Free idempotent generated semigroups and partial endomorphism monoids of free G-acts

Dandan Yang

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Biordered sets

Let S be a semigroup with $E = E(S)$ a set of idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Note $e \leq_{\mathcal{R}} f$ $(e \leq_{\mathcal{L}} f)$ implies both ef and fe are idempotents.

We say that (e, f) is **a basic pair** if

$$e \leq_{\mathcal{R}} f, f \leq_{\mathcal{R}} e, e \leq_{\mathcal{L}} f \text{ and } f \leq_{\mathcal{L}} e,$$

i.e.

$$\{e, f\} \cap \{ef, fe\} \neq \emptyset;$$

and ef, fe are said to be **basic products.**

Biordered sets

1. Under basic products, E satisfies a number of axioms; if S is regular, an extra axiom holds.
2. A **biordered set** is a partial algebra satisfying these axioms; if the extra one also holds it is a **regular biordered set**.
3. A biordered set is regular if and only if $E = E(S)$ for a regular semigroup S **Nambooripad (1979)**.
4. Any biordered set E is $E(S)$ for some semigroup S **Easdown (1985)**.
5. The category of inductive groupoids whose set of identities form a regular biordered set is equivalent to the category of regular semigroups **Nambooripad (1979)**.

Free Idempotent-generated semigroups

Let S be a semigroup with $E = E(S)$ a set of idempotents of S.

S is **idempotent-generated** if $S = \langle E \rangle$.

Example 1 (Howie 1966)

$\mathcal{T}_n$ - full transformation monoid, $S(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n : \operatorname{rank} \alpha < n\}$.

Example 2 (J.A. Erdös, 1967, Laffey, 1973)

$M_n(D)$ - full linear monoid, $S(M_n(D)) = \{A \in M_n(D) : \operatorname{rank} A < n\}$.

Example 3 (Fountain and Lewin, 1992)

A - independence algebra, $S(\operatorname{End} A) = \{\alpha \in \operatorname{End}(A) : \operatorname{rank} \alpha < n\}$.

Free idempotent-generated semigroups

Let E be a biordered set (equivalently, a set of idempotents E of a semigroup S).

The free idempotent generated semigroup $\mathrm{IG}(E)$ is a free object in the category of semigroups that are generated by E, defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts

1. $\text{IG}(E) = \langle \overline{E} \rangle$.
2. The natural map $\phi : \text{IG}(E) \to S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E(S) \rangle$.
3. The restriction of ϕ to the set of idempotents of $\text{IG}(E)$ is a bijection.
4. The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in $\text{IG}(E)$ and the corresponding set in $S' = \langle E(S) \rangle$.

Free idempotent-generated semigroups

Question: the structure of maximal subgroups of $\mathrm{IG}(E)$.

Maximal subgroups of IG(E)

There have been significant recent advances in the study of IG(E).

Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups. .

Brittenham, Margolis and Meakin (2009)

$\mathbb{Z} \oplus \mathbb{Z}$ can be a maximal subgroup of IG(E), for some E.

Gray, Ruskuc (2012)

Every group occurs as the maximal subgroup of IG(E) by using a general presentation and a special choice of E.

Maximal subgroups of $\mathrm{IG}(E)$

Gray and Ruskuc (2012)

$\mathcal{T}_n$ - full transformation monoid, $E = E(\mathcal{T}_n)$ - its biordered set.

$\operatorname{rank} e = r < n-1,\ n \geq 3,\ H_{\overline{e}} \cong H_e \cong \mathcal{S}_r.$

Dolinka (2013)

$\mathcal{PT}_n$ - partial transformation monoid, $E = E(\mathcal{PT}_n)$ - its biordered set.

$\operatorname{rank} e = r < n-1,\ n \geq 3,\ H_{\overline{e}} \cong H_e \cong \mathcal{S}_r.$

Brittenham, Margolis and Meakin (2010)

$M_n(D)$ - full linear monoid, $E = E(M_n(D))$ - its biordered set.

$\operatorname{rank} e = 1,\ n \geq 3,\ H_{\overline{e}} \cong H_e \cong D^*.$

Dolinka and Gray (2014)

$\operatorname{rank} e = r < n/3,\ n \geq 4,\ H_{\overline{e}} \cong H_e \cong GL_r(D).$

Maximal subgroups of $\mathrm{IG}(E)$

Dolinka, Gould and Yang (2015)

End $F_n(G)$ - the endomorphism monoid of a free G-act $F_n(G)$,
$E = E(\mathrm{End}\, F_n(G))$ - its biordered set.

$\mathrm{rank}\, e = r < n-1,\ n \geq 3,\ H_{\overline{e}} \cong H_e \cong G \wr \mathcal{S}_r$.

Gould and Yang (2016)

End $\mathbf{A}$ - the endomorphism monoid of an independence algebra $\mathbf{A}$ with no constants.

$E = E(\mathrm{End}\, \mathbf{A})$ - its biordered set.

$\mathrm{rank}\, e = 1,\ n \geq 3,\ H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.

Note $\mathrm{rank}\, e = n-1,\ H_{\overline{e}}$ is free; $\mathrm{rank}\, e = 0, n,\ H_{\overline{e}}$ is trivial.

Maximal subgroups of $\mathrm{IG}(E)$

A question posed by Vicky in 2015:

$\mathrm{PEnd}\, F_n(G)$ - the partial endomorphism monoid of a free G-act $F_n(G)$,
$E = E(\mathrm{PEnd}\, F_n(G))$ - its biordered set.

$0 < \mathrm{rank}\, e = r < n-1$, $n \geq 3$, $H_{\overline{e}} \cong H_e \cong G \wr \mathcal{S}_r$?

This was the task for Dandan and Tom in Sep. 2015 during Tom's visit in Xidian University, the answer to which is YES!

Free left G-acts

Sets and vector spaces over division rings are examples of independence algebras, as are free left G-acts.

Let G be a group, $n \in \mathbb{N}, n \geq 3$. Let $F_n(G)$ be a rank n **free left G-act.**

Recall that, as a set,

$$F_n(G) = \{gx_i : g \in G, i \in [1, n]\};$$

identify x_i with $1x_i$, where 1 is the identity of G;

$gx_i = hx_j$ if and only if $g = h$ and $i = j$;

the action of G is given by $g(hx_i) = (gh)x_i$.

The partial endomorphism monoid PEnd $F_n(G)$ of $F_n(G)$

A partial mapping α from $F_n(G)$ to itself is called a partial endomorphism of $F_n(G)$ if for each $i \in [1, n]$, we have that $x_i \in \operatorname{dom} \alpha$ if and only if for all $g \in G$, $gx_i \in \operatorname{dom} \alpha$ and $(gx_i)\alpha = g(x_i\alpha)$.

Let PEnd $F_n(G)$ be the partial endomorphism monoid of $F_n(G)$ with $E = E(\text{PEnd}\, F_n(G))$.

Note that the endomorphism monoid End $F_n(G)$ of $F_n(G)$ is a submonoid of PEnd $F_n(G)$.

The **rank** of an element of PEnd $F_n(G)$ is the minimal number of (free) generators in its image.

The partial endomorphism monoid PEnd $F_n(G)$ of $F_n(G)$

Let $\alpha \in \mathrm{PEnd}\, F_n(G)$ with $\mathrm{dom}\,\alpha = \{gx_i : i \in M, M \subseteq [1, n], g \in G\}$. We briefly write $\mathrm{dom}\,\alpha = \langle x_m \rangle_{m \in M}$. Then it is easy to see that α depends only on its action on the free generators $\{x_i : i \in M\}$. Suppose that $M = \{i_1, \cdots, i_s\} \subseteq [1, n]$ with $1 \leq s \leq n$. Then it is convenient to write

$$\alpha = \begin{pmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_s} \\ w_1^{\alpha} x_{i_1\overline{\alpha}} & w_2^{\alpha} x_{i_2\overline{\alpha}} & \ldots & w_s^{\alpha} x_{i_s\overline{\alpha}} \end{pmatrix}.$$

Note that α naturally induces a partial mapping $\overline{\alpha} : [1, n] \longrightarrow [1, n]$ with $\mathrm{dom}\,\alpha = M$.

Lemma For any $\alpha, \beta \in \mathrm{PEnd}\, F_n(G)$, we have the following:

(i) $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$ if and only if $\alpha \,\mathcal{L}\, \beta$;

(ii) $\ker \alpha = \ker \beta$ if and only if $\alpha \,\mathcal{R}\, \beta$;

(iii) $\mathrm{rank}\,\alpha = \mathrm{rank}\,\beta$ if and only if $\alpha \,\mathcal{D}\, \beta$ if and only if $\alpha \,\mathcal{J}\, \beta$.

The partial endomorphism monoid PEnd $F_n(G)$ of $F_n(G)$

Corollary For any $1 \leq r \leq n$, the maximal subgroup of PEnd $F_n(G)$ containing a rank r idempotent is isomorphic to that of End $F_n(G)$, and hence to the wreath product $G \wr \mathcal{S}_r$, where $\mathcal{S}_r$ is the symmetric group on r elements.

The $\mathcal{D}$-class of an arbitrary rank r element of PEnd $F_n(G)$ is given by

$$D_r = \{\alpha \in \mathrm{PEnd}\, F_n(G) : \mathrm{rank}\, \alpha = r\},$$

where $0 \leq r \leq n$.

The set $D_r^0 = D_r \cup \{0\}$ forms a completely 0-simple semigroup, where the binary operation on D_r^0 is defined as follows:

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \alpha, \beta \in D_r \text{ and } \mathrm{rank}\, \alpha\beta = r \\ 0 & \text{else} \end{cases}$$

A Rees representation for D_r^0

Put:

I the set of kernels;

$\Lambda = \{(u_1, u_2, \ldots, u_r) : 1 \leq u_1 < u_2 < \ldots < u_r \leq n\} \subseteq [1, n]^r$.

$H_{i\lambda} = R_i \cap L_\lambda$.

$K = \{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group.}\}$.

Assume $1 \in I \cap \Lambda$ with

$$1 = \langle (x_1, x_i) : r+1 \leq i \leq n \rangle \in I, 1 = (1, \cdots, r) \in \Lambda.$$

So $H = H_{11}$ is a group with identity $\varepsilon = \varepsilon_{11}$ defined as follows:

$$\varepsilon = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_1 & x_2 & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \ldots & x_r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}.$$

A Rees representation for D_r^0

For any $\alpha \in D_r$, $\ker \overline{\alpha}$ induces a partition

$$\{B_1^\alpha, \cdots, B_r^\alpha\}$$

on $\operatorname{dom} \alpha[1, n]$ with a set of minimum elements

$$l_1^\alpha, \cdots, l_r^\alpha \text{ such that } l_1^\alpha < \cdots < l_r^\alpha.$$

Note that $l_1^\alpha = 1$ if and only if $x_1 \in \operatorname{dom} \alpha$.
Put

$$\Theta = \{\alpha \in D_r : x_{l_j^\alpha} \alpha = x_j, j \in [1, r]\}.$$

Then it is a transversal of the $\mathcal{H}$-classes of L_1.

For each $i \in I$, define $\mathbf{r}_i$ as the unique element in $\Theta \cap H_{i1}$

For each $\lambda = (u_1, u_2, \ldots, u_r) \in \Lambda$, define

$$\mathbf{q}_\lambda = \mathbf{q}_{(u_1, \cdots, u_r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} & x_{u_1} & \cdots & x_{u_1} \end{pmatrix}.$$

A presentation for $H_{\overline{e}}$

We have that $D_r^0 = D_r \cup \{0\}$ is completely 0-simple, and hence

$$D_r^0 \cong \mathcal{M}^0(H; I, \Lambda; P),$$

where $P = (\mathbf{p}_{\lambda i})$ and

$$\mathbf{p}_{\lambda i} = (\mathbf{q}_\lambda \mathbf{r}_i) \text{ if } \operatorname{rank} \mathbf{q}_\lambda \mathbf{r}_i = r$$

and is 0 else.

Note that for all $i \in I$ and $\lambda = (u_1, u_2, \cdots, u_r) \in \lambda$, $\mathbf{p}_{\lambda i} \neq 0$ implies $\{x_{u_1}, x_{u_2}, \cdots, x_{u_r}\} \subseteq \operatorname{dom} \mathbf{r}_i$.

A presentation for $H_{\overline{e}}$

Define a **schreier system** of words $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ as follows:

put $\mathbf{h}_{(1,2,\cdots,r)} = 1$;

for any $(u_1, u_2, \ldots, u_r) > (1, 2, \cdots, r)$, take $u_0 = 0$ and i the largest such that $u_i - u_{i-1} > 1$. Then

$$(u_1, \ldots, u_{i-1}, u_i - 1, u_{i+1}, \ldots, u_r) < (u_1, u_2, \ldots, u_r).$$

Define

$$\mathbf{h}_{(u_1,\cdots,u_r)} = \mathbf{h}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}\alpha_{(u_1,\cdots,u_r)},$$

where

$$\alpha_{(u_1,\cdots,u_r)} = \begin{pmatrix} x_1 & \cdots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_{u_1} & \cdots & x_{u_1} & x_{u_2} & \cdots & x_{u_2} & \cdots & x_{u_r} & \cdots & x_{u_r} & x_{u_r} & \cdots & x_{u_r} \end{pmatrix}.$$

A presentation for $H_{\overline{e}}$

Lemma $\mathbf{h}_{(u_1,\cdots,u_r)}$ induces a bijection from $L_{(1,\cdots,r)}$ onto $L_{(u_1,\cdots,u_r)}$ in End $F_n(G)$, so does in PEnd $F_n(G)$ and IG(E).

$\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ forms the required schreier system.

Finally, define the function

$$\omega : I \longrightarrow \lambda, i \mapsto \omega(i) = (l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \ldots, l_r^{\mathbf{r}_i}).$$

Note $\mathbf{p}_{\omega(i),i} = \varepsilon$.

Lemma $\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu} \end{bmatrix}$ is a singular square $\Longleftrightarrow \mathbf{p}_{\lambda i}^{-1}\mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1}\mathbf{p}_{\mu k}$.

A presentation for $H_{\overline{e}}$

The maximal subgroup $\overline{H}$ of $\overline{e}$ in $\mathrm{IG}(E)$ is defined by the presentation

$$\mathcal{P} = \langle F : \Sigma \rangle$$

with generators:

$$F = \{f_{i,\lambda} : \quad (i, \lambda) \in K\}$$

and defining relations Σ:

$(R1)$ $f_{i,\lambda} = f_{i,\mu} \quad (\mathbf{h}_\lambda \varepsilon_{i\mu} = \mathbf{h}_\mu)$;

$(R2)$ $f_{i,\omega(i)} = 1 \quad (i \in I)$;

$(R3)$ $f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad \left(\begin{bmatrix} \varepsilon_{i\lambda} & \varepsilon_{i\mu} \\ \varepsilon_{k\lambda} & \varepsilon_{k\mu} \end{bmatrix} \text{ is singular i.e. } \mathbf{p}_{\lambda i}^{-1} \mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1} \mathbf{p}_{\mu k} \right)$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Put:

$I' = \{i \in I : H_{i1} \subseteq \text{End}\, F_n(G)\} \subseteq I$;

$P' = (\mathbf{p}_{\lambda i})$ where $i \in I'$ and $\lambda \in \Lambda$ be a submatrix of P.

Note that the structure of P' is exactly the same as that of the sandwich matrix we chose for the completely 0-simple semigroup related with the rank r $\mathcal{D}$-class of $\text{End}\, F_n(G)$.

By exactly the same argument as that of Dolinka, Gould, Yang (2015), we have:

Lemma For each $i, j \in I'$, $\lambda, \mu \in \Lambda$, we have

(i) $\mathbf{p}_{\lambda i} = \varepsilon$ implies $f_{i,\lambda} = 1$.

(ii) $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ implies $f_{i,\lambda} = f_{j,\mu}$.

To show the above result is true for all $i, j \in I$ and all $\lambda, \mu \in \Lambda$, we need the following lemma.

Lemma For each $i \in I$, there exists $j \in I'$ such that for all $\lambda \in \Lambda$ with $\mathbf{p}_{\lambda i} \neq 0$, we have $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$.

Proof If $i \in I'$, then we are fine. If not, we define $\alpha \in \operatorname{End} F_n(G)$ as follows:

$$x_s\alpha = x_s r_i \text{ if } x_s \in \operatorname{dom} \mathbf{r}_i ; x_v\alpha = x_1 \text{ if } x_v \notin \operatorname{dom} \mathbf{r}_i .$$

Suppose $\ker \overline{\mathbf{r}_i} = \{L_1^{\mathbf{r}_i}, \cdots, L_r^{\mathbf{r}_i}\}$ with minimum elements $l_1^{\mathbf{r}_i} < \cdots < l_r^{\mathbf{r}_i}$.

Then we must have $x_{l_j^{\mathbf{r}_i}} \mathbf{r}_i = x_j$ for all $j \in [1, r]$. Let $\ker \overline{\alpha} = \{L_1^{\alpha}, \cdots, L_r^{\alpha}\}$ with minimum elements $l_1^{\alpha} < \cdots < l_r^{\alpha}$. Then $L_1^{\alpha} = L_1^{\mathbf{r}_i} \cup \{v \in [1, n] : x_v \notin \operatorname{dom} \mathbf{r}_i\}, L_2^{\alpha} = L_2^{\mathbf{r}_i}, \cdots, L_r^{\alpha} = L_r^{\mathbf{r}_i}$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

If $x_1 \in \operatorname{dom} \mathbf{r}_i$, then $l_1^{\mathbf{r}_i} = 1$ and $l_1^{\alpha} = 1$, so that $x_{l_1^{\alpha}}\alpha = x_{l_1^{\mathbf{r}_i}}\alpha = x_{l_1^{\mathbf{r}_i}}\mathbf{r}_i = x_1$.
If $x_1 \notin \operatorname{dom} \mathbf{r}_i$, then $l_1^{\mathbf{r_i}} \neq l_1^{\alpha} = 1$, $x_{l_1^{\alpha}}\alpha = x_1\alpha = x_1$. Also, $x_{l_j^{\alpha}}\alpha = x_{l_j^{\mathbf{r}_i}} r_i = x_j$, for all $j \in [2, r]$. Hence there exist some $j \in I'$ such that $\mathbf{r}_j = \alpha$.

Let $\lambda = (u_1, \cdots, u_r) \in \Lambda$ with $\mathbf{p}_{\lambda i} \neq 0$. Then $\{x_{u_1}, \cdots, x_{u_r}\} \in \operatorname{dom} \mathbf{r}_i$. Since the restriction of $\mathbf{r}_j$ to $\operatorname{dom} \mathbf{r}_i$ is equal to $\mathbf{r}_i$, we have $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Lemma For all $i \in I, \lambda \in \Lambda$, $p_{\lambda i} = \varepsilon$ implies $f_{i,\lambda} = 1$.

Proof If $i \in I'$, then clearly $f_{i,\lambda} = 1$. Suppose that $i \notin I'$. First, we have $\mathbf{p}_{\omega(i)i} = \varepsilon$ and $f_{i,\omega(i)} = 1$ by $(R1)$.

There exists $j \in I'$ such that

$$\mathbf{p}_{\lambda j} = \mathbf{p}_{\lambda i} = \varepsilon, \mathbf{p}_{\omega(i)j} = \mathbf{p}_{\omega(i)i} = \varepsilon$$

so

$$\begin{pmatrix} \mathbf{p}_{\lambda i} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\omega(i)i} & \mathbf{p}_{\omega(i)j} \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Note that $f_{j,\lambda} = 1$ and $f_{j,\omega(i)} = 1$, and so, by $(R3)$, we have $f_{i,\lambda} = 1$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Lemma For all $i, j \in I$ and $\lambda, \mu \in \Lambda$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ implies $f_{i,\lambda} = f_{j,\mu}$.

Proof There exists $i' \in I'$ such that

$$\mathbf{p}_{\lambda i'} = \mathbf{p}_{\lambda i}, \mathbf{p}_{\omega(i)i} = \mathbf{p}_{\omega(i)i'} = \varepsilon$$

so that

$$\begin{pmatrix} \mathbf{p}_{\lambda i} & \mathbf{p}_{\lambda i'} \\ \mathbf{p}_{\omega(i)i} & \mathbf{p}_{\omega(i)i'} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{\lambda i} & \mathbf{p}_{\lambda i'} \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Since $f_{i,\omega(i)} = f_{i',\omega(i)} = 1$, we have $f_{i,\lambda} = f_{i',\lambda}$ by $(R3)$.

Similarly, there exists $j' \in I'$ such that $\mathbf{p}_{\mu j} = \mathbf{p}_{\mu j'}$ and $f_{j,\mu} = f_{j',\mu}$.

We know $f_{i',\lambda} = f_{j',\mu}$, and hence $f_{i,\lambda} = f_{j,\mu}$.

We now denote all generators $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \alpha$ by f_α, where $(i, \lambda) \in K$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

We will say that for $\phi, \varphi, \psi, \sigma \in P$ the quadruple $(\phi, \varphi, \psi, \sigma)$ is singular in P if $\phi^{-1}\psi = \varphi^{-1}\sigma$ and we can find $i, j \in I, \lambda, \mu \in \Lambda$ with $\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j}$ and $\sigma = \mathbf{p}_{\mu j}$.

Exactly as in Dolinka, Gould and Yang (2015), we obtain the following lemma.

Lemma Let $\overline{\overline{H}}$ be the group given by the presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ with generators:

$$S = \{f_\phi : \ \phi \in P\}$$

and with the defining relations Γ :

$(P1)$ $f_\phi^{-1} f_\varphi = f_\psi^{-1} f_\sigma$ where $(\phi, \varphi, \psi, \sigma)$ is singular in P;

$(P2)$ $f_\epsilon = 1$.

Then $\overline{\overline{H}}$ is isomorphic to $\overline{H}$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Let I', Λ and D_r be defined as above and $P' = (\mathbf{p}_{\lambda i})$ where $i \in I'$ and $\lambda \in \Lambda$.

Let D'_r be the $\mathcal{D}$-class of ε in $\operatorname{End} F_n(G)$. Then

$$D'_r = \{\alpha \in \operatorname{End} F_n(G) : \operatorname{rank} \alpha = r\} \subseteq D_r.$$

We may use I' and Λ to denote the set of $\mathcal{R}$-classes and $\mathcal{L}$-classes of D'_r, respectively.

By defining $\omega' : I' \longrightarrow \Lambda$ as the restriction of ω on I' and K' as the restriction of K on $I' \times \Lambda$ we have the following result.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Lemma Let E be the biordered set of idempotents of End $F_n(G)$. Then the maximal subgroup $\overline{H'}$ of $\overline{\varepsilon}$ in $\mathrm{IG}(E)$ is defined by the presentation

$$\mathcal{P}' = \langle F' : \Sigma' \rangle$$

with generators:

$$F' = \{g_{i,\lambda} : \quad (i,\lambda) \in K'\}$$

and defining relations Σ':

$(R1')$ $g_{i,\lambda} = g_{i,\mu} \quad (\overline{\mathbf{h}}_\lambda \overline{\varepsilon}_{i\mu} = \overline{\mathbf{h}}_\mu)$;

$(R2')$ $g_{i,\omega(i)} = 1 \quad (i \in I')$;

$(R3')$ $g_{i,\lambda}^{-1} g_{i,\mu} = g_{k,\lambda}^{-1} g_{k,\mu} \quad \left(\begin{bmatrix} \varepsilon_{i\lambda} & \varepsilon_{i\mu} \\ \varepsilon_{k\lambda} & \varepsilon_{k\mu} \end{bmatrix} \text{ is singular i.e. } \mathbf{p}_{\lambda i}^{-1} \mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1} \mathbf{p}_{\mu k} \right)$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

The following two lemmas are taken from Dolinka, Gould, Yang(2015):

Lemma For each $i \in I', \lambda \in \Lambda$, $\mathbf{p}_{\lambda i} = \varepsilon$ implies $g_{i,\lambda} = 1$.

Lemma For each $i, j \in I'$, $\lambda, \mu \in \Lambda$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ implies $g_{i,\lambda} = g_{j,\mu}$.

We now denote all generators $g_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \alpha$ by g_α, where $(i, \lambda) \in K'$.

We will say that for $\phi, \varphi, \psi, \sigma \in P'$ the quadruple $(\phi, \varphi, \psi, \sigma)$ is *singular* in P' if $\phi^{-1}\psi = \varphi^{-1}\sigma$ and we can find $i, j \in I', \lambda, \mu \in \Lambda$ with $\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j}$ and $\sigma = \mathbf{p}_{\mu j}$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Lemma Let $\overline{\overline{H'}}$ be the group given by the presentation $\mathcal{Q}' = \langle S' : \Gamma' \rangle$ with generators:

$$S' = \{g_\phi : \ \phi \in P'\}$$

and with the defining relations Γ' :

$(P1)$ $g_\phi^{-1} g_\varphi = g_\psi^{-1} g_\sigma$ where $(\phi, \varphi, \psi, \sigma)$ is singular in P';

$(P2)$ $g_\epsilon = 1$.

Then $\overline{\overline{H'}}$ is isomorphic to $\overline{H'}$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Lemma The group $\overline{\overline{H}}$ with a presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ is isomorphic to the presentation $\mathcal{Q}' = \langle S' : \ \Gamma' \rangle$ of $\overline{\overline{H'}}$, and the later is isomorphic to the wreath product $G \wr \mathcal{S}_r$.

Proof Let $\widetilde{S}$ be the free group generated by S. We define a map

$$\boldsymbol{\theta} : \widetilde{S} \longrightarrow \overline{\overline{H'}}, f_{\phi}\boldsymbol{\theta} = g_{\phi}$$

for all $\phi \in P$.

First, $\boldsymbol{\theta}$ is well-defined, for which we need to show $\phi \in P'$. There must exist $i \in I, \lambda \in \Lambda$ such that $\mathbf{p}_{\lambda i} = \phi$. By the result we obtained, there exits $i' \in I'$ such that $\mathbf{p}_{\lambda i'} = \mathbf{p}_{\lambda i} = \phi$, so that $\phi \in P'$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Now we show $\Gamma \subseteq \ker \boldsymbol{\theta}$. Suppose that $(\phi, \varphi, \psi, \sigma)$ is singular in P and $f_\phi^{-1} f_\varphi = f_\psi^{-1} f_\sigma$ in $\overline{\overline{H}}$. Then there exists $i, j \in I, \lambda, \mu \in \Lambda$ with

$$\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j}, \sigma = \mathbf{p}_{\mu j}$$

Again, we know there exists $i', j' \in I'$ such that

$$\mathbf{p}_{\lambda i'} = \mathbf{p}_{\lambda i} = \phi, \mathbf{p}_{\mu i'} = \mathbf{p}_{\mu i} = \varphi, \mathbf{p}_{\lambda j'} = \mathbf{p}_{\lambda j} = \psi, \mathbf{p}_{\mu j'} = \mathbf{p}_{\mu j} = \sigma$$

so that $(\phi, \varphi, \psi, \sigma)$ is also singular in P', and hence $g_\phi^{-1} g_\varphi = g_\psi^{-1} g_\sigma$ in $\overline{\overline{H'}}$.

Therefore $\Gamma \subseteq \ker \boldsymbol{\theta}$, and so there is a well-defined morphism from $\overline{\boldsymbol{\theta}} : \overline{\overline{H}} \longrightarrow \overline{\overline{H'}}$ given by $f_\phi \overline{\theta} = g_\phi$, where $\phi \in P$.

The relationship between $f_{i,\lambda}$ and $p_{\lambda i}$

Conversely, we define a map

$$\boldsymbol{\psi} : \widetilde{S'} \longrightarrow \overline{\overline{H}}, g_\phi \boldsymbol{\psi} = f_\phi$$

for all $\phi \in P'$, where $\widetilde{S'}$ is the free group generated by S'. Clearly, $\boldsymbol{\psi}$ is well-defined and $\Gamma' \subseteq \ker \boldsymbol{\psi}$, so that there is a well-defined morphism $\overline{\boldsymbol{\psi}} : \overline{\overline{H'}} \longrightarrow \overline{\overline{H}}$ given by $g_\phi \overline{\psi} = f_\phi$, where $\phi \in P'$.

Also, $g_\phi \boldsymbol{\psi} \boldsymbol{\theta} = f_\phi \boldsymbol{\theta} = g_\phi$ for all $\phi \in P'$ and $f_\phi \boldsymbol{\theta} \boldsymbol{\psi} = g_\phi \boldsymbol{\psi} = f_\phi$ for all $\phi \in P$. Therefore $\overline{\overline{H'}} \simeq \overline{\overline{H}}$.

So, $\mathcal{Q} = \langle S : \Gamma \rangle$ is isomorphic to $\mathcal{Q}' = \langle S' : \Gamma' \rangle$, and the later is isomorphic to $G \wr \mathcal{S}_r$ by Dolinka, Gould and Yang(2015).

The main theorem

Theorem Let E be the biordered set of idempotents of the partial endomorphism monoid PEnd $F_n(G)$ of a rank n free G-act $F_n(G)$, where G is a group, $n \in \mathbb{N}$ and $n \geq 3$. Let $\varepsilon \in E$ be a rank r idempotent with $1 \leq r \leq n-2$. Then the maximal subgroup of $\mathrm{IG}(E)$ containing $\overline{\varepsilon}$ is isomorphic to the wreath product $G \wr \mathcal{S}_r$.

If ε is an idempotent with rank n or 0, that is, the identity map or empty map, then $\overline{H}$ is the trivial group, since it is generated (in $\mathrm{IG}(E)$) by idempotents of the same rank.

If the rank of ε is $n-1$, then $\overline{H}$ is the free group as there are no non-trivial singular squares in the $\mathcal{D}$-class of ϵ in PEnd $F_n(G)$.

The main theorem

If G is trivial, then PEnd $F_n(G)$ is essentially $\mathcal{PT}_n$, so we deduce the following result of Dolinka(2013)

Corollary Let IG(E) be the free idempotent generated semigroup over the biordered set E of idempotents of the partial transformation monoid $\mathcal{PT}_n$, where $n \in \mathbb{N}$ and $n \geq 3$. Let $\varepsilon \in E$ be a rank r idempotent with $1 \leq r \leq n-2$. Then the maximal subgroup $\overline{H}$ of IG(E) containing $\overline{\varepsilon}$ is isomorphic to the maximal subgroup H of $\mathcal{PT}_n$ containing ε, and hence to $\mathcal{S}_r$.

Thank you!