Arens regularity of semigroups and semigroup algebras

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Semigroups

Let S be a semigroup.

The **centre** of S is the sub-semigroup $\mathfrak{Z}(S)$, where

$$\mathfrak{Z}(S) = \{t \in S : st = ts \ \ (s \in S)\}\,,$$

and S is **abelian** when $\mathfrak{Z}(S) = S$

An element $p \in S$ is **idempotent** if $p^2 = p$, and S is **idempotent** if every element is idempotent.

The semigroup is **weakly cancellative** if the equations $xs = t$ and $sx = t$ have only finitely-many solutions for x for each $s, t \in S$. For $s \in S$, set $L_s(t) = st$ and $R_s(t) = ts$ for $t \in S$. An element $s \in S$ is **cancellable** if both L_s and R_s are injective, and S is **cancellative** if each $s \in S$ is cancellable.

Semilattices

Let $(S, \leq)$ be a non-empty, partially ordered set, and suppose that

$$s \wedge t = \min\{s, t\}$$

exists for all $s, t \in S$. Then $(S, \wedge)$ is an abelian, idempotent semigroup, called a **semilattice**.

Conversely, suppose that S is an abelian, idempotent semigroup. Take $s, t \in S$, and set $s \leq t$ if $st = s$. Then $(S, \leq)$ is a semilattice and $s \wedge t = st$ $(s, t \in S)$. Hence $(S, \wedge)$ is a semigroup that can be identified with S.

Stone–Čech compactifications

The **Stone–Čech compactification** of a set S is denoted by βS; we regard S as a subset of βS, and set $S^* = \beta S \setminus S$; this is the **growth** of S.

The space βS is each of the following:

• - abstractly characterized by a universal property: βS is a compactification of S such that each bounded function from S to a compact space K has an extension to a continuous map from βS to K;

• - the space of ultrafilters on S;

• - the Stone space of the Boolean algebra $\mathcal{P}(S)$, the power set of S;

• - the character space of the commutative C^*-algebra $\ell^\infty(S)$, so that $\ell^\infty(S) = C(\beta S)$.

The space βS is big: if $|S| = \kappa$ (infinite), then $|\beta S| = 2^{2^\kappa}$.

Semigroup compactifications

Let S be a semigroup. Then βS becomes a semigroup, as follows.

For each $s \in S$, the map $L_s : S \to \beta S$ has an extension to a continuous map $L_s : \beta S \to \beta S$. For $u \in \beta S$, define $s \,\Box\, u = L_s(u)$.

Next, the map $R_u : s \mapsto s \,\Box\, u,\ \ S \to \beta S$, has an extension to a continuous map $R_u : \beta S \to \beta S$ for each $u \in \beta S$. Define

$$u \,\Box\, v = R_v(u) \quad (u, v \in \beta S)\,.$$

Similarly we have $(\beta S, \diamond)$ (exchanging right and left).

In the case where S is abelian, $u \diamond v = v \,\Box\, u$ for all $u, v \in \beta S$.

A book on this is [HS].

Compact, right topological semigroup

Definition A semigroup X with a topology τ is a **compact, right topological semigroup** if (X, τ) is a compact space and the map R_v is continuous with respect to τ for each $v \in X$.

For example, let S be a semigroup. Then $X = \beta S$ is compact, right topological semigroup wrt $\Box$. The map L_s is continuous on X when $s \in S$, but maybe not more generally.

The **left topological centre** of X is:

$$\mathfrak{Z}_t^{(\ell)}(X) = \{u \in X : u \Box v = u \Diamond v \ \ (v \in X)\},$$

and we have

$$\mathfrak{Z}_t^{(\ell)}(X) = \{u \in X : L_u \text{ is continuous on } X\},$$

so $S \subset \mathfrak{Z}_t^{(\ell)}(X) \subset X$.

In the case where S is abelian, $\mathfrak{Z}_t^{(\ell)}(X) = \mathfrak{Z}(X)$.

Arens regularity

The semigroup S is **Arens regular** (AR) if $\mathfrak{Z}_t^{(\ell)}(X) = X$ and **strongly Arens irregular** (SAI) if $\mathfrak{Z}_t^{(\ell)}(X) = \mathfrak{Z}_t^{(r)}(X) = S$.

A subset V of X is **determining for the left topological centre** (a DLTC set) if $u \in \mathfrak{Z}_t^{(\ell)}(X)$ whenever $u \in X$ and $u \,\Box\, v = u \Diamond v \ (v \in V)$.

The following theorem is in [DLS]; see also [PaS].

Theorem Let S be an infinite, weakly cancellative semigroup. Then S is SAI, and there is a two-point subset of S^* that is a DLTC set for X. $\Box$

Banach algebras

Let A be an algebra such that $(A, \|\cdot\|)$ is also a Banach space. Then A is a **Banach algebra** if also $\|ab\| \le \|a\|\,\|b\|$ for all $a, b \in A$.

Example Let S be a non-empty set. Consider the linear space of functions $f : S \to \mathbb{C}$ such that $\sum_{s\in S} |f(s)| < \infty$. This is the space $\ell^1(S)$. It is a Banach space for the norm

$$\|f\|_1 = \sum_{s\in S} |f(s)| \,.$$

Now suppose that S is a semigroup, and write δ_s for the characteristic function of $\{s\}$. Then $\ell^1(S)$ is a Banach algebra, where the **convolution product** is specified by $\delta_s \star \delta_t = \delta_{st}$ for all $s, t \in S$. It is the **semigroup algebra** of S.

This algebra is commutative if and only if S is abelian.

Banach spaces

Let E be a Banach space. The closed unit ball is $E_{[1]}$. The **dual space** of E is E'; it is the Banach space of all bounded linear functionals on E.

Write $\langle x, \lambda \rangle = \lambda(x)$ for $x \in E$ and $\lambda \in E'$. Thus $\langle \cdot\ ,\ \cdot \rangle$ gives the **duality**.

The **weak-$*$ topology** on E' is such that $\lambda_\alpha \to 0$ iff $\langle x, \lambda_\alpha \rangle \to 0$ for each $x \in E$. The closed unit ball of E' is weak-$*$ compact.

The **bidual** is $E'' = (E')'$. The map

$$\kappa : E \to E'' ,$$

where $\langle \kappa(x), \lambda \rangle = \langle x, \lambda \rangle$, is an isometric embedding, so that E is a closed subspace of E''.

$M(\beta S)$

Example Start with a non-empty set S and $E = \ell^1(S)$. Then we can identify E' with $\ell^\infty(S) = C(X)$, where $X = \beta S$. The bidual E'' is $C(X)' = M(X)$, the Banach space of all complex-valued, regular Borel measures μ on X, with

$$\|\mu\| = |\mu|(X)\,.$$

Then $M(X) = \ell^1(S) \oplus M(S^*)$ as Banach spaces.

Biduals of Banach algebras

Let A be a Banach algebra. Then there are two natural products, $\Box$ and $\Diamond$, on the bidual A'' of A; they are called the **Arens products**.

There are formal definitions, but we shall use:

$$\mathsf{M} \Box \mathsf{N} = \lim_{\alpha} \lim_{\beta} a_\alpha b_\beta \,, \qquad \mathsf{M} \Diamond \mathsf{N} = \lim_{\beta} \lim_{\alpha} a_\alpha b_\beta$$

for $\mathsf{M}, \mathsf{N} \in A''$, where (a_α) and (b_β) are nets in A with $\lim_\alpha a_\alpha = \mathsf{M}$ and $\lim_\beta b_\beta = \mathsf{N}$ in the weak-$*$ topology, $\sigma(A'', A')$.

Set $A'' = (A'', \Box\,)$. We regard A as a closed subalgebra of A''.

In the case where A is abelian, $\mathsf{M} \Diamond \mathsf{N} = \mathsf{N} \Box \mathsf{M}$.

Example Let S be a semigroup. Then we identify $\ell^{\,1}(S)''$ with $M(X)$, where $X = \beta S$, and this gives two products on $M(X)$. $\Box$

See the book [DU] for an account of Arens regularity.

Arens regularity

Let A be a Banach algebra

The algebra A is **Arens regular** (AR) if $\Box$ and $\Diamond$ coincide on A''. All C^*-algebras are AR.

There are definitions of $\mathfrak{Z}_t^{(\ell)}(A'')$ and $\mathfrak{Z}_t^{(r)}(A'')$, and A is **strongly Arens irregular** (SAI) if both are equal to A.

In the case where A is commutative, both are equal to $\mathfrak{Z}(A'')$, the centre of A'', and $A \subset \mathfrak{Z}(A'') \subset A''$.

Example Let S be a semigroup. We identify $u \in X$ with the point mass at u, and then the definitions of $u \,\Box\, v$ and $\delta_u \,\Box\, \delta_v$ are consistent for $u, v \in X$. Suppose that S is abelian. Then

$$\mathfrak{Z}(M(\beta S)) \cap \beta S \subset \mathfrak{Z}(\beta S)\,.$$

We do not have an example where the above inclusion is proper. Can you see one? $\Box$

DLTC sets

Definition Let A be a Banach algebra. A non-empty subset V of A'' is **determining for the left topological centre** (a DLTC set)for A'' if $\mathsf{M} \in \mathfrak{Z}_t^{(\ell)}(A'')$ whenever $\mathsf{M} \in A''$ and $\mathsf{M} \,\Box\, \mathsf{N} = \mathsf{M} \diamond \mathsf{N}$ $(\mathsf{N} \in V)$.

The following is Theorem 12.15 of [DLS].

Theorem Let S be an infinite, cancellative semigroup. Then the semigroup algebra $\ell^1(S)$ is SAI, and there exist a and b in S^* such that the two-point set $\{\delta_a, \delta_b\}$ is a DLTC set for $M(X)$. $\Box$

A related theorem:

Theorem Let G be a locally compact group, and consider the group algebra $A = (L^1(G), \star)$. Then A is SAI, and there is a two-element DLTC set in A''. $\Box$

Totally ordered semigroups

Let T be an infinite, totally ordered space. For $s, t \in T$, we set

$$s \wedge t = \min\{s, t\}$$

so that T is a lattice and a semigroup with respect to the operation $\wedge$. Also T is an abelian, idempotent semigroup, i.e., a semilattice.

We further suppose that T has a minimum element, called 0, and a maximum element, called ∞, and that T is complete, in the sense that every non-empty subset of T has a supremum and an infimum. We give T its interval topology, so that the closed intervals provide a subbase for the closed sets

Then T is then a compact topological semigroup.

The semigroup S

We take S to be an arbitrary, infinite subset of T, so that S is a sub-semigroup of $(T, \wedge)$ that is also an abelian, idempotent semigroup.

[In fact, each totally ordered semigroup $(S, \wedge)$ can be embedded in a complete, compact, totally ordered topological semigroup $(T, \wedge)$; this is well-known.]

We let E denote the set of accumulation points of S in T.

Examples Set $T = \mathbb{N} \cup \{\infty\}$ and $S = \mathbb{N}$, so that $E = \{\infty\}$; set $T = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and $S = \mathbb{Q}$, so that $E = T$; set $T = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, and $S = [0, \omega_1)$, so that E is the collection of limit ordinals in T. These examples are not weakly cancellative. $\square$

The space X

We regard S as having the discrete topology. Set $X = \beta S$, and $X^* = X \setminus S$, as before.

The closures of a subset A of S in T and X are $\mathrm{cl}_T A$ and $\mathrm{cl}_X A$, respectively. The map

$$\pi : X \to T$$

denotes the continuous extension of the inclusion map of S into T, so that $\pi(X) = \mathrm{cl}_T S$. We shall write F_t for the fibre $\{x \in X : \pi(x) = t\}$ for $t \in \mathrm{cl}_T S$. We set $F_t^* = F_t \cap X^*$, so that

$$F_t^* = F_t \quad (t \in T \setminus S) \quad \text{and} \quad F_t^* = F_t \setminus \{t\} \quad (t \in S)\,.$$

Take $t \in T$. Then F_t^* is a closed, and hence compact, subset of X, and clearly $F_t^* \neq \emptyset$ if and only if $t \in E$.

We recall the standard facts, that, for every subset A of S, the set $\mathrm{cl}_X A$ is clopen in X, and that, for every subsets A and B of S that are disjoint, the two sets $\mathrm{cl}_X A$ and $\mathrm{cl}_X B$ are disjoint in X.

A character

Let S be a semigroup. Take a subset A of S, and suppose that $\mu \in M(X)_{[1]}$ with $\operatorname{supp}\mu \subset \mathrm{cl}_X A$. Then the measure μ belongs to the weak-$*$ closure of $\operatorname{aco}\{\delta_s : s \in A\}$.

The **augmentation character** on $(M(X), \Box)$ is the map

$$\varphi_0 : \mu \mapsto \langle 1_X, \mu \rangle = \mu(X) = \int_X \mathrm{d}\mu \,.$$

Clearly $\varphi_0(\mu \Box \nu) = \varphi_0(\mu)\varphi_0(\nu) \;\; (\mu, \nu \in M(X))$, so φ_0 is indeed a character on $(M(X), \Box)$, and φ_0 is weak-$*$ continuous.

Lemmas

Lemma 1 Suppose that A and B are subsets of S such that $A \leq B$. Take $\mu, \nu \in M(X)_{[1]}$ with $\operatorname{supp}\mu \subset \mathrm{cl}_{\mathrm{X}}\mathrm{A}$ and $\operatorname{supp}\nu \subset \mathrm{cl}_{\mathrm{X}}\mathrm{B}$. Then

$$\mu \,\Box\, \nu = \nu \,\Box\, \mu = \varphi_0(\nu)\mu \,.$$

Proof Take $\sigma \in \mathrm{aco}\{\delta_s : s \in A\}$, $\tau \in \mathrm{aco}\{\delta_t : t \in B\}$. We have $\delta_s \star \delta_t = \delta_t \star \delta_s = \delta_s$ $(s \in A,\, t \in B)$, and so $\sigma \star \tau = \tau \star \sigma = \varphi_0(\tau)\sigma$. Take weak-$*$ limits. $\Box$

Take $t \in E$, and now set $A = S \cap [0, t)$ and $B = S \cap (t, \infty]$. Then $F_t^* \cap \mathrm{cl}_X A$ and $F_t^* \cap \mathrm{cl}_X B$ are disjoint, compact subsets of F_t^* whose union is F_t^*.

Lemma 2 Suppose that $p, q \in F_t^* \cap \mathrm{cl}_X A$. Then $p \,\Box\, q = p$. Suppose that $p \in F_t^* \cap \mathrm{cl}_X A$ and that $q \in F_t^* \cap \mathrm{cl}_X B$. Then $p \,\Box\, q = q \,\Box\, p = p$. $\Box$

A theorem

Theorem The semigroup $(S, \wedge)$ is SAI, and the semigroup algebra $(\ell^1(S), \star)$ is not AR.

Proof Consider a point $p \in X^*$, say $p \in F_t^*$, where $t \in E$. We may suppose that $p \in \mathrm{cl}_X A$, and so there exists $q \in F_t^* \cap \mathrm{cl}_X A$ with $q \neq p$. Hence $p \,\Box\, q \neq q \,\Box\, p$, and so $p \notin \mathfrak{Z}(X)$. Hence S is SAI.

Clearly $\delta_p \notin \mathfrak{Z}(M(X))$, and so $\ell^1(S)$ is not AR. $\square$

Another theorem

Lemma Let $\mu \in M(X)$, and take $t \in E$. Suppose that $\mu \mid (F_t^* \cap A) = 0$. Then $\mu \,\Box\, p = p \,\Box\, \mu$ for each $p \in F_t^* \cap A$.

Proof Take $\varepsilon > 0$. Then there exists $u \in S$ with $u < t$ and $|\mu|\,(\mathrm{cl}_X(S \cap (u,t))) < \varepsilon$. This shows by previous lemmas that $\|\mu \,\Box\, p - p \,\Box\, \mu\| < \varepsilon$. Hence $\mu \,\Box\, p = p \,\Box\, \mu$. $\Box$

Theorem Let $\mu \in M(X)$. Then $\mu \in \mathfrak{Z}(M(X))$ if and only if $\mu \mid F_t^* = 0$ $(t \in E)$.

Proof If $\mu \mid F_t^* \neq 0$, use $p, q \in F_t^*$. If $\mu \mid F_t^* = 0$ for each $t \in E$, use the lemma. $\Box$

Scattered spaces

A compact space K is **scattered** if each non-empty subset of K contains a point that is isolated in the subset. This happens if and only if $f(K)$ is countable for each $f \in C(K)$ if and only if $\ell^1(K) = M(K)$.

Theorem The semigroup algebra $(\ell^1(S), \star)$ is SAI if and only if $\mathrm{cl}_T S$ is scattered. $\square$

For example, $\mathbb{N} \cup \{\infty\}$ and $[0, \omega_1]$ are scattered, and so $(\mathbb{N}, \wedge)$ and $([0, \omega_1), \wedge)$ are SAI, but $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is not scattered, and so $(\mathbb{Q}, \wedge)$ is not SAI.

DTC sets for $M(X)$

For each $t \in E$, choose two distinct points in $F_t^* \cap A$ and two distinct points in $F_t^* \cap B$ whenever the respective sets are non-empty. The collection of these points is called V. Now take $\mu \in M(X)$, and suppose that $\mu \,\Box\, p = p \,\Box\, \mu$ for each $p \in V$. For each $t \in E$, it follows that $\mu \mid (F_t^* \cap A) = 0$ and that $\mu \mid (F_t^* \cap B) = 0$. Thus $\mu \mid F_t^* = 0$. This implies that $\mu \in \mathfrak{Z}(M(X))$, and hence V is a DTC set for $M(X)$.

This shows that $M(X)$ has a DTC set consisting of at most 2^κ points of X, where $\kappa = |E|$; this is a small subset of X because $|X| = 2^{2^\kappa}$.

Suppose that E is infinite. Then there cannot be a finite DTC set for the SAI semigroup S. For suppose that V is a finite subset in X, and choose $t \in E \setminus \pi(V)$, and then choose $p \in F_t^*$. We have $p \,\Box\, v = v \,\Box\, p$ $(v \in V)$, but $p \notin S$, and so V is not a DTC set for S.

DTC sets for $M(X)$, bis

Theorem Suppose that the set E is finite. Then there is a finite DTC set for S.
Suppose that the set E is infinite or uncountable. Then there is no finite or countable DTC set for S, respectively. $\square$

Theorem Suppose that the set E is countable. Then the semigroup algebra $\ell^1(S)$ is SAI and has a DTC set consisting of at most four measures in $M(X^*)^+$.

Proof Suppose that (s_n) is a sequence in S such that $F^*_{s_n} \cap \mathrm{cl}_X(S \cap [0, s_n)) \neq \emptyset$. Choose two points $p_{1,n}$ and $p_{2,n}$ in $F^*_{s_n} \cap \mathrm{cl}_X(S \cap [0, s_n))$, and set

$$\mu_j = \sum \frac{1}{2^n}\delta_{p_{j,n}} \quad (j = 1, 2)\,.$$

We obtain two measures in $M(X^*)^+$. Similarly on the other side, so we have four measures. These work. $\square$

Examples 1

(I) Consider the semigroup $S = (\mathbb{N}, \wedge)$. Then $\ell^{\,1}(S)$ is SAI and any two distinct points in $\mathbb{N}^*$ form a DTC set for $M(\beta\,\mathbb{N})$.

(II) Consider the semigroup $S = (\mathbb{Q}, \wedge)$. Then S is SAI, but there is no countable DTC set for S. The semigroup algebra $\ell^{\,1}(S)$ is neither AR nor SAI, and we know $\mathfrak{Z}(M(X))$. There is no countable DTC set for $M(X)$.

(III) Consider the subset S of

$$T := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

that consists of numbers of the form $n - x$, where $n \in \mathbb{Z}$ and $x \in \{1/2, 1/4, 1/8, \dots\}$. Then the corresponding set E is $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, a countable set. There is no finite DTC set for the semigroup S. However $\ell^{\,1}(S)$ is SAI, and there is a two-element DTC set in $M(X^*)^+$.

Examples 2

(IV) Consider the semigroup $S = T = ([0, \kappa], \wedge)$, where κ is a cardinal with $|\kappa| \geq \aleph_1$, so that the corresponding set E has cardinality κ. Since T is scattered, the algebra $\ell^1(S)$ is SAI. But there is no DTC set for $M(X)$ with cardinality strictly less than κ. This shows that the cardinality of a DTC set can be arbitrarily large, even when $\ell^1(S)$ is SAI.

A question

Set $S = T = \{0, 1\}^\kappa$, where κ is an infinite cardinal. Then $(S, \wedge)$ is a lattice, but S is not totally ordered. Are S and $\ell^1(S)$ both SAI?