Quasivarieties, Adequate Monoids and Expansions

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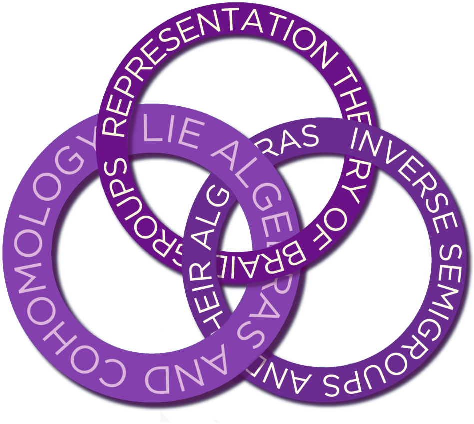
Some pre-announcements...

1. The next **NBSAN** will be in Manchester on Friday 11th April 2025.
2. **Introduction to Modern Advances in Algebra 2025** in Manchester 9th – 11th April.



https://sites.google.com/view/itmaia2025/

Universal Algebra

Definition

The *type* of an algebraic structure A (in the sense of universal algebra) is a tuple $\sigma = (\sigma_1, \ldots, \sigma_k)$ of functions $\sigma_i : A^{n_i} \to A$ each corresponding to some n_i-ary operation on A. The corresponding tuple of arities $(n_1, \ldots, n_k)$ is called the *signature* of A.

Examples.

1. Semigroups have type $(\cdot)$ and signature (2).
2. Monoids have type $(\cdot, \mathbf{1})$ and signature $(2, 0)$.
3. Groups have type $(\cdot, ^{-1}, \mathbf{1})$ and signature $(2, 1, 0)$.

Definition

For a fixed set of symbols X and type σ, an *identity* is a formal equation $u = v$ where u, v are formal terms in $\mathcal{T}(X)$.

Definition

For a fixed set of symbols X and type σ, a *quasi-identity* is a formal implication $(\bigwedge_{i=1}^{n} u_i = v_i) \to u = v$ where $u_1, \ldots, u_n, v_1, \ldots, v_n, u, v$ are formal terms in $\mathcal{T}(X)$.

Varieties and Quasivarieties

Definition

A σ-*variety* of algebraic structures $\mathcal{Q}$ is a class of all models of algebraic structures of type σ satisfying a defining list of identities.

Examples.

1. Semigroups: $a(bc) = (ab)c$.
2. Monoids: $a(bc) = (ab)c \quad \mathbf{1}a = a \quad a\mathbf{1} = a$.
3. Groups: $a(bc) = (ab)c \quad \mathbf{1}a = a \quad a\mathbf{1} = a \quad aa^{-1} = \mathbf{1} \quad a^{-1}a = \mathbf{1}$.

Definition

A σ-*quasivariety* of algebraic structures of type σ is a class of all models of algebraic structures of type σ satisfying a defining list of quasi-identities.

Examples.

1. Any variety!
2. Right Cancellative Monoids: $a(bc) = (ab)c \quad \mathbf{1}a = a \quad a\mathbf{1} = a \quad ac = bc \to a = b$.

Birkhoff's Theorem

Theorem (Birkhoff 1935)

Let $\mathcal{V}$ be a class of algebraic structures of the same type. Then $\mathcal{V}$ forms a variety if and only if $\mathcal{V}$ is closed under:

- *Taking homomorphic images (quotients).*
- *Taking subalgebras.*
- *Arbitrary direct products.*

Theorem (Adámek, Rosicky 1994)

Let $\mathcal{Q}$ be a class of algebraic structures of the same type. Then $\mathcal{Q}$ forms a quasi-variety if and only if $\mathcal{Q}$ is closed under:

- *Taking subalgebras.*
- *Arbitrary direct products.*
- *Filtered colimits.*

In particular, quasivarieties might not be closed under quotients!

Presentation Problems

What does $\textbf{Mon}\langle a, b \mid ab = 1\rangle$ actually mean?
Formally: Take the free monoid on $\{a, b\}$ and quotient by the smallest $(2,0)$-congruence σ s.t.

$$(ab, 1) \in \sigma.$$

If we do the same in a quasivariety $\mathcal{Q}$, then:

- Free objects always exist! ☺
- But $\mathcal{Q}\langle X \mid R\rangle$ might not be in $\mathcal{Q}$... ☹

Example. Take $\textbf{RC}\langle a \mid a^2 = a\rangle$.
The free right cancellative monoid on $\{a\}$ is $\{a\}^*$.
The smallest $(2,0)$-congruence containing (a^2, a) is $\sigma = \{(1,1), (a^n, a) : n \in \mathbb{N}\}$.
The quotient is ${\{a\}^*}/{\sigma} \cong (\{0,1\}, \vee)$.
Not right cancellative!

Intuitively, $\textbf{RC}\langle a \mid a^2 = a\rangle$ should be the trivial monoid...

The Fix

Definition

Let $\mathcal{Q}$ be a quasivariety of signature $\mathfrak{s}$. Let X be a set and let R be a collection of identities. By $\mathcal{Q}\langle X \mid R\rangle$, we mean ${}^{F}\!/_{\sigma}$ where:

- F is the free object of rank $|X|$ in $\mathcal{Q}$.
- σ is the smallest $\mathfrak{s}$-congruence such that:
  1. $R \subseteq \sigma$.
  2. ${}^{F}\!/_{\sigma} \in \mathcal{Q}$.

  (Note that such a σ always exists.)

Example. $\mathbf{RC}\langle a \mid a^2 = a\rangle \cong \{1\}$. ☺

Left Adequate Monoids

Definition

Left adequate monoids form the quasivariety with type $(\cdot, ^+, 1)$ and signature $(2, 1, 0)$ with defining quasi-identities:

$$a(bc) = (ab)c, \quad a1 = a = 1a,$$

$$a^+a = a, \quad (a^+b^+)^+ = a^+b^+, \quad a^+b^+ = b^+a^+, \quad (ab)^+ = (ab^+)^+,$$

$$a^2 = a \rightarrow a = a^+ \quad \text{and} \quad ac = bc \rightarrow ac^+ = bc^+.$$

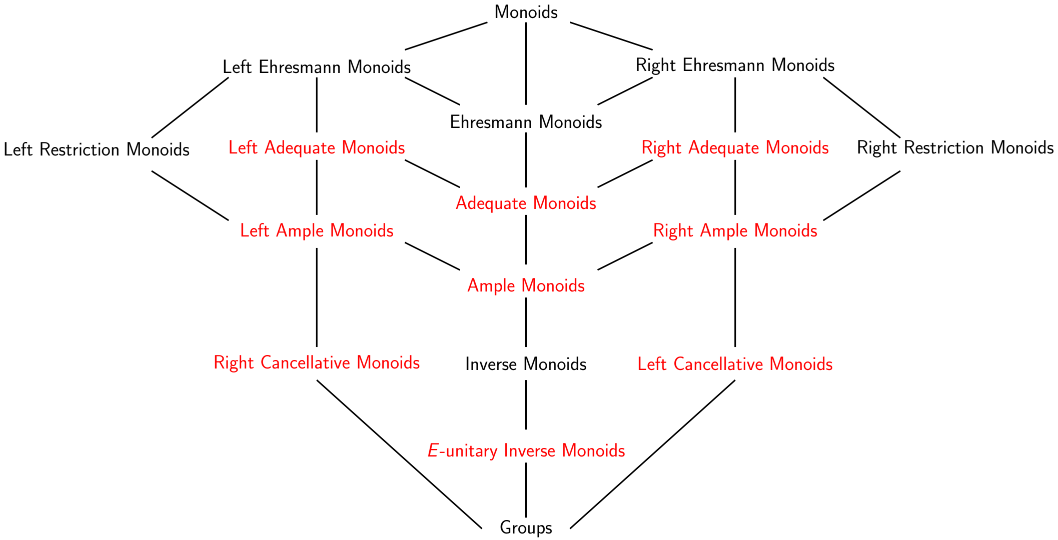
Definition

Equivalently, a monoid M is **left adequate** if:

1. Idempotents of M commute;
2. For all $a \in M$, there exists a unique idempotent $a^+ \in E(M)$ such that

$$\forall x, y \in M \quad xa = ya \iff xa^+ = ya^+.$$

A Big Diagram

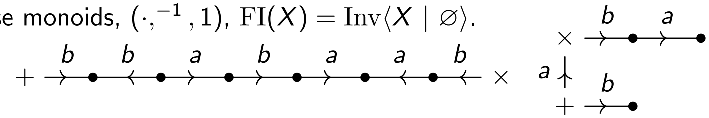


Free Objects

Many of these (quasi)varieties have free objects described by operations on directed graphs.

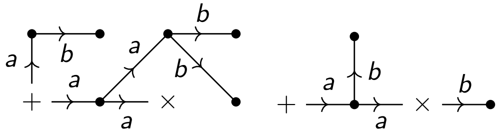
Munn 1974: Free inverse monoids, $(\cdot, ^{-1}, 1)$, $\mathrm{FI}(X) = \mathrm{Inv}\langle X \mid \varnothing \rangle$.

$bb^{-1}abaa^{-1}b^{-1}$



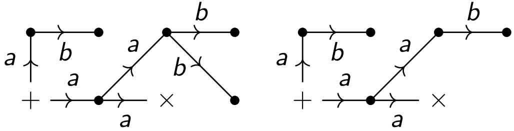
Fountain, Gomes, Gould 2009: Free (left) ample / restriction monoids, $(\cdot, ^{+}, 1)$, $\mathrm{FLAm}(X) = \mathrm{LAm}\langle X \mid \varnothing \rangle$.

$(ab)^{+}a(ab^{+}b)^{+}a$



Kambites 2011: Free (left) adequate / Ehresmann monoids, $(\cdot, ^{+}, 1)$, $\mathrm{FLAd}(X) = \mathrm{LAd}\langle X \mid \varnothing \rangle$.

$(ab)^{+}a(ab^{+}b)^{+}a$



Semigroup Expansions

Definition (Birget, Rhodes 1984)

Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbf{Sgp}$. An *expansion of* $\mathcal{C}$ *to* $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there is a natural transformation $\eta : F \implies \mathrm{Id}_{\mathcal{C}}$ whose components η_S are all surjective.

I.e. for all $S \in \mathcal{C}$, there is a semigroup $F(S) \in \mathcal{D}$ and a surjective morphism $\eta_S : F(S) \to S$ such that whenever $\tau : S \to T$ is a morphism, there is a morphism $F(\tau) : F(S) \to F(T)$ making the following diagram commute:

$$\begin{array}{ccc} F(S) & \xrightarrow{F(\tau)} & F(T) \\ {\scriptstyle \eta_S}\downarrow & & \downarrow{\scriptstyle \eta_T} \\ S & \xrightarrow{\quad\tau\quad} & T \end{array}$$

Theorem (Birget, Rhodes 1984 / Szendrei 1989)

There is an expansion $\mathrm{Sz} : \mathbf{Gp} \to \mathbf{FInv}$ *given by* $\mathrm{Sz}(G) = \{(H, g) : H \subseteq G$ *finite and* $1, g \in H\}$.

Theorem (Szendrei 1989)

Sz *is left adjoint to the maximal group image functor* $\sigma^\natural : \mathbf{FInv} \to \mathbf{Gp}$.

Expansions of other Categories

Recall that $\mathrm{FI}(X)$ was constructed by 'tracing' the Cayley graph of $\mathrm{FG}(X)$... what about other Cayley graphs?

Theorem (Margolis, Meakin 1989)

*Let X be a set and let G be an X-generated group. There is an expansion $\mathcal{M}$: **XGp** $\to$ **XEInv** given by $\mathcal{M}(G) = \{(\Gamma, g) : \Gamma$ is a finite connected subgraph of $\mathrm{Cay}(G), 1, g \in V(\Gamma)\}$. $\mathcal{M}$ is left adjoint to the maximal group image functor $\sigma^\natural$: **XEInv** $\to$ **XGp**.*

Theorem (Gould 1996, + Gomes 2000)

Let X be a set and let M be an X-generated monoid. Define

$$\mathcal{G}(M) = \{(\Gamma, m) : \Gamma \text{ is a finite connected subgraph of } \mathrm{Cay}(M), 1, m \in V(\Gamma)\}.$$

*Then $\mathcal{G}$ forms expansions **XRC** $\to$ **XPLAm** and **XU** $\to$ **XPWLAm**. Moreover, $\mathcal{G}$ is left adjoint to taking the maximal right cancellative image and maximal unipotent image respectively.*

Question

Can we find an expansion **XRC** $\to$ **XLAd**? Preferably with some graphical interpretation?

Pretzels!

Fix a set X and an X-generated right cancellative monoid C.

Definition

An *idempath* in an X-labelled digraph Γ is a path labelled by a word $x_1 x_2 \cdots x_n$ which is equal to the identity in C. We take the empty path with label ϵ to have $\epsilon =_C 1$.
An *idempath identification* in Γ is the process of 'cycling up' an idempath.

Lemma (H., Kambites, Szakács 2024)

Given a tree $T \in \mathrm{FLAd}(X)$, there exists a unique graph obtainable by sequentially performing all non-trivial idempath identifications (in any order) to T.

Definition

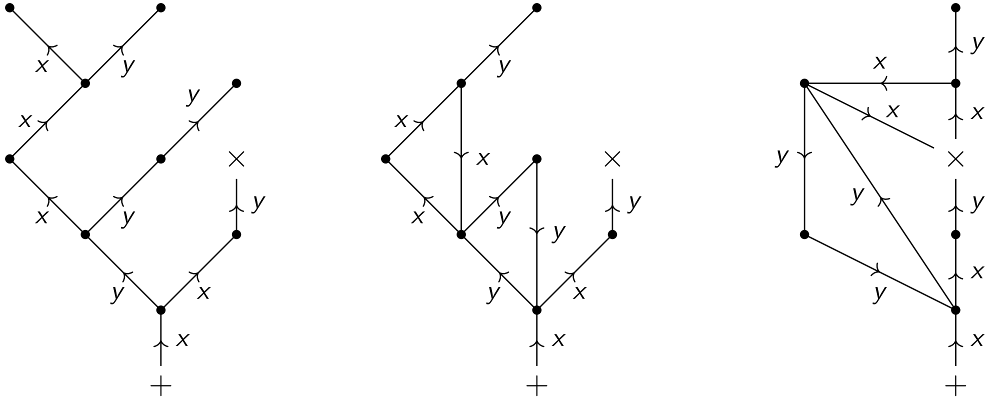
Given any tree $T \in \mathrm{FLAd}(X)$, perform the following:

1. Idempath identify as far as possible...
2. ...then retract anything in the result which can retract (take minimal image under idempotent graph endomorphisms).

We call the (uniquely obtained) result the *pretzel* of T, denoted $\overline{\widetilde{\widetilde{T}}}$.

Example

Take $X = \{x, y\}$ and $C = \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathrm{Mon}\langle x, y\rangle$.



$(2, 1, 0)$-algebras

Define a multiplication $\cdot$ on pretzels as follows:

1. Glue $\overline{\widetilde{T}}$ to $\overline{\widetilde{S}}$, start-to-end.
2. Pretzel-ify the result (note that new idempaths could have been created!).

Define a unary operation $+$ on pretzels as follows:

1. Move the end vertex of $\overline{\widetilde{T}}$ to the start vertex.
2. Pretzel-ify the result (note that new retractions might be possible!).

Theorem (H., Kambites, Szakács 2024)

The set of all pretzels $\mathcal{PT}(C)$ forms an X-generated left adequate monoid.

Theorem (H., Kambites, Szakács 2024)

$\mathcal{PT}(C; X) \cong \mathbf{LAd}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle.$

Margolis-Meakin Expansions vs. Pretzels

Properties of $\mathcal{M}(G)$

1. $\mathcal{M}(\mathrm{FG}(X)) \cong \mathrm{FI}(X)$.
2. $\mathcal{M}(G)$ is finite $\iff$ G is finite.
3. Elements are subgraphs of $\mathrm{Cay}(G)$.
4. $\mathcal{M}(G) \cong \mathbf{Inv}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_G 1\rangle$.
5. $\mathcal{M}$ defines an expansion **XGp** $\to$ **XEInv**.

Properties of $\mathcal{PT}(C)$

1. $\mathcal{PT}(X^*) \cong \mathrm{FLAd}(X)$.
2. $\mathcal{PT}(C)$ is finite $\iff$ C is finite $\implies$ C is a group.
3. Elements are trees of strongly connected subgraphs of $\mathrm{Cay}(C)$.
4. $\mathcal{PT}(C; X) \cong \mathbf{LAd}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle$.

Theorem (H., Kambites, Szakács 2024)

$\mathcal{PT}$ defines an expansion **XRC** $\to$ **XLAd**.

Open Questions and What's Next

- Can we describe other presentations using similar combinatorial methods?
  E.g. can we describe:

$$\mathbf{LAm}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_C 1\rangle \text{ for right cancellative } C \text{ ?}$$

$$\mathbf{FInv}\langle X \mid w^2 = w \text{ for } w \in X^* \text{ s.t. } w =_G 1\rangle \text{ for group } G \text{ ?}$$

- What about right adequate and two-sided adequate pretzel monoids?
- Can we find geometric interpretations of other analogues of Margolis-Meakin expansions in the left adequate setting, perhaps one such that $M(C)$ has maximal right cancellative image C?
- Can we apply similar pretzel-style techniques in F-inverse land? In particular for the free F-inverse monoid...?
- What about other interesting presentations of (left) adequate monoids?