Subsemigroups of the semigroup of all mappings on an infinite set

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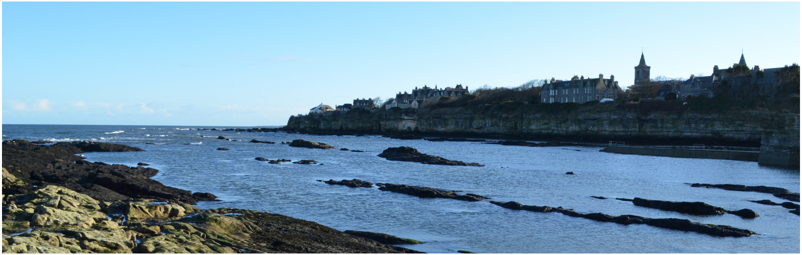

Joint work with...


... James East and James Mitchell

Joint work with...

... Michał Morayne and Zak Mesyan

Your 3rd & 4th favourite semigroups

We will talk about these two semigroups:

- S_Ω - the symmetric group on a set Ω;
- Ω^Ω - the full transformation semigroup on Ω.

Throughout, Ω will be infinite.

The **sub(semi)group lattice** of a (semi)group is just the sub(semi)groups ordered by inclusion where

$$A \vee B = \langle A, B \rangle \qquad \text{and} \qquad A \wedge B = A \cap B.$$

The sub(semi)group lattice of a (semi)group is an **algebraic lattice**:

complete: the supremum $\bigvee A$ and infimum $\bigwedge A$ of any subset A exists;

compact: every sub(semi)group is a supremum of f.g. sub(semi)groups.

Complete answers are unlikely

Let Ω be any infinite set. Then there are $2^{2^{|\Omega|}}$ subsemigroups of Ω^Ω and subgroups of S_Ω.

Theorem (Pinsker 2005)

If $|\Omega| = \aleph_\alpha$ and $\lambda = \max\{|\alpha|, \aleph_0\}$, then there are 2^{2^λ} subsemigroups of Ω^Ω containing S_Ω.

The algebraic lattices with 2^λ compact elements are, up to isomorphism, the subalgebra lattices of algebras whose domain has 2^λ elements.

Theorem (Pinsker and Shelah 2011)

The complete sublattices of the lattice of submonoids of Ω^Ω are, up to isomorphism, the algebraic lattices with at most $2^{|\Omega|}$ compact elements.

Maximal sub(semi)groups

Let S be a (semi)group and let T be a proper sub(semigroup) of S. Then T is **maximal** if it is not contained in any other proper sub(semi)group.

Theorem (Baumgartner, Shelah, Thomas '93)

It is consistent and independent of ZFC that $\exists G \leqslant S_{\mathbb{N}}$ not contained in any maximal subgroup.

Theorem (Zorn's Lemma)

Let G be any (semi)group and let $H \leqslant G$ such that $\exists K \subseteq G$ with $|K| < \infty$ and $\langle H, K \rangle = G$. Then H is contained in a maximal sub(semi)group of G.

Theorem (Macpherson, Praeger '90)

Let $G \leqslant S_{\mathbb{N}}$ such that G is not highly transitive, or $|G| \leqslant \aleph_0$. Then G is contained in a maximal subgroup.

Maximal subgroups of S_Ω

Theorem (Ball '66)

The setwise stabiliser of a finite set in S_Ω is a maximal subgroup of S_Ω.

Moiety: a subset Σ of Ω such that $|\Sigma| = |\Omega \setminus \Sigma| (= |\Omega|)$.

Finite partition: a partition $\mathcal{P}$ of Ω into moieties $\Sigma_1, \ldots, \Sigma_n$.

Stabiliser: $\mathrm{Stab}(\mathcal{P}) = \{\, f \in S_\Omega \;:\; \forall i\ \exists j\ (\Sigma_i)f = \Sigma_j \,\}$.

Almost equal: $\Sigma \approx \Gamma$ if $|\Sigma \setminus \Gamma| + |\Gamma \setminus \Sigma| < |\Omega|$.

Almost stabiliser: $\mathrm{AStab}(\mathcal{P}) = \{\, f \in S_\Omega \;:\; \forall i\ \exists j\ (\Sigma_i)f \approx \Sigma_j \,\}$.

Theorem (Ball '66)

$\mathrm{AStab}(\mathcal{P})$ *is maximal for all* $n \geq 2$.

More (a lot more) maximal subgroups of S_Ω: ultrafilters

If $\mathcal{A}$ is a family of subsets of Ω, then the **stabiliser** of $\mathcal{A}$ in S_Ω is

$$S_{\{\mathcal{A}\}} = \{\, f \in S_\Omega \,:\, (\forall A \subseteq \Omega)(A \in \mathcal{A} \leftrightarrow f(A) \in \mathcal{A}) \,\}.$$

Theorem (Richman '67)

If $\mathcal{F}$ is an ultrafilter on Ω, then:

(i) $S_{\{\mathcal{F}\}}$ *has two orbits on moieties of* Ω*;*

(ii) $S_{\{\mathcal{F}\}}$ *is a maximal subgroup of* S_Ω*;*

(iii) $S_{\{\mathcal{F}\}} = \{\, f \in S_\Omega \,:\, \mathrm{fix}(f) \in \mathcal{F} \,\}$.

Corollary

There are $2^{2^{|\Omega|}}$ *non-conjugate maximal subgroups of* S_Ω.

Maximal subsemigroups of Ω^Ω ... containing S_Ω

If $f \in \Omega^\Omega$, then a **transversal** of f is any $\Sigma \subseteq \Omega$ such that $f|_\Sigma$ is injective and $\Sigma f = \Omega f$.

$$\begin{aligned} d(f) &= |\Omega \setminus f(\Omega)| && \text{(defect)} \\ c(f) &= |\Omega \setminus \Sigma| \text{ where } \Sigma \text{ is any transversal of } f && \text{(collapse)} \\ k(f,\mu) &= |\{\, \alpha \in \Omega : |f^{-1}(\alpha)| \geq \mu \,\}| && \text{where } \mu \leqslant |\Omega|. \end{aligned}$$

Theorem (Heindorf '02 & Pinsker '05)

If $|\Omega|$ is regular, then the maximal subsemigroups of Ω^Ω containing S_Ω are:

$$\{\, f \in \Omega^\Omega : c(f) < \mu \text{ or } d(f) \geq \mu \,\} \text{ for some } \aleph_0 \leqslant \mu \leqslant |\Omega|;$$

$$\{\, f \in \Omega^\Omega : c(f) = 0 \text{ or } d(f) > 0 \,\};$$

$$\{\, f \in \Omega^\Omega : c(f) \geq \mu \text{ or } d(f) < \mu \,\} \text{ for some } \aleph_0 \leqslant \mu \leqslant |\Omega|;$$

$$\{\, f \in \Omega^\Omega : c(f) > 0 \text{ or } d(f) = 0 \,\};$$

$$\{\, f \in \Omega^\Omega : k(f, |\Omega|) < |\Omega| \,\}.$$

Maximal subsemigroups of Ω^Ω ... containing $\mathrm{AStab}(\mathcal{P})$

Let $\mathcal{P} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ where $n \geq 2$ be a finite partition of Ω and let $f \in \Omega^\Omega$. Then define

$$\rho_f = \{\, (i,j) \;:\; |f(\Sigma_i) \cap \Sigma_j| = |\Omega| \,\} \text{ and } \rho_f^{-1} = \{\, (i,j) \;:\; (j,i) \in \rho_f \,\}.$$

A binary relation σ is **total** if $\forall i\ \exists j$ such that $(i,j) \in \sigma$.

Theorem (East, Mitchell, P '11)

The maximal subsemigroups of Ω^Ω containing $\mathrm{Stab}(\mathcal{P})$ but not S_Ω are:

$$\{\, f \in \Omega^\Omega \;:\; \rho_f \in S_n \textit{ or } \rho_f \textit{ is not total} \,\}$$
$$\{\, f \in \Omega^\Omega \;:\; \rho_f \in S_n \textit{ or } \rho_f^{-1} \textit{ is not total} \,\}.$$

Theorem (East, Mitchell, P '11)

The maximal subsemigroups of Ω^Ω *containing* $\mathrm{Stab}(\mathcal{P})$ *but not* S_Ω *are:*

$$\{\, f \in \Omega^\Omega \;:\; \rho_f \in S_n \text{ or } \rho_f \text{ is not total} \,\}$$

$$\{\, f \in \Omega^\Omega \;:\; \rho_f \in S_n \text{ or } \rho_f^{-1} \text{ is not total} \,\}.$$

Proof. Let $\mathrm{Stab}(\mathcal{P}) \leqslant M \leqslant \Omega^\Omega$ but M is not contained in any of the semigroups in the above or previous theorem.

- $\exists f, g \in M$ such that ρ_f and ρ_g^{-1} are total and $\rho_f, \rho_g \notin \mathrm{Sym}(n)$
- $\exists t \in M$ such that $\rho_t = n \times n$
- $\exists f', g' \in M$ such that f' is injective, g' is surjective, and $d(f') = c(g') = |\Omega|$
- $\mathrm{Sym}(\Omega)$ is contained in M and so $M = \Omega^\Omega$. □

... containing stabilisers of finite sets

Theorem (East, Mitchell, P '11)

Let Γ be a non-empty finite subset of Ω. Then the maximal subsemigroups of Ω^Ω containing the pointwise stabiliser of Γ but not S_Ω are:

$$\{f \in \Omega^\Omega : \ \big(f^{-1}(\Sigma) = \Sigma \text{ and } c(f) < \mu\big) \text{ or } d(f) \geq \mu \text{ or } \Sigma \not\subseteq f(\Omega)\} \cup \mathfrak{F}$$

$$\{f \in \Omega^\Omega : \ \big(f(\Sigma) = \Sigma \text{ and } d(f) < \nu\big) \text{ or } c(f) \geq \nu \text{ or } |f(\Sigma)| < |\Sigma|\} \cup \mathfrak{F}$$

where $\emptyset \neq \Sigma \subseteq \Gamma$, $\aleph_0 \leqslant \mu, \nu \leqslant |\Omega^+|$, and if $|\Sigma| = 1$, then $\nu = |\Omega|^+$.

… containing stabilisers of singleton sets

Corollary

Let $a \in \Omega$. Then the maximal subsemigroups of Ω^Ω containing the stabiliser of $\{a\}$ but not S_Ω are:

$$\{\, f \in \Omega^\Omega \;:\; \big(c(f) < \mu \text{ and } f(a) = a\big) \text{ or } d(f) \geq \mu \text{ or } a \notin f(\Omega) \,\} \cup \mathfrak{F}$$

where $\aleph_0 \leqslant \mu \leqslant |\Omega^+|$ and

$$\{\, f \in \Omega^\Omega \;:\; f(a) = a \,\} \cup \mathfrak{F}$$

The stabiliser of the singleton $\{a\}$ is identical to the stabiliser of the corresponding principal ultrafilter $\{\, A \subseteq \Omega \;:\; a \in A \,\}$ for some fixed $a \in A$. So the above corollary deals with stabilisers of principal ultrafilters.

... containing stabilisers of non-principal ultrafilters

Theorem (East, Mitchell, P '11)

Let $\mathcal{F}$ be a non-principal ultrafilter on Ω and let κ be the least cardinality of an element of $\mathcal{F}$. Then the maximal subsemigroups of Ω^Ω containing the stabiliser of $\mathcal{F}$ but not S_Ω are

- $U_1(\mathcal{F}, \mu)$:= *all* $f \in \Omega^\Omega$ *satisfying any of:*
  - $\Sigma \notin \mathcal{F}$ *for all* $\Sigma \notin \mathcal{F}$ *and* $c(f) < \mu$
  - $d(f) \geq \mu$
  - $f(\Omega) \notin \mathcal{F}$
- $U_2(\mathcal{F}, \mu) :=$ *all* $f \in \Omega^\Omega$ *satisfying any of:*
  - $\Sigma \in \mathcal{F}$ *for all* $\Sigma \in \mathcal{F}$ *and* $d(f) < \mu$
  - $c(f) \geq \mu$
  - $c(f|_\Sigma) > 0$ *for all* $\Sigma \in \mathcal{F}$

where $\kappa < \mu \leqslant |\Omega|^+$.

... containing stabilisers of uniform ultrafilters

A filter is **uniform** if every set in the filter has the same cardinality. Every non-principal ultrafilter on a countable set is uniform.

Corollary

Let $\mathcal{F}$ be a uniform ultrafilter on Ω. Then the maximal subsemigroups of Ω^Ω containing the stabiliser of $\mathcal{F}$ but not S_Ω are
$U_1(\mathcal{F}) = \{ f \in \Omega^\Omega : (f(\Sigma) \notin \mathcal{F})(\forall \Sigma \notin \mathcal{F}) \}$ *and*
$U_2(\mathcal{F}) = \{ f \in \Omega^\Omega : (f(\Sigma) \in \mathcal{F})(\forall \Sigma \in \mathcal{F})$ *or* $(c(f|_\Sigma) > 0)(\forall \Sigma \in \mathcal{F}) \}$.

Corollary

There are $2^{2^{|\Omega|}}$ non-conjugate maximal subsemigroups of Ω^Ω.

Maximal subsemigroups of the symmetric group

We have classified the maximal subsemigroups of Ω^Ω containing:

- the symmetric group;
- the stabilizer of a finite partition;
- the pointwise stabilizer of a finite set;
- the stabilizer of an ultrafilter.

The known maximal subsemigroups M of Ω^Ω satisfy $M \cap S_\Omega$ is a **maximal subsemigroup**.

Is $M \cap S_\Omega$ a maximal sub**semi**group of S_Ω if M is maximal in Ω^Ω?

Can $M \cap S_\Omega$ be trivial?

The man who wasn't there

$\mathbb{N}^{\mathbb{N}}$ is a **Polish semigroup** - a complete, separable, topological semigroup (i.e. the map $(x, y) \mapsto xy$ is continuous)

$S_{\mathbb{N}}$ is a **Polish group** - a complete, separable, topological group (i.e. the maps $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ is continuous)

Theorem (Folklore)

A subgroup of $S_{\mathbb{N}}$ is closed if and only if it is the group of automorphisms of a first-order structure.

A submonoid of $\mathbb{N}^{\mathbb{N}}$ is closed if and only if it is the monoid of endomorphisms of a first-order structure.

For example, G could be the automorphisms of $(\mathbb{Q}, <)$, the random graph, ...

Bergman and Shelah's Theorem

If $U, V \subseteq S_{\mathbb{N}}$, then $U \preccurlyeq V$ if $\exists$ countable $C \subseteq S_{\mathbb{N}}$ such that

$$U \subseteq \langle\, V, C \,\rangle.$$

We write $U \approx V$ if $U \preccurlyeq V$ and $V \preccurlyeq U$.

Theorem (Bergman and Shelah '06)

Let G be a closed subgroup of $S_{\mathbb{N}}$. Then:

(i) $G_{(\Sigma)}$ *has an infinite orbit for all finite* $\Sigma \subseteq \mathbb{N}$ *and* $G \approx S_{\mathbb{N}}$*;*

(ii) $G_{(\Sigma)}$ *has only finite orbits for some finite* $\Sigma \subseteq \mathbb{N}$*,* $G_{(\Gamma)}$ *has orbits of unbounded length for all finite* $\Gamma \subseteq \mathbb{N}$*, and* $G \approx S_2 \times S_3 \times S_4 \times \cdots$*;*

(iii) $G_{(\Sigma)}$ *has orbits of bounded length for some finite* $\Sigma \subseteq \mathbb{N}$ *and* $G \approx S_2 \times S_2 \times \cdots$*;*

(iv) $G \approx \{1_{\mathbb{N}}\}$*.*

A simpler question we also cannot solve

Consider the analogue of Bergman & Shelah's relations for Ω^Ω:
If $U, V \subseteq \mathbb{N}^\mathbb{N}$, then $U \approx V$ if there is a countable $C \subseteq \mathbb{N}^\mathbb{N}$ such that

$$U \subseteq \langle\, V, C \,\rangle \quad \text{and} \quad V \subseteq \langle\, U, C \,\rangle.$$

Zak considering $\preccurlyeq$ on $\mathbb{N}^\mathbb{N}$

The relations $\preccurlyeq$ and $\approx$ on $\mathbb{N}^{\mathbb{N}}$

Theorem (Z. Mesyan '07)

The following subsemigroups of $\mathbb{N}^{\mathbb{N}}$ are not $\approx$-equivalent.

- $\mathfrak{F}_i = \{\, f \in \mathbb{N}^{\mathbb{N}} \;:\; |f(\mathbb{N})| = i \,\}$ *for* $i \in \mathbb{N}$
- $\mathfrak{F} = \bigcup_{i \in \mathbb{N}} \mathfrak{F}_i$
- $S_{0,x} = \{\, f \in \mathbb{N}^{\mathbb{N}} \;:\; f(x) \in \{0, x\}$ *for all* $x \in \mathbb{N} \,\}$
- $S_2 = \{\, f \in \mathbb{N}^{\mathbb{N}} \;:\; f(\{2n, 2n+1\}) = \{2n, 2n+1\} \,\}$
- $S_{\leqslant} = \{\, f \in \Omega^{\Omega} \;:\; f(x) \leqslant x$ *for all* $x \in \mathbb{N} \,\}$
- $\mathbb{N}^{\mathbb{N}}$

In fact,

$$\emptyset \prec \mathfrak{F}_2 \prec \mathfrak{F}_3 \prec \cdots \prec \mathfrak{F} \prec S_{x,0} \prec S_2 \prec S_{\leqslant} \prec \mathbb{N}^{\mathbb{N}}.$$

Could $\mathfrak{F}_2$ be minimal?

The Cantor-Bendix theorem

Is it maybe true that $2^{\mathbb{N}} \preccurlyeq S$ for every uncountable subsemigroup S?

Theorem

Let X be a perfect Polish space. Then there exists an embedding of the Cantor set $2^{\mathbb{N}}$ into X.

Theorem (Cantor-Bendixon)

Let X be a Polish space. Then X can be written uniquely as $X = P \cup C$ where P is perfect and C is countable and open.

A closed subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is a Polish space and so if it has cardinality $2^{\aleph_0}$, then S contains a copy of the Cantor set $2^{\mathbb{N}}$.

A minimal class of subsemigroups?

Lemma

Let S be a closed subsemigroup of $\mathbb{N}^{\mathbb{N}}$ with $|S| = 2^{\aleph_0}$. Then there exist $U \subseteq S$ and $f \in 2^{\mathbb{N}}$ such that U is homeomorphic to $2^{\mathbb{N}}$ and $\lambda : U \longrightarrow 2^{\mathbb{N}}$ defined by $\lambda(g) = f \circ g$ for all $g \in U$ is a homeomorphism from U to $\lambda(U)$.

Theorem

Let S be a closed subsemigroup of $\mathbb{N}^{\mathbb{N}}$ of cardinality $2^{\aleph_0}$. Then there exists a closed subsemigroup T of $2^{\mathbb{N}}$ with $|T| = 2^{\aleph_0}$ such that $T \preccurlyeq S$.

Proof.

- $\lambda(U)$, being homeomorphic to $2^{\mathbb{N}}$, is compact.
- $\mathbb{N}^{\mathbb{N}}$ is Hausdorff $\Rightarrow \lambda(U) \subseteq 2^{\mathbb{N}}$ is closed.
- let $T := \langle\, \lambda(U), (0\ 1)\,\rangle$
- $T \leqslant 2^{\mathbb{N}}$ is closed, $|T| = 2^{\aleph_0}$, and $T \approx \lambda(U)$.
- $\lambda(U) = \{\, f \circ g \ :\ g \in U \,\} \subseteq \langle\, U, f \,\rangle$ and so

$$T \approx \lambda(U) \subseteq \langle\, U, f \,\rangle \approx U \subseteq S. \quad \square$$

The lost monoid

If A is a subset of $\mathbb{N}$, then define $s_A \in \mathbb{N}^{\mathbb{N}}$ by

$$s_A(n) = \begin{cases} n & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

If $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$, then we define $S_{\mathcal{A}} = \{\, s_A \in \mathbb{N}^{\mathbb{N}} \,:\, A \in \mathcal{A} \,\}$ and we say that $\mathcal{A}$ **almost disjoint** if

$$|A \cap B| < \infty \text{ for all } A, B \in \mathcal{A}.$$

The subset $S_{\mathcal{A}}$ is closed if and only if $\mathcal{A}$ is closed in $2^{\mathbb{N}}$.

Theorem

If $\mathcal{A}$ is an almost disjoint family of cardinality $2^{\aleph_0}$, then $S_{\mathcal{A}}$ is incomparable under $\preccurlyeq$ to $\mathfrak{F}$ and $\mathfrak{F}_n$ for all $n \geq 2$.

There exists $T \preccurlyeq S_{\mathcal{A}}$ such that $\emptyset \prec T \preccurlyeq \mathfrak{F}_2$ and so $T \not\approx \mathfrak{F}_2$.

Theorem

There exists a chain, having length $\aleph_1$, of $\approx$-classes containing (not necessarily closed) subsemigroups of $2^{\mathbb{N}}$.

Theorem

The following are equivalent:

(i) $\aleph_1 = 2^{\aleph_0}$;

(ii) *there exists a subsemigroup S of $\mathbb{N}^{\mathbb{N}}$ such that $S \approx \mathbb{N}^{\mathbb{N}}$ and for all subsemigroups T of S either $T \approx \mathbb{N}^{\mathbb{N}}$ or $T \approx \{1_{\mathbb{N}}\}$.*

Theorem

For all $i \in \mathbb{N}$, there exist i distinct closed subsemigroups contained in $\mathfrak{F}$ that are mutually incomparable under $\preccurlyeq$.

The key message

Thank you for listening!