Semigroup Identities of Tropical Matrices

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Semigroup Identities

Let $\mathcal{A}$ be a countably infinite set of "letters" – an alphabet.

$\mathcal{A}^+ := (\mathcal{A}^+, \circ)$ is the free semigroup generated by $\mathcal{A}$, where $\circ$ is concatenation. Its elements are called words.

Def. *A* **semigroup identity** *is a formal equality*

$$\Pi : u = v,$$

where u and v are elements (words) of the free semigroup $\mathcal{A}^+$.

(For a monoid identity, we allow u and v to be the empty word.)

Semigroup Identities

A semigroup $S := (S, \cdot\ )$ **satisfies** a semigroup identity $\Pi : u = v$, if $\varphi(u) = \varphi(v)$ for every morphism $\varphi : \mathcal{A}^+ \to S$.

Prop. A semigroup that satisfies an n-letter identity, $n \geqslant 2$, also satisfies a 2-letter identity of the same length.

For $\mathcal{A} = \{a, b\}$, S satisfies an identity $\Pi : u = v$, written $\langle u, v \rangle \in \mathrm{Id}(S)$, if

$$u\,[\![s', s'']\!] = v\,[\![s', s'']\!]\,, \qquad a := s',\ b := s'',$$

for any $s', s'' \in S$.

Semigroup Identities: Examples

- A commutative semigroup satisfies the identity $\Pi : ab = ba$, written $ab\,[\![s', s''\,]\!] = ba\,[\![s', s''\,]\!]$.
- Any idempotent semigroup satisfies the identity $\Pi : a^n = a^m$ for any $n, m \in \mathbb{N}$.
- The semigroup

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

of 2×2 boolean matrices satisfies the identity $\Pi : a^2b^2 = b^2a^2$, written $a^2b^2\,[\![s', s''\,]\!] = b^2a^2\,[\![s', s''\,]\!]$.

Semigroup Identities: Groups

- Gromov's theory implies that every finitely generated group having polynomial growth satisfies a nontrivial semigroup identity (since it is virtually nilpotent).
- Shneerson gave examples that show that this does not hold for semigroups.

Semigroup Identities: Semigroups

Qu. *Given an identity $\Pi : u = v$ and a semigroup $S := (S, \cdot\ )$, does S satisfy the identity Π?*

Possible approach for solution:
Use a faithful linear representation

$$\rho : S \longrightarrow M_n(\mathbb{T}),$$

where $M_n(\mathbb{T})$ is the monoid of $n \times n$ tropical matrices, and then prove the identity for the image of S.

Qu. *Does the monoid $M_n(\mathbb{T})$ satisfy a nontrivial identity?*

The Tropical Semiring

A **semiring** $(R, +, \cdot\,)$ is a set R equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that $(R, \cdot\,)$ is a monoid and $(R, +)$ is a commutative monoid, with distributivity of multiplication over addition on both sides.

The **tropical semiring** $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ is the set of real numbers, equipped with the operations of maximum and summation

$$a \vee b := \max\{a, b\}, \quad a \cdot b := a + b,$$

providing respectively the addition and multiplication.

$\mathbb{T} := (\mathbb{T}, \vee\ , \cdot\ )$ is a commutative idempotent semiring whose unit is "1" $:= 0$ and whose zero is "0" $:= -\infty$.

Tropical Matrices

Matrices over the tropical semiring $\mathbb{T}$ are defined in the standard way. They form the semiring $M_n(\mathbb{T})$, whose addition and multiplication are induced by the operations of $\mathbb{T}$.

The unit matrix of $M_n(\mathbb{T})$ is the matrix

$$I = \begin{pmatrix} 0 & \dots & -\infty \\ \vdots & \ddots & \vdots \\ -\infty & \dots & 0 \end{pmatrix}.$$

$M_n(\mathbb{T})$ is referred to as a multiplicative monoid.

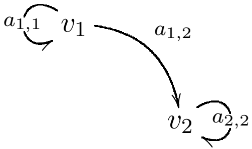
Tropical Matrices

Any $n \times n$ tropical matrix $A := (a_{i,j})$ corresponds uniquely to the weighted digraph $G_A = (V, E)$ defined to have vertex set $V = \{1, \ldots, n\}$, and an edge $(i, j) \in E$ from i to j, of weight $a_{i,j}$, whenever $a_{i,j} \neq -\infty$.

For example:

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} \\ -\infty & a_{2,2} \end{pmatrix}$$

corresponds to



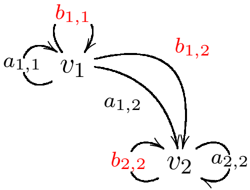
The (i, j)-entry of A^k gives the highest weight of a walk from i to j of length k.

Tropical Matrices

The entries of a matrix product AB correspond to labeled-weighted walks on the digraph $G_{AB} = G_A \cup G_B$.

For example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ -\infty & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ -\infty & b_{2,2} \end{pmatrix} \quad \Rightarrow$$



Tropical Matrices

- The **trace** $\operatorname{tr}(A) = \sum_i a_{i,i}$ is the usual trace taken with respect to summation, although it corresponds to the tropical product of diagonal entries.
- The **permanent** of a matrix $A := (a_{i,j})$ is defined as:

  $$\operatorname{per}(A) = \bigvee_{\pi \in \mathrm{S}_n} \sum_i a_{i,\pi(i)},$$

  where S_n is the set of all permutations over $\{1, \dots, n\}$.

  The **weight** of a permutation $\pi \in \mathrm{S}_n$ is $\omega(\pi) = \sum_i a_{i,\pi(i)}$, so that $\operatorname{per}(A) = \bigvee_{\pi \in \mathrm{S}_n} \omega(\pi)$.
- A is **nonsingular**, if there exists a unique permutation $\tau_A \in \mathrm{S}_n$ that reaches $\operatorname{per}(A)$; that is, $\operatorname{per}(A) = \omega(\tau_A) = \sum_i a_{i,\tau_A(i)}$. Otherwise, A is said to be **singular**.

Tropical Matrices

The permanent is not multiplicative!

Ex. Take the nonsingular matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad \textit{for which} \quad A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Then $\mathrm{per}(A)^2 = 4$, $\mathrm{per}(A^2) = 5$, *and* $\mathrm{per}(A^2) \neq \mathrm{per}(A)^2$.

Thm. (I., 2007) *For any matrices* $A, B \in M_n(\mathbb{T})$,

$$\mathrm{per}(AB) \geqslant \mathrm{per}(A)\mathrm{per}(B).$$

If AB *is nonsingular, then* $\mathrm{per}(AB) = \mathrm{per}(A)\,\mathrm{per}(B)$ *and* $\tau_{AB} = \tau_A \circ \tau_B$.

Tropical Matrices

- The **tropical rank** $\mathrm{rk}_{\mathrm{tr}}(A)$ of A is the largest k for which A has a $k \times k$ nonsingular submatrix.

  Equivalently, $\mathrm{rk}_{\mathrm{tr}}(A)$ is the maximal number of independent columns (or rows) of A for an adequate notion of independence.
- The **factor rank** $\mathrm{rk}_{\mathrm{fc}}(A)$ of A is the smallest k for which A can be written as $A = BC$ with $B \in M_{n,k}$ and $C \in M_{k,n}$.

  Equivalently, $\mathrm{rk}_{\mathrm{fc}}(A)$ is the minimal number of vectors whose tropical span contains the span of the columns (or rows) of A, or the minimal number of rank-one matrices A_i needed to write A additively as $A = \bigvee_i A_i$.

Tropical Matrices

By definition, a matrix $A \in M_n(\mathbb{T})$ is nonsingular iff $\mathrm{rk}_{\mathrm{tr}}(A) = n$.

Tropical nonsingularity (and dependence) does not coincide with spanning.

For example, the vectors

$$\mathbf{v}_1 = (0, 0, -\infty), \quad \mathbf{v}_2 = (0, -\infty, 0), \quad \mathbf{v}_3 = (-\infty, 0, 0),$$

are dependent, but none of them can be written in terms of the others.

Tropical Matrices

It is well known that the above notions of rank do not coincide. Nevertheless, the inequality

$$\mathrm{rk}_{\mathrm{tr}}(A) \leqslant \mathrm{rk}_{\mathrm{fc}}(A)$$

holds for every $A \in M_n(\mathbb{T})$.

Thm. (I., Merlet, 2018)

$$\mathrm{rk}_{\mathrm{fc}}(A^t) \leqslant \mathrm{rk}_{\mathrm{tr}}(A)$$

for any $A \in M_n(\mathbb{T})$ *and* $t \geqslant (n-1)^2 + 1$.

Tropical Linear Representation

A finite dimensional **tropical linear representation** of a semigroup $S := (S, \cdot\ )$ is a semigroup homomorphism

$$\rho: \ S \longrightarrow M_n(\mathbb{T}), \qquad \rho(s's'') = \rho(s')\rho(s''), \ \forall s', s'' \in S,$$

where $M_n(\mathbb{T})$ is realized as an associative semialgebra of linear operators acting on the space $\mathbb{T}^n$.

ρ is a **faithful representation**, if it is injective.

To prove that S satisfies a semigroup identity $\Pi : u = v$:

- Find a faithful tropical linear representation ρ of S,
- Use “tropical techniques” to prove the semigroup identity for $\rho(S) \subset M_n(\mathbb{T})$.

Proving Identities of Tropical Matrices

To prove that a matrix subsemigroup $\mathcal{M}_n \subset M_n(\mathbb{T})$ admits a given identity $\Pi : u = v$ we have different approaches:

- To use generic matrices whose entries are variables, treated as functions.
- To realize matrices as labeled-weighted digraphs, and to compare walks on such graphs.
- To consider matrices as linear operators and to analyze their actions on the space.

Identities of Matrices

A matrix semigroup $\mathcal{M}_n \subset M_n(\mathbb{T})$ **satisfies** the semigroup identity $\Pi : u = v$,

$$\text{if } \varphi(u) = \varphi(v) \text{ for every morphism } \varphi : \mathcal{A}^+ \longrightarrow \mathcal{M}_n,$$

written $u \llbracket A, B \rrbracket = v \llbracket A, B \rrbracket$ for any $A, B \in \mathcal{M}_n$.

Any semigroup identity $\Pi : u = v$ of $M_n(\mathbb{T})$

- is **balanced** – the number of occurrences of each letter is the same in u and in v;
- is k-**uniform** – each letter appears in u and v exactly k times;
- is **uniform** – it is k-uniform for some k.

The Bicyclic Monoid

The **bicyclic monoid** $\mathcal{B} = \langle p, q \rangle$ is the monoid generated by two elements p and q, satisfying the relation

$$pq = e.$$

Each element w of $\mathcal{B}$ can be written uniquely as

$$w = q^i p^j, \qquad i, j \in \mathbb{Z}_+ \ .$$

$\mathcal{B}$ is faithfully represented by the map $\rho : \mathcal{B} \to M_2(\mathbb{T})$, given by

$$p \mapsto \begin{pmatrix} -1 & -1 \\ -\infty & 1 \end{pmatrix}, \quad q \mapsto \begin{pmatrix} 1 & -1 \\ -\infty & -1 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & -2 \\ -\infty & 0 \end{pmatrix}.$$

The Bicyclic Monoid

Thm. (Adjan 67) The bicyclic monoid $\mathcal{B}$ satisfies the semigroup identity

$$ab^2a\,\underline{ab}\,ab^2a = ab^2a\,\underline{ba}\,ab^2a.$$

Therefore, $M_2(\mathbb{T})$ has a nontrivial submonoid that admits a semigroup identity.

Combinatorial Approach

Given two matrices $A := (a_{i,j})$ and $B := (b_{i,j})$, we write

$$A \sim_{\text{diag}} B \quad \Leftrightarrow \quad a_{i,i} = b_{i,i}, \quad \text{for all } i = 1, \dots, n.$$

A and B are said to be **diagonally equivalent**, if $A \sim_{\text{diag}} B$.

The products AB and BA of any two triangular matrices A and B is always diagonally equivalent:

$$AB \sim_{\text{diag}} BA.$$

Rem. The digraph of a triangular matrix is acyclic.

Combinatorial Approach

Let $x = ab$ and $y = ba$. The Adjan's identity

$$ab^2a\,\underline{ab}\,ab^2a = ab^2a\,\underline{ba}\,ab^2a.$$

can be written as

$$xy\,\underline{x}\,xy = xy\,\underline{y}\,xy.$$

Let $X \sim_{\mathrm{diag}} Y$ be diagonally equivalent matrices in $M_2(\mathbb{T})$. The (i,j)-entry of the matrix product $XY\underline{X}XY$ corresponds to a labeled walk $\gamma_{i,j}$ from i to j of highest weight and length 5 on $G_{XY\underline{X}XY}$.

Lem. (I., 2013) If the contribution of $G_{\underline{X}}$ to a walk $\gamma_{i,j}$ on $G_{XY\underline{X}XY}$ is not a loop, then there is another walk $\gamma'_{i,j}$ on $G_{XY\underline{X}XY}$ from i to j of the same length and the same weight for which the contribution of $G_{\underline{X}}$ is a loop.

2×2 Tropical Matrices

Thm. (I., Margolis, 2009) *The submonoid $U_2(\mathbb{T})$ of upper triangular tropical matrices admits the Adjan's identity*

$$ab^2a \, \underline{ab} \, ab^2a = ab^2a \, \underline{ba} \, ab^2a \,,$$

i.e., $xy \, \underline{x} \, xy \, [\![AB, BA]\!] = xy \, \underline{y} \, xy \, [\![AB, BA]\!]$ *for any* $A, B \in U_2(\mathbb{T})$.

Thm. (I., Margolis, 2009) *The monoid $M_2(\mathbb{T})$ admits the identity*

$$a^2b^4a^2 \, \underline{a^2b^2} \, a^2b^4a^2 = a^2b^4a^2 \, \underline{b^2a^2} \, a^2b^4a^2 \,,$$

i.e., $xy \, \underline{x} \, xy \, [\![A^2B^2, B^2A^2]\!] = xy \, \underline{y} \, xy \, [\![A^2B^2, B^2A^2]\!]$ *for any* $A, B \in M_2(\mathbb{T})$.

Power Words

Let $\mathcal{A}$ be a finite (nonempty) alphabet, and let $p, n \in \mathbb{N}_+$.

Def. (I., 2013) *A* (p,n)-**power word** $\widetilde{w}_{(p,n)}$ *is a word in* $\mathcal{A}^+$ *such that:*

(a) *Each letter* $a_i \in \mathcal{A}$ *may appear in* $\widetilde{w}_{(p,n)}$ *at most* p*-times sequentially, i.e.,* $a_i^q \nmid \widetilde{w}_{(p,n)}$ *for any* $q > p$ *and* $a_i \in \mathcal{A}$*;*

(b) *Every word* $u \in \mathcal{A}^+$ *of length* n *that satisfies rule (a) is a factor of* $\widetilde{w}_{(p,n)}$*.*

$\widetilde{w}_{(p,n)}$ *is called an* n-**power word**, *if* $p = n$*.*

A (p,n)*-power word is* **uniform**, *if it is uniform as a word.*

A (p,n)-power word needs not be unique in $\mathcal{A}^+$. Different (p,n)-power words may have different length, and they can be concatenated to a new (p,n)-power word.

Power Words

Ex. *Let* $\mathcal{A} = \{x, y\}$.

1. *The* 2*-power word* $\widetilde{w}_{(2,2)} = x^2y^2x$ *is not uniform, while*

   $$\widetilde{w}_{(2,2)} = yx^2y^2x$$

   is uniform of length 6.

2. $\widetilde{w}_{(2,3)} = xy^2xyx^2y$ *is a uniform* $(2,3)$*-power word of length* 8.

   $$\widetilde{w}_{(3,3)} = xy^3xyx^3y$$

   is a uniform 3*-power word of length* 10.

3. *The word*

   $$\widetilde{w}_{(2,4)} = xyx^2y^2x^2yxy^2xyxy$$

   is a uniform $(2,4)$*-power word of length* 16.

Semigroup Identities

Let $\widetilde{w}_{(p,n)}$ be a uniform (p,n)-power word over $\mathcal{A} = \{x, y\}$ such that the words

$$\widetilde{w}_{(p,n)}\ \underline{x}\ \widetilde{w}_{(p,n)} \quad \text{and} \quad \widetilde{w}_{(p,n)}\ \underline{y}\ \widetilde{w}_{(p,n)}$$

are both (p,n)-power words.

Define the 2-letter identity

$$\Pi_{(p,n)}: \quad \widetilde{w}_{(p,n)}\ \underline{x}\ \widetilde{w}_{(p,n)} = \widetilde{w}_{(p,n)}\ \underline{y}\ \widetilde{w}_{(p,n)}.$$

To refine $\Pi_{(p,n)}$ to a uniform identity, substitute
$x := ab$ and $y := ba$.

Semigroup Identities

Ex. *Let* $\mathcal{A} = \{x, y\}$.

1. *Using the uniform* 2*-power word* $\widetilde{w}_{(2,2)} = yx^2y^2x$, *we obtain the identity*

$$\Pi_{(2,2)} : \; yx^2y^2x \; \underline{x} \; yx^2y^2x = yx^2y^2x \; \underline{y} \; yx^2y^2x \; .$$

2. *Taking the uniform* 3*-power word* $\widetilde{w}_{(3,3)} = xy^3xyx^3y$, *we obtain the identity*

$$\Pi_{(3,3)} : \; xy^3xyx^3y \; \underline{x} \; xy^3xyx^3y = xy^3xyx^3y \; \underline{y} \; xy^3xyx^3y \; .$$

These identities become uniform by substituting $x := ab$, $y := ba$,

Triangular Matrices

Think of $\langle U \rangle = \widetilde{w}_{(n-1,n-1)} [\![X, Y]\!]$ as a word with letters $X, Y \in M_n(\mathbb{T})$, and let $G_{\langle Z \rangle}$ be the labeled-weighted digraph of $\langle Z \rangle = \langle U \rangle \underline{X} \langle U \rangle$.

The (i,j)-entry of the matrix Z corresponds to a labeled walk $\gamma_{i,j}$ on $G_{\langle Z \rangle}$ from i to j of highest weight and length $\ell(\langle Z \rangle)$.

Lem. (I., 2013) Let $X \sim_{\mathrm{diag}} Y$ be diagonally equivalent matrices in $M_n(\mathbb{T})$. If the contribution of $G_{\underline{X}}$ to $\gamma_{i,j}$ is not a loop, then there is another walk $\gamma'_{i,j}$ from i to j on $G_{\langle Z \rangle}$ of the same length and the same weight for which the contribution of $G_{\underline{X}}$ is a loop.

Tropical Triangular Matrices

Thm. (I., 2013) Any two diagonally equivalent matrices $X \sim_{\mathrm{diag}} Y$ *in* $U_n(\mathbb{T})$ *satisfy the identity:*

$$\Pi_{(m,m)}: \ \widetilde{w}_{(m,m)} \ \underline{x} \ \widetilde{w}_{(m,m)} = \widetilde{w}_{(m,m)} \ \underline{y} \ \widetilde{w}_{(m,m)},$$

where $\mathcal{A} = \{x, y\}$ *and* $m = n - 1$.

Ex. Diagonally equivalent matrices admit the following identities.

- *In* $U_2(\mathbb{T})$

$$xy \ \underline{x} \ xy = xy \ \underline{y} \ xy.$$

- *In* $U_3(\mathbb{T})$

$$yx^2y^2x \ \underline{x} \ yx^2y^2x = yx^2y^2x \ \underline{y} \ yx^2y^2x.$$

Tropical Triangular Matrices

Thm. (I., 2013) The submonoid $U_n(\mathbb{T}) \subset M_n(\mathbb{T})$ of upper triangular tropical matrices satisfies the semigroup identity

$$\Pi_{(m,m)} : \ \widetilde{w}_{(m,m)} \ \underline{x} \ \widetilde{w}_{(m,m)} = \widetilde{w}_{(m,m)} \ \underline{y} \ \widetilde{w}_{(m,m)},$$

where $\mathcal{A} = \{x, y\}$, $m = n - 1$, by letting $x = AB$, $y = BA$, i.e.,

$$\widetilde{w}_{(m,m)} \ \underline{x} \ \widetilde{w}_{(m,m)} \, [\![AB, BA]\!] = \widetilde{w}_{(m,m)} \ \underline{y} \ \widetilde{w}_{(m,m)} \, [\![AB, BA]\!] \, .$$

Nonsingular Matrix Subsemigroups

A matrix subsemigroup $\mathcal{M}_n \subset M_n(\mathbb{T})$ is **nonsingular**, if each $X \in \mathcal{M}_n$ is nonsingular.

Thm. (I., 2014) Any nonsingular subsemigroup $\mathcal{M}_n \subset M_n(\mathbb{T})$ of tropical matrices satisfies the semigroup identities

$$\Pi_{(m,m)} : \; \widetilde{w}_{(m,m)} \; \underline{x} \; \widetilde{w}_{(m,m)} = \widetilde{w}_{(m,m)} \; \underline{y} \; \widetilde{w}_{(m,m)},$$

where $\mathcal{A} = \{x, y\}$, $m = n - 1$, by letting $x = A^{n!}B^{n!}$, $y = B^{n!}A^{n!}$.

General Matrices

Lem. (Shitov, 2015) Let $A, B, C \in M_n(\mathbb{T})$ such that $A = PQ$, where $P \in M_{n\times k}(\mathbb{T})$, $Q \in M_{k\times n}(\mathbb{T})$, $k < n$, and let $w \in \{a, b\}^+$. Then

$$(wa)\,[\![AB, AC]\!] = P\big(w\,[\![QBP, QCP]\!]\big)QB.$$

Thm. (I., Merlet, 2018) $\operatorname{rk}_{\mathrm{fc}}(A^t) \leqslant \operatorname{rk}_{\mathrm{tr}}(A)$ *for any $A \in M_n(\mathbb{T})$ and $t \geqslant (n-1)^2 + 1$.*

Thm. (I., Merlet, 2018) The monoid $M_n(\mathbb{T})$ satisfies a nontrivial semigroup identity for every $n \in \mathbb{N}$:

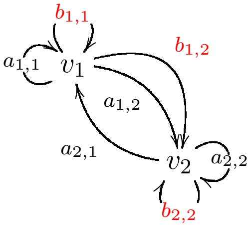
$$\langle ua, va\rangle \left[\left((qr)^t\right)\left[a^{\overline{n}}, b^{\overline{n}}\right], \left((qr)^t r\right)\left[a^{\overline{n}}, b^{\overline{n}}\right] \right],$$

with $\overline{n} = \operatorname{lcm}(1, \dots, n)$, $\langle q, r\rangle \in \operatorname{Id}(U_n(\mathbb{T}))$, $\langle u, v\rangle \in \operatorname{Id}(M_{n-1}(\mathbb{T}))$.

The length of this identity grows with n as $e^{Cn^2 + o(n^2)}$ for some $C \leqslant 1/2 + \ln(2)$.

Digraph View

Cor. (I., Merlet, 2018) *For any labeled-weighted digraph G, having (parallel) arcs labeled by $\{a, b\}$, there exist two different labeling sequences $u, v \in \{a, b\}^+$, each determines a different labeled walk of length $\ell(u) = \ell(v)$ between any pair of vertices of G, but with the same highest weight*



The Plactic Monoid

The **plactic monoid** $\mathcal{P}_n$ is the presented monoid $\mathcal{A}_n^*/_{\equiv_{\mathrm{knu}}}$, i.e., the free monoid $\mathcal{A}_n^*$ over an ordered alphabet $\mathcal{A}_n$ modulo the congruence $\equiv_{\mathrm{knu}}$ determined by the **Knuth relations**

$$\begin{aligned} a\,c\,b &= c\,a\,b \quad \text{if} \quad a \leqslant b < c\,, \\ b\,a\,c &= b\,c\,a \quad \text{if} \quad a < b \leqslant c\,. \end{aligned} \tag{KNT}$$

Thm. (I., 2016) *$\mathcal{P}_n$ has a tropical linear representation, which is faithful for $n = 3$.*

Therefore, $\mathcal{P}_3$ admits a nontrivial semigroup identity, for example

$$\Pi_{(2,2)}:\; yx^2y^2x\,\underline{x}\,yx^2y^2x = yx^2y^2x\,\underline{y}\,yx^2y^2x$$

with $x = pq$, $y = qp$, for $p, q \in \mathcal{P}_3$.