Undecidability of the word problem for one-relator inverse monoids

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University of East Anglia

Word problem for one-relator groups and monoids

	$\mathrm{Gp}\langle A \mid w = 1\rangle$ $\mathrm{FG}(A)/\langle\langle w\rangle\rangle$	$\mathrm{Mon}\langle A \mid w = 1\rangle$ $A^*/\langle\langle (w,1)\rangle\rangle$	$\mathrm{Inv}\langle A \mid w = 1\rangle$ $\mathrm{FIM}(A)/\langle\langle (w,1)\rangle\rangle$
Word problem decidable	Magnus (1932) ✓	Adjan (1966) ✓	?

Theorem (Scheiblich (1973) & Munn (1974))

Free inverse monoids have decidable word problem.

Conjecture (Margolis, Meakin, Stephen (1987))

If $M = \mathrm{Inv}\langle A \mid w = 1\rangle$, then the word problem for M is decidable.

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Proved when w is: an idempotent word, a sparse word, a one-relator surface group relation, a Baumslag–Solitar relation, a relation of Adjan type,...
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Theorem (Ivanov, Margolis, Meakin (2001))

If the word problem is decidable for all inverse monoids of the form $\mathrm{Inv}\langle A \mid w = 1\rangle$ then the word problem is also decidable for every one-relator monoid $\mathrm{Mon}\langle A \mid u = v\rangle$.

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Theorem (RDG (2019))

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

Ingredients for the proof:

- Submonoid membership problem for one relator groups.
- HNN-extensions and free products of groups.
- Right-angled Artin groups (RAAGs).
- Right units of special inverse monoids

$$\mathrm{Inv}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1\rangle$$

  and Stephen's procedure for constructing Schützenberger graphs.
- Properties of E-unitary inverse monoids.

Inverse monoid presentations

An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

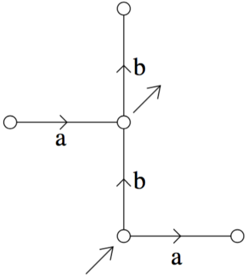
For all $x, y \in M$ we have

$$x = xx^{-1}x, \ (x^{-1})^{-1} = x, \ (xy)^{-1} = y^{-1}x^{-1}, \ xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \qquad (\dagger)$$

$$\mathrm{Inv}\langle A \mid u_i = v_i \ (i \in I) \rangle = \mathrm{Mon}\langle A \cup A^{-1} \mid u_i = v_i \ (i \in I) \cup (\dagger) \rangle$$

where $u_i, v_i \in (A \cup A^{-1})^*$ and x, y range over all words from $(A \cup A^{-1})^*$.

Free inverse monoid $\mathrm{FIM}(A) = \mathrm{Inv}\langle A \mid \ \rangle$



Munn (1974)

Elements of $\mathrm{FIM}(A)$ can be represented using Munn trees. e.g. in $\mathrm{FIM}(a, b)$ we have $u = w$ where

$u = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$
$w = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b$

The word problem

M - a finitely generated monoid with a finite generating set A.

$\pi : A^* \to M$ – the canonical monoid homomorphism.

The monoid M has decidable word problem if there is an algorithm which solves the following decision problem:

INPUT: Two words $u, v \in A^*$.

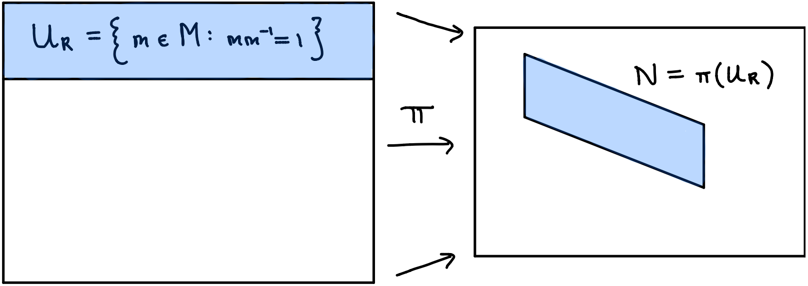
QUESTION: $\pi(u) = \pi(v)$? i.e. do u and v represent the same element of the monoid M?

For a group or an inverse monoid with generating set A the word problem is defined in the same way except the input is two words $u, v \in (A \cup A^{-1})^*$.

Example. The bicyclic monoid $\mathrm{Inv}\langle a \mid aa^{-1} = 1 \rangle$ has decidable word problem.

Proof strategy

$M = \mathrm{Inv}\langle A \mid r = 1\rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1\rangle$



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

since for $w \in (A \cup A^{-1})^*$ $\boxed{w \in U_R \iff ww^{-1} = 1}$

(sometimes) $\rightsquigarrow$ membership problem for $N \leq G$ is decidable

RAAGs induced subgraphs and subgroups

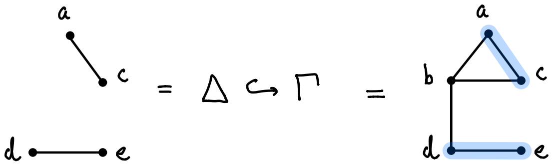
Definition

The right-angled Artin group $A(\Gamma)$ associated with the graph Γ is the group defined by the presentation

$$\mathrm{Gp}\langle V\Gamma \mid uv = vu \text{ if and only if } \{u, v\} \in E\Gamma\rangle.$$

Fact: If Δ is an induced subgraph of Γ then the embedding $\Delta \to \Gamma$ induces an embedding $A(\Delta) \to A(\Gamma)$.

Example



$$A(\Delta) = \mathrm{Gp}\langle a, c, d, e \mid ac = ca,\ de = ed\rangle$$

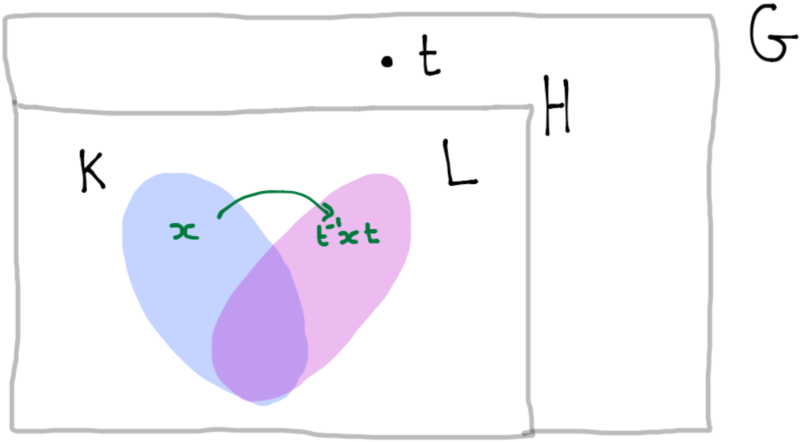
$$\hookrightarrow A(\Gamma) = \mathrm{Gp}\langle a, b, c, d, e \mid ac = ca,\ de = ed,\ ab = ba,\ bc = cb,\ bd = db\rangle$$

HNN-extensions of groups

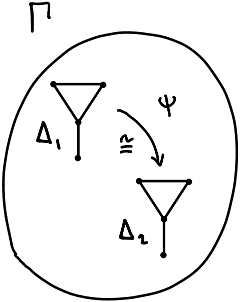
$H \cong \mathrm{Gp}\langle A \mid R \rangle$, $\quad K, L \leq H$ with $K \cong L$. Let $\phi : K \to L$ be an isomorphism.
The HNN-extension of H with respect to ϕ is

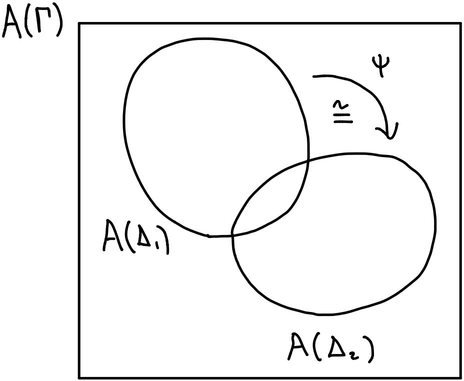
$$G = \mathrm{HNN}(H, \phi) = \mathrm{Gp}\langle A,\ t \mid R,\ t^{-1}kt = \phi(k)\ (k \in K) \rangle$$

Fact: H embeds naturally into the HNN extension $G = HNN(H, \phi)$.



HNN-extensions of RAAGs





Definition

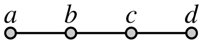
Γ - finite graph, $\psi : \Delta_1 \to \Delta_2$ an isomorphism between finite induced subgraphs.

$A(\Gamma, \psi)$ is defined to be the HNN-extension of $A(\Gamma)$ with respect to the isomorphism $A(\Delta_1) \to A(\Delta_2)$ induced by ψ.

Fact: $A(\Gamma)$ embeds naturally into $A(\Gamma, \psi)$.

HNN-extension of $A(P_4)$ over $A(P_3)$

Let P_4 be the graph



$$A(P_4) = \mathrm{Gp}\langle a, b, c, d \mid ab = ba, bc = cb, cd = dc\rangle.$$

Δ_1 - subgraph induced by $\{a, b, c\}$, Δ_2 subgraph induced by $\{b, c, d\}$,
$\psi : \Delta_1 \to \Delta_2$ - the isomorphism $a \mapsto b$, $b \mapsto c$, and $c \mapsto d$.

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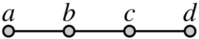

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Then the HNN-extension $A(P_4, \psi)$ of $A(P_4)$ with respect to ψ is

$$\begin{aligned} & A(P_4, \psi) \\ = \quad & \mathrm{Gp}\langle a, b, c, d, t \mid ab = ba, bc = cb, cd = dc, tat^{-1} = b, tbt^{-1} = c, tct^{-1} = d \rangle \end{aligned}$$

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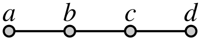

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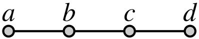

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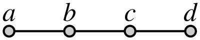

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Conclusion

$A(P_4)$ embeds into the one-relator group

$$A(P_4, \psi) = \mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle.$$

Submonoid membership problem

G - a finitely generated group with a finite group generating set A.

$\pi : (A \cup A^{-1})^* \to G$ – the canonical monoid homomorphism.

T – a finitely generated submonoid of G.

The membership problem for T within G is decidable if there is an algorithm which solves the following decision problem:

INPUT: A word $w \in (A \cup A^{-1})^*$.

QUESTION: $\pi(w) \in T$?

Theorem B

Let G be the one-relator group $\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle$. Then there is a fixed finitely generated submonoid N of G such that the membership problem for N within G is undecidable.

Proof of Theorem B

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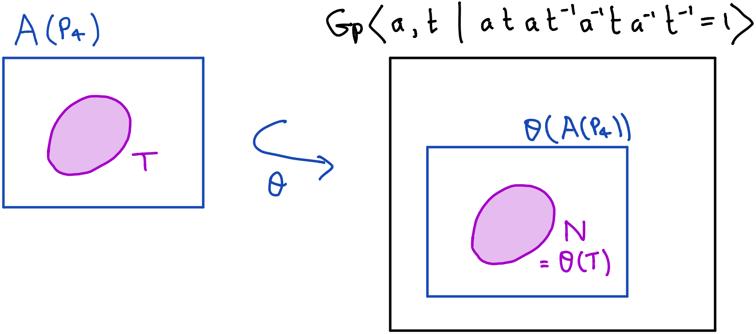
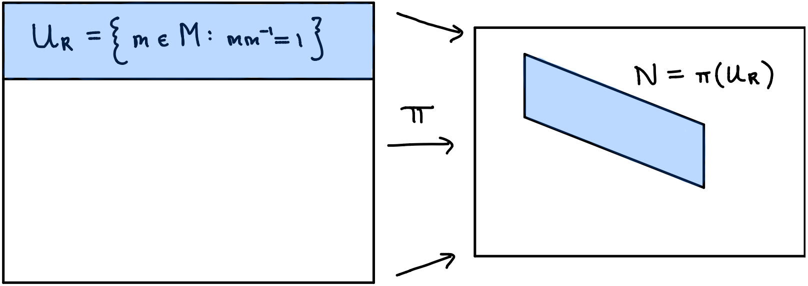
Proof. By [Lohrey & Steinberg, 2008] there is a finitely generated submonoid T of $A(P_4)$ such that the membership problem for T within $A(P_4)$ is undecidable. Let $\theta : A(P_4) \to G$ be an embedding. Then $N = \theta(T)$ is a finitely generated submonoid of G such that the membership problem for N within G is undecidable.



Proof strategy

$M = \mathrm{Inv}\langle A \mid r = 1\rangle \longrightarrow G = \mathrm{Gp}\langle A \mid r = 1\rangle$



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General observations about inverse monoids

S – an inverse monoid generated by A, $E(S)$ – set of idempotents,

$U_R \leq S$ – right units = submonoid if right invertible elements.

- If $e \in E(S)$ and $e \in U_R$ then $e = 1$.
- Two relations for the price of one: If e is an idempotent in $\mathrm{FIM}(A)$ and $r \in (A \cup A^{-1})^*$ then

$$\mathrm{Inv}\langle A \mid er = 1 \rangle = \mathrm{Inv}\langle A \mid e = 1, r = 1 \rangle.$$

- $e \in (A \cup A^{-1})^*$ is an idempotent in $\mathrm{FIM}(A)$ if and only if e freely reduces to 1 in the free group $\mathrm{FG}(A)$. e.g.

$$x^{-1}y^{-1}xx^{-1}yzz^{-1}x \in E(\mathrm{FIM}(x, y, z)).$$

A general construction

Let $A = \{a_1, \ldots, a_n\}$ and let $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$. Let M be the inverse monoid

$$\mathrm{Inv}\langle A, t \mid er = 1\rangle$$

where e is the idempotent word

$$a_1a_1^{-1} \ldots a_na_n^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1}) \ldots (tw_kt^{-1})(tw_k^{-1}t^{-1})a_n^{-1}a_n \ldots a_1^{-1}a_1.$$

So M is equal to

$$\mathrm{Inv}\langle A, t \mid r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A), \\ (tw_it^{-1})(tw_it^{-1})^{-1} = 1 \ (1 \leq i \leq k)\rangle$$

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$$a_1 a_1^{-1} \ldots a_n a_n^{-1} (t w_1 t^{-1})(t w_1^{-1} t^{-1})(t w_2 t^{-1})(t w_2^{-1} t^{-1}) \ldots (t w_k t^{-1})(t w_k^{-1} t^{-1}) a_n^{-1} a_n \ldots a_1^{-1} a_1.$$

So M is equal to

$$\mathrm{Inv}\langle A, t \mid r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A), \\ (t w_i t^{-1})(t w_i t^{-1})^{-1} = 1 \ (1 \leq i \leq k) \rangle$$

Which is the free product of inverse monoids

$$\mathrm{Gp}\langle A \mid r = 1 \rangle * \mathrm{FIM}(t) = \mathrm{Inv}\langle A \mid \ r = 1, aa^{-1} = 1, a^{-1}a = 1 \ (a \in A) \rangle * \mathrm{Inv}\langle t \mid \ \rangle$$

modulo the relations

$$(t w_i t^{-1})(t w_i t^{-1})^{-1} = 1$$

which say "$t w_i t^{-1}$ is right invertible" for all $1 \leq i \leq k$.

A general construction

For any $r, w_1, \ldots w_k \in (A \cup A^{-1})^*$, the one-relator inverse monoid $M = \mathrm{Inv}\langle A, t \mid er = 1 \rangle$ is equal to

$$\mathrm{Inv}\langle A, t \mid \; r = 1, aa^{-1} = 1, a^{-1}a = 1 \;(a \in A), (tw_it^{-1})(tw_it^{-1})^{-1} = 1 \;(1 \le i \le k)\rangle$$

which is equal to

$$\mathrm{Gp}\langle A \mid r = 1\rangle * \mathrm{FIM}(t) \;\; / \;\; \{(tw_it^{-1})(tw_it^{-1})^{-1} = 1 \;(1 \le i \le k)\}.$$

Key claim

Let T be the submonoid of $G = \mathrm{Gp}\langle A \mid r = 1\rangle$ generated by $\{w_1, w_2, \ldots, w_k\}$. Then for all $u \in (A \cup A^{-1})^*$ we have

$$u \in T \text{ in } G \iff tut^{-1} \in U_R \text{ in } M.$$

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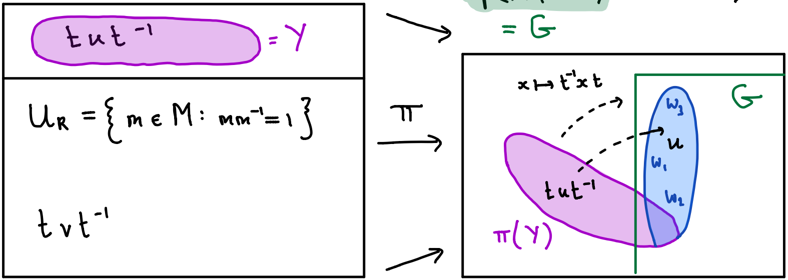
$$u \in T \text{ in } G \Longleftrightarrow tut^{-1} \in U_R \text{ in } M.$$

Theorem

If M has decidable word problem then the membership problem for T within G is decidable.

Proof strategy refined

$M = \mathrm{Inv}\langle A, t \mid er = 1\rangle \longrightarrow \mathrm{Gp}\langle A, t \mid r = 1\rangle = \mathrm{Gp}\langle A \mid r = 1\rangle * FG(t) = G$



If M has decidable word problem

$\Longrightarrow$ membership problem for $U_R \leq M$ is decidable

$\Longrightarrow$ $\forall\, u \in (A \cup A^{-1})^*$ can decide $tut^{-1} \in U_R$?

$\Longrightarrow$ (by key claim) can decide $u \in T = \mathrm{Mon}\langle w_1, \ldots, w_k\rangle \leq G$

Tying things together

Thoerem A

There is a one-relator inverse monoid $\mathrm{Inv}\langle A \mid w = 1\rangle$ with undecidable word problem.

Proof.

Let $A = \{a, z\}$ and let G be the one-relator group

$$\mathrm{Gp}\langle a, z \mid azaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle.$$

Let $W = \{w_1, \ldots, w_k\}$ be a finite subset of $(A \cup A^{-1})^*$ such that the membership problem for $T = \mathrm{Mon}\langle W\rangle$ within G is undecidable. Such a set W exists by Theorem B. Set e to be the idempotent word

$$aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})(tw_2t^{-1})(tw_2^{-1}t^{-1})\ldots(tw_kt^{-1})(tw_k^{-1}t^{-1})z^{-1}za^{-1}a.$$

Then by the above theorem the one-relator inverse monoid

$$\mathrm{Inv}\langle a, z, t \mid eazaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle$$

has undecidable word problem. This completes the proof. □

Open problems

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For which words $w \in (A \cup A^{-1})^*$ does $\mathrm{Inv}\langle A \mid w = 1 \rangle$ have decidable word problem? In particular is the word problem always decidable when w is (a) reduced or (b) cyclically reduced?

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Something more speculative...

Free adequate semigroups exist and have decidable word problem Branco, Gomes & Gould (2011) and Kambites (2011). Does it make sense to talk about "one-relator adequate monoids"? If yes, what can be said about the word problem for these monoids?

Further notes / ideas for the proof

All of the following facts are also used in the proof:

- If $w \in (A \cup A^{-1})^*$ represents a right invertible element of S then $w = \mathrm{red}(w)$ in S.
- If $\mathrm{Inv}\langle A \mid R \rangle$ is E-unitary then the natural projection $\pi : \mathrm{Inv}\langle A \mid R \rangle \to \mathrm{Gp}\langle A \mid R \rangle$ is injective on $\mathcal{R}$-classes.
- The right units of

$$\mathrm{Inv}\langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle$$

are finitely generated by the set of all prefixes of all the words w_i.