Representations and Identities of Plactic-like Monoids

Marianne Johnson
University of Manchester
Joint work with: Alan J. Cain, Mark Kambites, and António Malheiro

York Semigroups Seminar
9th March 2022

Overview

Given a semigroup P, we aim to give information about the identities satisfied by P by studying certain representations of P.

Semigroups we are interested in are: finite rank "plactic-like" monoids...

P	Elements of P viewed combinatorially...
Plactic monoids	Young tableaux
Hypoplactic monoids	Quasi-ribbon tableaux
Stalactic monoids	Stalactic tableaux
Taiga monoids	Binary search trees with multiplicities
Sylvester monoids	Right strict binary search trees
Baxter monoids	Pairs of twin binary search trees
Right patience sorting monoids	Right patience sorting tableaux

Theorems we prove concern: the semigroup variety generated by P.

Representations we study are: matrix representations of P over certain commutative semirings.

Finite rank plactic-like monoids

Each of the 'plactic-like monoids' P listed on the previous slide...

- ...is really a family of monoids, indexed by the size of a finite generating set $[n] = \{1, \ldots, n\}$
- ...can be defined as a quotient $[n]^* / \equiv_P$, where $\equiv_P$ is the congruence defined by $w \equiv_P w'$ if and only if w and w' produce the same combinatorial object (of 'type' P) as output when successively applying a certain insertion algorithm to the strings w and w'.
- ...has elements in 1-1 correspondence with some nice combinatorial objects (certain types of tableaux/trees)...

Plactic

5			
4	4		
2	3	4	6
1	2	3	3

Hypoplactic

1	1	2			
		3	4	4	
				5	
				6	6

Stalactic

3	1	2	6	5
3	1		6	5
	1			5
	1			

Right patience sort

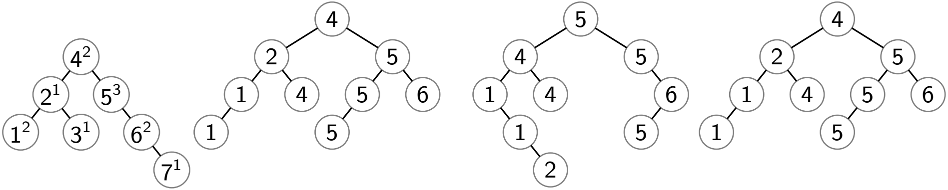
3			6	
2		6	7	
2	4	6	5	
1	3	4	5	7

Finite rank plactic-like monoids

Each of the 'plactic-like monoids' P listed on the previous slide...

- ...is really a family of monoids, indexed by the size of finite generating set $[n] = \{1, \ldots, n\}$
- ...can be defined as a quotient $[n]^* / \equiv_P$, where $\equiv_P$ is the congruence defined by $w \equiv_P w'$ if and only if w and w' produce the same combinatorial object (of 'type' P) as output when successively applying a certain insertion algorithm to the strings w and w'.
- ...has elements in 1-1 correspondence with some nice combinatorial objects (certain types of tableaux/trees)

Taiga Sylvester Baxter



Finite-rank plactic-like monoids in this talk...

- . . . arise from the study of insertion algorithms (such as **Schensted's algorithm**, 1961) which constructs tableaux/trees from words.
- . . . are of interest to researchers in algebraic combinatorics, semigroup theory, computer science and representation theory.
- . . . are $\mathcal{J}$-trivial.
- . . . have polynomial growth.
- . . .

The name "*plaxique*" (later translated as plactic) was given by Lascoux and Schützenberger (1981), who extensively studied the (original) plactic monoid.

Later researchers noticed similarities between the plactic monoids and the other families of monoids mentioned on the previous slides.

Varieties and identities of semigroups

A semigroup identity is a pair of non-empty words, usually written $u = v$ over some alphabet Σ.

A semigroup P satisfies the identity $u = v$ if for all morphisms ϕ from the free semigroup Σ^+ to P we have $\phi(u) = \phi(v)$.

For example, a semigroup satisfies ...

- ... $AB = BA$ if and only if it is commutative;
- ... $A^2 = A$ if and only if it is idempotent;
- ... $AB = A$ if and only if it is a left-zero semigroup.

The variety of semigroups generated by P, denoted $\mathrm{var}(P)$, is the class of semigroups satisfying every identity satisfied by P.

[Birkhoff's theorem: Equivalently, $\mathrm{var}(P)$, is the class of semigroups obtained from P by taking subsemigroups, direct products and homomorphic images.]

Matrix representations over semirings

Let S be a commutative unital semiring containing an element of infinite multiplicative order.

For example, the tropical semiring $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with binary operations defined by:

$$x \oplus y := \max(x, y), \quad x \otimes y := x + y.$$

Write $UT_n(S)$ to denote the semigroup of all $n \times n$ upper triangular matrices over S with respect to matrix multiplication.

A semigroup P admits an upper triangular matrix representation over S if there is a morphism from P to $UT_n(S)$ for some n.
Say that the representation is faithful if the morphism is injective.

Clearly, if P admits a faithful upper triangular matrix representation of size n over S, then P satisfies every identity satisfied by $UT_n(S)$...

Identities for upper triangular tropical matrices

Theorem (Izhakian 2013–16, Okniński 2015, Taylor 2016)

For each fixed n, there exists a semigroup identity satisfied by $UT_n(\mathbb{T})$.

Okniński's construction

Let $u(A,B) = ABBA\ AB\ ABBA$ and $v(A,B) = ABBA\ BA\ ABBA$.
Set $u_1 = u(A,B)$ and $v_1 = v(A,B)$ and for $j \geq 1$ set

$$u_{j+1} = u(u_j, v_j), \quad v_{j+1} = v(u_j, v_j).$$

Then $UT_n(\mathbb{T})$ satisfies $u_{n-1} = v_{n-1}$; each word has length 10^{n-1}

Theorem (Aird 2021)

For all n, one has $\mathrm{var}(UT_n(\mathbb{T})) \neq \mathrm{var}(UT_{n+1}(\mathbb{T}))$.

Theme of the talk

If you can find a tropical matrix representation of your favourite semigroup, then you can conclude that this semigroup satisfies non-trivial identities...

(Some of) everyone's favourite semigroups...

- **Finite semigroups:** Every finite semigroup can be represented by tranformations on a finite set (Cayley's theorem), and hence by Boolean (or indeed tropical) matrices of the size of this finite set.
- **The natural numbers:** Clearly any subsemigroup of $(\mathbb{R}, +)$ has a tropical representation of size 1.
- **Free semigroups of rank at least 2:** Do not satisfy non-trivial semigroups, and so do not have tropical matrix representations.
- **Full matrix semigroups over an infinite field K:** The semigroup of $n \times n$ matrices over K does not satisfy non-trivial semigroup identities for $n \geq 2$, and so these semigroups do not have tropical matrix representations.
- **Bicyclic monoid:** Izhakian and Margolis have shown that the bicyclic monoid has a tropical representation by upper triangular 2×2 matrices.

For the rest of the talk: Your new favourite semigroups are the plactic-like monoids.

Plactic Monoids

The **plactic monoid** plac_n of rank n is the monoid generated by $[n] = \{1, 2, \ldots, n\}$ subject to the **Knuth relations**:

$$bca = bac \quad (a < b \leq c) \qquad\qquad acb = cab \quad (a \leq b < c)$$

Elements are in bijective correspondence (via row reading or column reading) with **(semistandard) Young tableaux** over $[n]$:

4	4		
2	3	4	
1	2	3	3

$= 442341233 = 421432433 = \cdots$

4			
2	3	4	4
1	2	3	3

$= 423441233 = 421324343 = \cdots$

(Entries in each column strictly decreasing, entries in each row weakly increasing, row lengths weakly increasing. Multiplication determined by Schensted's insertion algorithm.)

Identities for plactic monoids

Question (Kubat & Okniński 2013)

Does plac_n satisfy a non-trivial semigroup identity?

- "Yes" when $n \leq 3$ (Kubat & Okniński 2013)
- Conjectured "yes" for all finite n (Kubat & Okniński 2013)
- "No" when n infinite (Cain, Klein, Kubat, Malheiro & Okniński 2017)
- Again conjectured "yes" for all finite n (Cain & Malheiro 2018)
- Preprint of Okniński (2019) on $n \geq 4$ withdrawn.

Corresponding answer is...

- ... "yes" for hypoplactic, stalactic, taiga, sylvester, and Baxter monoids (Cain & Malheiro 2018)
- ... "yes" for right patience sorting monoids (Cain, Malheiro & F. M. Silva 2018)

Tropical representations of plactic monoids

Question (Izhakian 2017)

Does each plac_n *have a* **faithful** *tropical representation?*

Theorem (Izhakian 2017)

The plactic monoid plac_3 *has a faithful representation in* $UT_3(\mathbb{T}) \times UT_3(\mathbb{T})$.

Cain, Klein, Kubat, Malheiro & Okniński 2017

Alternative faithful tropical representation for plac_3.

Both the above representations generalise naturally to higher rank but do **not** remain faithful. e.g. in plac_4 they do not separate:

4	4		
2	3	4	
1	2	3	3

and

4			
2	3	4	4
1	2	3	3

Tropical representations of plactic monoids

Theorem (J. & Kambites)

For every finite n, plac_n has a faithful upper triangular tropical representation. Thus every finite rank plactic monoid satisfies a non-trivial semigroup identity.

In general the size of our representation is of order 2^n but by using a result of Daviaud, J. & Kambites, 2018 we can show ...

Theorem (J. & Kambites)

plac_n satisfies all identities satisfied by $UT_d(\mathbb{T})$ where $d = \lfloor \frac{n^2}{4} + 1 \rfloor$

By studying the image of our representation we also show:

Theorem (J. & Kambites)

$UT_n(\mathbb{T})$ satisfies all identities satisfied by plac_n.

Varieties of plactic monoids

Putting the previous results together we have...
$\mathrm{var}(\mathrm{UT}_n(\mathbb{T})) \subseteq \mathrm{var}(\mathrm{plac}_n) \subseteq \mathrm{var}(\mathrm{UT}_d(\mathbb{T}))$, $d = \lfloor \frac{n^2}{4} + 1 \rfloor$

In particular,

- $\mathrm{var}(\mathrm{UT}_1(\mathbb{T})) = \mathrm{var}(\mathrm{plac}_1)$
- $\mathrm{var}(\mathrm{UT}_2(\mathbb{T})) = \mathrm{var}(\mathrm{plac}_2)$
- $\mathrm{var}(\mathrm{UT}_3(\mathbb{T})) = \mathrm{var}(\mathrm{plac}_3)$
- $\mathrm{var}(\mathrm{UT}_4(\mathbb{T})) \subseteq \mathrm{var}(\mathrm{plac}_4) \subseteq \mathrm{var}(\mathrm{UT}_5(\mathbb{T}))$

Question

For each n, does there exist k such that $\mathrm{var}(\mathrm{UT}_k(\mathbb{T})) = \mathrm{var}(\mathrm{plac}_n)$?
Do we always have $\mathrm{var}(\mathrm{UT}_n(\mathbb{T})) = \mathrm{var}(\mathrm{plac}_n)$?

Theorem (Aird 2021)

For $k \neq 4$, $\mathrm{var}(\mathrm{UT}_k(\mathbb{T})) \neq \mathrm{var}(\mathrm{plac}_4)$.

Varieties of plactic-like monoids

What can we say about the variety generated by other plactic-like monoids? (As mentioned earlier: it is already known that each of these monoids satisfies a non-trivial semigroup identity.)
By studying matrix representations over commutative unital semirings S containing an element of infinite multiplicative order we show:

Theorem (Cain, J, Kambites & Malheiro)

P	$P \leq \mathrm{UT}_k(S)$?	P in $\mathrm{var}(\mathrm{UT}_d(S))$?	$\mathrm{var}(P) = ?$
hypo_n	✓ $k = n^2$	✓ $d = 2$	**Comm** $\vee\ \mathrm{var}(\mathcal{C}_3)$
stal_n	✓ $k = n^2$	✓ $d = 2$	**Comm** $\vee$ **RRB**
taig_n	✓ $k = 3n^2 - 2n$	✓ $d = 2$	**Comm** $\vee$ **RRB**
sylv_n	✓ $k = n^2$	✓ $d = 2$	$\mathrm{var}(\mathcal{M})$
baxt_n	✓ $k = 2n^2 - n$	✓ $d = 2$	$\mathrm{var}(\mathcal{M}) \vee \mathrm{var}(\mathcal{M}^\sharp)$
rPS_n	✓ $k = 2^{n-1}(n^2 + n)$	✓ $d = \binom{n+1}{2} + 1$	?

where $\mathcal{C}_3$ is a finite monoid and $\mathcal{M}$, and $\mathcal{M}^\sharp$ are certain other finitely presented monoids.

Overall strategy

Let S be a commutative semiring with an element α of infinite multiplicative order.

Idea

For each plactic-like monoid P, construct morphisms with image in a given (finite) subsemigroup of $\mathrm{UT}_n(S)$ to "count" (using α) and "detect" certain characteristics/configurations of the combinatorial objects of P.

For example:

- The morphism $c_n : [n]^* \to \mathrm{UT}_n(S)$ determined by extending the map sending $x \in [n]$ to the matrix with (p, q)th entry given by:
$$c_n(x)_{p,q} = \begin{cases} \alpha & \text{if } p = q = x \\ 1_S & \text{if } p = q \neq x \\ 0_S & \text{else} \end{cases}$$
can clearly be used to determine the *content* of a word.
- Since for each plactic-like monoid $P := [n]^* / \equiv_P$, words in the same $\equiv_P$-class have the same content, this restricts to a morphism on P.

Overall strategy

Let S be a commutative semiring with an element α of infinite multiplicative order.

Idea

For each plactic-like monoid P, construct morphisms with image in a given (finite) subsemigroup of $\mathrm{UT}_n(S)$ to "count" (using α) and "detect" certain characteristics/configurations of the combinatorial objects of P.

In each case...

- Construct morphisms $\phi : P \to \mathrm{UT}_k(S)$ that capture some crucial data about the structure of the combinatorial objects in P.
- Construct "enough" morphisms, so that each crucial aspect of the structure can be determined by looking at the image of one such.
- Create a faithful upper triangular representation of P by glueing these smaller representations together to create a block-diagonal representation that captures all information.

T H A N K
Y O U
F O R
Y O U R
A T T E N T I O N

For more details...

- M. Johnson & M. Kambites,
  Tropical representations and identities of plactic monoids,
  Transactions of the American Mathematical Society, **374** (2021), 4423–4447.

- A. J. Cain, M. Johnson, M. Kambites & A. Malheiro,
  Representations and identities of plactic-like monoids,
  `arXiv:2107.04492` (2021).