Transformation Semigroups and their Automorphisms

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Joint work with
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York Semigroup

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- Note that T_X contains the symmetric group over X (as its group of units)
- We will only consider finite X, and more specifically $T_n = T_{X_n}$, $S_n = S_{X_n}$, where $X_n = \{1, \ldots, n\}$ for $n \in \mathbb{N}$
- For $S \leq T_n$, we are looking at the interaction of $S \cap S_n$ and S

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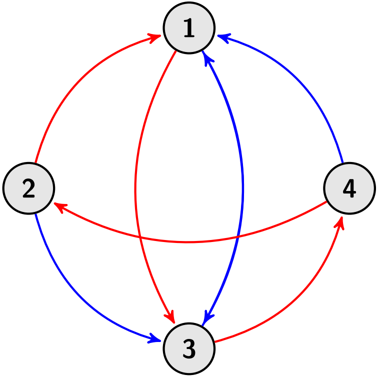
- Our research examines on properties of transformation semigroups with “well-behaved” groups of units
- As we only consider finite underlying sets, we also have access to graph theoretic tools

Synchronisation



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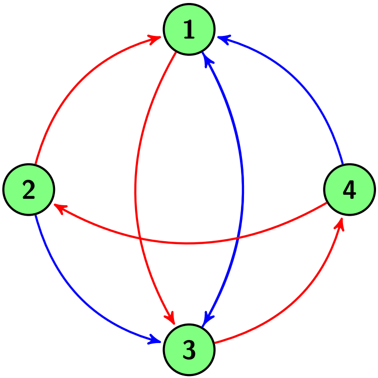

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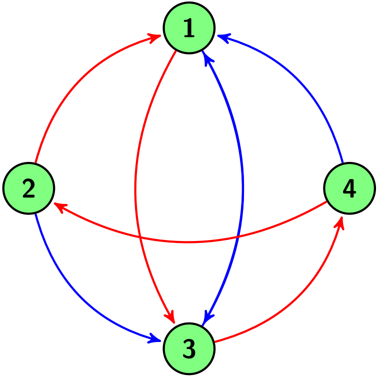

A synchronising word is BLUE

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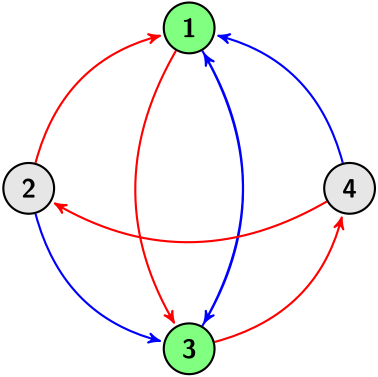

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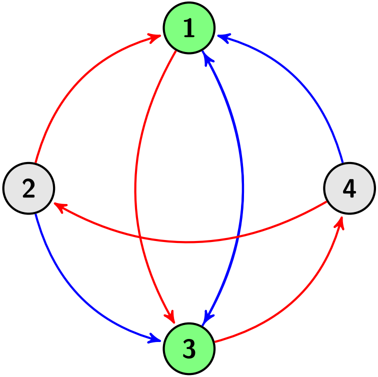

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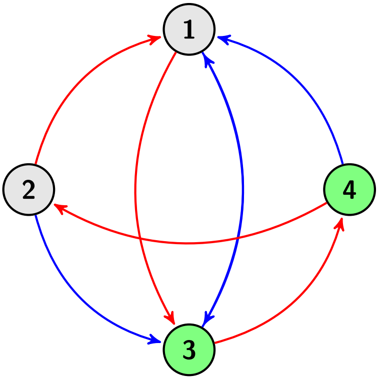

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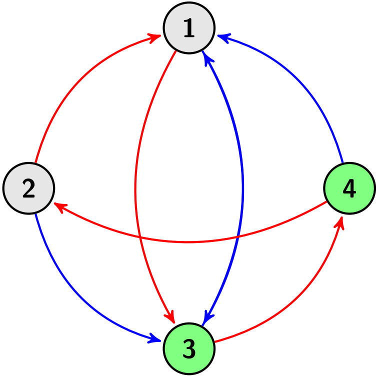

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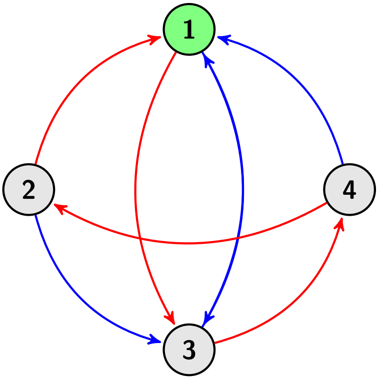

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Synchronisation for Automata and Semigroups

Definition

A subsemigroup of T_X is *synchronising* if it contains a constant map
A subset $S \subseteq T_X$ of transpositions is *synchronising* if the semigroup $\langle S \rangle$ contain a constant map

Synchronisation for Groups

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Definition

A subgroup G of S_n is a *synchronising group* if it synchronises every non-permutation in T_n

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- "Pure" group theoretic definition (Araújo):
  A permutation group G on X is *synchronising* if no (proper, non-trivial) partition of X has a transversal S, such that S^g is also a transversal for all $g \in G$

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- The converse is wrong (example with 36 elements)
- Note that there is a "quasi-classification" of finite primitive permutation groups, as well as powerful theoretic results

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- "Good" means either k-homogeneous or primitive
- k-homogeneity is the "unordered" version of k-transitivity, ie. a k-homogeneous group induces a transitive action on k-sets
- There is a complete classification of k-homogeneous groups (for $2 \leq k \leq n-2$)

Ranks of transformation semigroups

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- We showed that in order to generate S, we only need to generate G and all kernels of rank $n-k$ maps
- So to get an overall bound, we need to determine the number of orbits G has on $(n-k)$-partitions of X, and the rank of G

The case $k = 3$

If G is 3-homogeneous, then one of the following holds:

- $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$, for some prime power q;
- $G = \mathrm{AGL}(d, 2)$ for some d;
- G is one of finitely many sporadic groups

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Degree	2^d $(d \geq 5)$	16	8
Group	$\mathrm{AGL}(d, 2)$	$\mathrm{AGL}(4, 2)$	$\mathrm{AGL}(3, 2)$
$(4, 1, \ldots)$	2	2	2
$(3, 2, 1, \ldots)$	3	3	2
$(2, 2, 2, 1, \ldots)$	7	6	3
Total	12	11	7

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Sporadic groups can be calculated directly

Degree	Group	$(4,1,\ldots)$	$(3,2,1,\ldots)$	$(2,2,2,1,\ldots)$	Total
8	$\mathrm{AGL}(1,8)$	2	10	11	23
	$\mathrm{A\Gamma L}(1,8)$	2	4	5	11
	$\mathrm{PSL}(2,7)$	3	4	7	14
	$\mathrm{PGL}(2,7)$	2	3	5	10
9	$\mathrm{PSL}(2,8)$	1	4	7	12
	$\mathrm{P\Gamma L}(2,8)$	1	2	3	6
10	$\mathrm{PGL}(2,9)$	2	5	12	19
	M_{10}	2	5	9	14
	$\mathrm{P\Gamma L}(2,9)$	2	4	8	14
11	M_{11}	1	2	4	7
12	M_{11}	2	4	6	12
	M_{12}	1	1	3	5
16	$2^4 : A_7$	2	4	10	16
22	M_{22}	2	5	11	18
	$M_{22} : 2$	2	4	10	16

Maximal ranks of primitive groups

Homogeneity k	$\|A\|$	Maximal example	Maximal rank for primitive G
1	$\frac{(n-1)}{2}$	C_p, D_p (n odd prime)	$\frac{C \log n}{\sqrt{\log \log n}}$
2	$O(n^2)$	Example 2.1	2
3	$O(n^3)$	$\mathrm{PSL}(2, q), \mathrm{P\Gamma L}(2, q)$	2
4	12160	$\mathrm{P\Gamma L}(2, 32)$ ($n = 33$)	2
5	77	M_{24} ($n = 24$)	2
> 5	$p(k)$	S_n, A_n	2

Table: Number $|A|$ of rank $n - k$ maps needed to together with a k-homogeneous group G generate all the maps of rank not larger than $n - k$.

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- This seems to have been completely unknown
- For example, GAP provides 2-element generating sets for these groups in only about one-third of all cases

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- Proving this in general is likely very difficult

A small step

Theorem (Araújo, WB, Cameron)

Let $G \leq S_n$ be a primitive permutation group, $S \supset G$ a transformation monoid, such that some $t \in S$ is a singular transformation of image size at most 3. Then

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where we identify elements of S_n with their conjugation action.

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- We need a different approach to construct h from F

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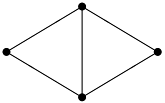
- Every element of J is a triangle in Γ
- Γ is vertex-primitve with clique and chromatic number 3
- By a previous results on vertex-primitve graphs (also independently by Spiga and Verret), Γ does not contain this subgraph:



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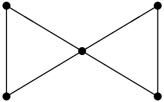

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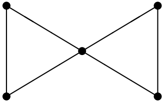

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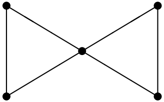

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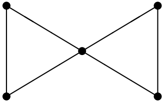

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- Moreover, this carries forward to intersection of more than two triangles in the same point
- The mapping induced on the intersection points give us the map we need

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- Work for the future!

Thank you!