Groups and groupoids constructed from higher rank graphs

Mark V Lawson
Heriot-Watt University, Edinburgh
and the
Maxwell Institute for Mathematical Sciences

In collaboration with

Aidan Sims (Wollongong), Alina Vdovina (Newcastle, UK).

References

1. M. V. Lawson, A. Vdovina, Higher dimensional generalizations of the Thompson groups, *Adv. Math.* **369** (2020), 107191.

2. M. V. Lawson, A. Vdovina, A generalization of higher rank graphs, *Bull. Austral. Math. Soc.*, appeared online 26 July 2021.

3. M. V. Lawson, A. Sims, A. Vdovina, Higher dimensional generalizations of the Thompson groups via higher rank graphs, arXiv:2010:08960, submitted.

In this talk, I shall refer only to the results of paper (1) above.

What we shall do

In this talk, I shall show how to construct a family of groups from what we term k-monoids.

k-monoids are a class of one-vertex higher rank graphs and generalize free monoids.

The groups we construct generalize the Thompson groups $G_{n,1}$ and Brin's higher dimensional analogues nV.

The groups will be topological full groups of a class of étale groupoids.

How we shall do it

We shall construct both groups and groupoids from inverse monoids.

The group will be constructed using the minimum group congruence.

The groupoid will be constructed using non-commutative Stone duality.

Why inverse monoids? Because they provide the link between groups and groupoids.

Motivation: Free monoids

Let $A_n = \{a_1, \ldots, a_n\}$ be a finite alphabet where $n \geq 2$. The set of all finite strings over A_n is denoted by A_n^*. This is a monoid under concatenation of strings with the empty string ε as the identity. In fact, A_n^* is the free monoid on A_n. The key property of the free monoid that we shall need is the following. This is an arithmétic property.

Lemma *The free monoid A_n^* is singly aligned. This means that for any strings x and y we have that $xA_n^* \cap yA_n^*$ is either empty or again a principal right ideal.*

The free monoid A_n^* comes equipped with a monoid homomorphism $A_n^* \to \mathbb{N}$ given by $x \mapsto |x|$ which measures the length of the string.

Strings are 1-dimensional entities.

If $|x| = m + n$ then there are unique strings u and v such that $x = uv$ where $|u| = m$ and $|u| = n$. We call this the *unique factorization property* of the length function.

We now generalize free monoids to higher dimensions.

k-monoids

A countable monoid S is said to be a k-monoid if there is a monoid homomorphism $d\colon S \to \mathbb{N}^k$ satisfying the unique factorization property (UFP): if $d(x) = \mathbf{m} + \mathbf{n}$ then there are unique elements $x_1, x_2 \in S$ such that $x = x_1 x_2$ and $d(x_1) = \mathbf{m}$ and $d(x_2) = \mathbf{n}$.

Let $\mathbf{e}_i \in \mathbb{N}^k$ have a 1 in the ith position and 0's elsewhere. Put $X_i = d^{-1}(\mathbf{e}_i)$ for each $1 \leq i \leq k$. We call $(X_1, \ldots, X_k)$ the set of k-*alphabets* associated with S.

In what follows, we assume that not all alphabets contain exactly one element.

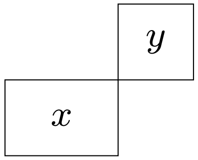
A couple of facts:

1. k-monoids are cancellative and have no non-trivial invertible elements.

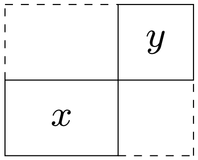
2. If S is a k-monoid and T is an l-monoid then $S \times T$ is a $(k+l)$-monoid.

Picture: multiplication in 2-monoids

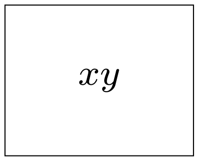
Given x and y to calculate xy



use UFP



and now we get the result xy



Remarks

1. The 1-monoids are precisely the countable free monoids by Levi's Theorem (see Corollary 1.6 of Lallement, *Semigroups and combinatorial applications*, John Wiley & Sons, 1979.)

2. Direct products of k free monoids are no longer free but are k-monoids.

3. The elements of S in the case of 2-monoids should be regarded as rectangles, in the case of 3-monoids cuboids, and so on. This justifies us in regarding the elements of k-monoids as being higher-dimensional strings.

4. Our definition of k-monoid is simply the one-vertex case of the usual definition of a higher rank graph introduced by Kumjian and Pask, Higher rank graph C^*-algebras, *New York J. Math.* **6** (2000), 1–20.

5. The theory of free monoids leads to automata theory; where might the theory of k-monoids lead? (Real question)

6. How can we construct examples of k-monoids? (Another real question)

Finite alignment

Let S be a k-monoid. Suppose that x and y are dependent. We say that S is m-aligned if $xS \cap yS = \bigcup_{i=1}^{m} x_i S$.

If $m = 1$, then we refer to singly aligned.

We say that S is finitely aligned if it is m-aligned for some m.

Lemma *If S is k-aligned and T is l-aligned then $S \times T$ is kl-aligned.*

WE SHALL ALWAYS ASSUME THAT OUR k-MONOIDS ARE FINITELY ALIGNED.

Codes

Let S be a k-monoid. Elements $x, y \in S$ are said to be independent if $xS \cap yS = \varnothing$, otherwise they are said to be dependent.

A finite subset $X \subseteq S$ is called a code if the elements are pairwise independent.

A code is maximal if every element of S is dependent on an element of X.

Example In free monoids, a code is a prefix code and a maximal code is a maximal prefix code.

Aperiodicity

There is a technical condition that we shall mention now and assume from this point on. (It will ensure that the groupoid we construct is effective).

Let S be a k-monoid. We say that S is effective if for all distinct $a, b \in S$ there exists $u \in S$ such that au and bu are independent.

Ideals and morphisms

Let S be any monoid. A subset $R \subseteq S$ is called a right ideal if $RS \subseteq R$.

If X is any subset then XS is a right ideal; this is finitely generated if X is a finite set.

Let R_1 and R_2 be right ideals of S. Then a function $\theta\colon R_1 \to R_2$ is said to be a morphism if $\theta(rs) = \theta(r)s$.

(Just think right-module morphisms).

Inverse semigroups

Groups are the abstract versions of groups of bijections whereas inverse semigroups are the abstract versions of inverse semigroups of partial bjiections.

A semigroup S is an inverse semigroup if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Inverse semigroups contain lots of idempotents since $a^{-1}a$ and aa^{-1} are idempotents. (An inverse semigroup with exactly one idempotent is a group).

The idempotents of an inverse semigroup commute with each other. If e is an idempotent so too is aea^{-1} for any $a \in S$.

Observe that $ae = a(a^{-1}ae) = a(ea^{-1}a) = (aea^{-1})a$. Thus idempotents can 'pass through elements' whilst still remaining idempotents.

The natural partial order on an inverse semigroup

Let S be an inverse semigroup. Define $a \leq b$ iff $a = be$ for some idempotent e.

- $\leq$ is a partial order called the natural partial order.
- If $a \leq b$ then $a^{-1} \leq b^{-1}$.
- If $a \leq b$ and $c \leq d$ then $ac \leq bd$.

If e and f are idempotents then $e \leq f$ iff $e = ef$.

Observe that if $a, b \leq c$ then both $a^{-1}b$ and ab^{-1} are idempotents.

More generally, we say that a and b are compatible if both $a^{-1}b$ and ab^{-1} are idempotents.

It follows, that a necessary condition for a and b to have a join is that they be compatible.

An inverse semigroup S is said to be distributive if the join of a and b exists whenever a and b are compatible and multiplication distributes over such joins from the left and the right.

An inverse semigroup is a meet-semigroup if every pair of elements has a meet.

Two inverse semigroups constructed from a k-monoid

Let S be a k-monoid. Then there are in general two inverse semigroups associated with S; as we shall see, they will give rise to the same group.

In the case of free monoids these are the same.

One of these inverse semigroups gives us more insight into the group we shall construct, whereas the other is important in constructing our groupoid.

Let S be a k-monoid. Define R(S) to be the set of all bijective morphisms between finitely generated right ideals.

Theorem R(S) *is a distributive inverse meet monoid.*

Define P(S) to be the set of all bijective morphisms between right ideals generated by codes. (Suggested by John Fountain).

Theorem P(S) *is an inverse submonoid of* R(S)*.*

How to get a group from an inverse semigroup

Let S be any inverse semigroup. The congruence σ is defined on S by $a \, \sigma \, b$ if and only if there exists an element c such that $c \leq a, b$.

Observe that S/σ is an inverse semigroup with a unique idempotent and so is a group.

Not only is S/σ is a group but if ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$.

Thus σ is the minimum group congruence on S.

The essential part of an inverse semigroup

Let S be any inverse monoid with zero. Then S/σ is trivial. So, we look for the 'big' elements in S.

A non-zero idempotent e in S is said to be essential if $ef \neq 0$ for all non-zero idempotents f.

An element a of S is said to be essential if both $a^{-1}a$ and aa^{-1} are essential.

Define S^e, the essential part of S, to consist of all essential elements of S. It is an inverse monoid (without zero). The group S^e/σ will be interesting to us.

The group associated with a k-monoid

Theorem *Let S be a k-monoid. Then*

$$\mathcal{G}(S) = \mathsf{R}(S)^e/\sigma \cong \mathsf{P}(S)^e/\sigma.$$

We call $\mathcal{G}(S)$ the group associated with the k-monoid S.

The most natural way to think about this group is as

$$\mathsf{P}(S)^e/\sigma$$

where $\mathsf{P}(S)^e$ is the inverse semigroup of all bijective morphisms between right ideals generated by maximal code.

But, we shall need the inverse semigroup $\mathsf{R}(S)$ to get our groupoid.

Boolean inverse monoids

A distributive inverse monoid is said to be Boolean if its semilattice of idempotents is a Boolean algebra.

Boolean inverse monoids should be regarded as non-commutative generalizations of Boolean algebras.

Boolean algebras are associated with Boolean spaces (= compact, Hausdorff spaces with a basis of clopen sets) via classical Stone duality. We call the unique atomless countable Boolean algebra the *Tarski algebra*. Its Stone dual is the Cantor space.

Boolean inverse monoids are associated with Boolean groupoids (= étale topological groupoids the identity space of which is a Boolean space) via non-commutative Stone duality.

An important congruence

We shall need a definition from the following paper:

D. Lenz, An order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

Let S be an inverse semigroup with zero. We shall define a congruence $\equiv$ on S which we call the Lenz congruence.

Define $\equiv$ on S as follows: $s \equiv t$ if and only if the following two conditions hold:

1. If $0 < x \leq s$ then there exists a non-zero element x' such that $x' \leq x, t$.

2. If $0 < y \leq t$ then there exists a non-zero element y' such that $y' \leq y, s$.

A Boolean inverse monoid

Theorem *Let S be a k-monoid. Put $\mathsf{C}(S) = \mathsf{R}(S)/\equiv$. Then this is a countably infinite Boolean inverse meet-monoid whose group of units is isomorphic to $\mathcal{G}(S)$. Its idempotents form the Tarski Boolean algebra. As an inverse semigroup it is congruence-free.*

The proof of this result is non-trivial. In particular, we have to use the structures determined by S, called k-*tilings*, to show that the set of idempotents forms a Boolean algebra. k-tilings are the analogues of the right-infinite strings in the case of free monoids.

Groups from groupoids

Let G be a Boolean groupoid.

A subset $A \subseteq G$ is called a local bisection if $g, h \in A$ and $g^{-1}g = h^{-1}h$ (respectively, $gg^{-1} = hh^{-1}$) implies that $g = h$.

This subset is called a bisection if it is a local bisection and every identity of G is of the form $g^{-1}g$ where $g \in A$ (respectively, hh^{-1} where $h \in A$).

The set of compact-open bisections of the groupoid G forms a group called the topological full group of G.

The topological full group of a Boolean groupoid is isomorphic to the group of units of the Boolean inverse monoid associated with the groupoid under non-commutative Stone duality.

The synthesis

We now apply non-commutative Stone duality to the Boolean inverse monoid $\mathsf{C}(S)$.

Theorem *The Boolean groupoid* $\mathscr{G}(S)$ *associated with the Boolean inverse monoid* $\mathsf{C}(S)$ *is second countable, Hausdorff and its topological full group is isomorphic to* $\mathcal{G}(S)$.

By Theorem 4.16 of the paper by H. Matui, Topological full groups of one-sided shifts of finite type, *J. Reine Angew. Math.* **705**, 35–84 combined with the above theorem we obtain the following.

Theorem *Let S be a k-monoid. Then the group $\mathcal{G}(S)$ is countable with a simple commutator subgroup.*

Our groups therefore generalize the finite symmetric groups (on at least 5 letters).

Two concrete examples.

Theorem $\mathcal{G}(A_n^*) \cong G_{n,1}$.

Theorem $\mathcal{G}((A_2^*)^n) \cong nV$.

Concluding remarks

1. In this talk, I have concentrated on constructing groups from one-vertex higher rank graphs. The generalization to arbitrary higher rank graphs is carried out in paper (3).

2. Higher rank graphs are themselves generalized in paper (2). We show there how this extension of the definition accommodates the theory of graphs of groups.

3. We construct groups, but say very little about them. We would like to construct presentations. We would like to construct invariants — see paper (3) for some information on these.