Leftarrow$ Non-Boolean Separating

$\not\Nwarrow$ $\not\Swarrow$

Abelian

Thank you for listening