# Solving semigroup questions through combinatorics:

## extended Green's relations in the Stylic monoid

Ambroise Grau
*Universidade Nova de Lisboa*

(joint work with Duarte Ribeiro)

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# The Stylic Monoid

Throughout this talk, we let $n \in \mathbb{N}$, $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$, $w \in \mathcal{A}_n^*$ and $\varepsilon$ be the empty word on $\mathcal{A}_n^*$. Set $\mathcal{S} = \big\{(a^2, a) \mid a \in \mathcal{A}_n\big\}$ and $\mathcal{K} = \{(acb, cab) \mid a \leq b < c \in \mathcal{A}_n\} \cup \{(bac, bca) \mid a < b \leq c \in \mathcal{A}_n\}$.

## Definition

The following are equivalent definitions of the *Stylic monoid of rank $n$*, denoted by $\mathrm{styl}_n$ or $\mathrm{Styl}(\mathcal{A}_n)$:

- $\mathrm{styl}_n$ is given by the monoid presentation $\langle \mathcal{A}_n \mid \mathcal{K} \cup \mathcal{S} \rangle$;
- $\mathrm{styl}_n$ is the finite quotient of the plactic monoid of rank $n$ by the relations in $\mathcal{S}$.

*Remarks:*

1. For $u, v \in \mathcal{A}_n^*$, $[u]$ denotes the equivalence class of $u$ in $\mathrm{styl}_n$, and $u \equiv v$ if and only if $[u] = [v]$.
2. Idempotents correspond to the classes of strictly decreasing words.
3. The stylic monoid $\mathrm{styl}_n$ is a finite monoid with a zero, which is the idempotent $[n(n-1)\ldots 1]$.

# Extended Green's relations

Since $\mathrm{styl}_n$ is $\mathcal{J}$-trivial, it is not regular and we can get more insight into its structure by looking at some *extended Green's relations*:

## Definition

Let $a, b \in \mathrm{styl}_n$, then:

- $a \,\widetilde{\mathcal{L}}\, b$ if and only if $\big(ae = a \Leftrightarrow be = b \quad \forall e = e^2 \in \mathrm{styl}_n\big)$;
- $a \,\widetilde{\mathcal{R}}\, b$ if and only if $\big(ea = a \Leftrightarrow eb = b \quad \forall e = e^2 \in \mathrm{styl}_n\big)$;

  i.e. *a* and *b* have the same idempotents as right/left identities
- $a \,\mathcal{L}^*\, b$ if and only if $\big(ax = ay \Leftrightarrow bx = by \quad \forall x, y \in \mathrm{styl}_n\big)$;
- $a \,\mathcal{R}^*\, b$ if and only if $\big(xa = ya \Leftrightarrow xb = yb \quad \forall x, y \in \mathrm{styl}_n\big)$;

  i.e. *a* and *b* have similar cancellative properties
- $\widetilde{\mathcal{D}} = \widetilde{\mathcal{L}} \vee \widetilde{\mathcal{R}}$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$.

Remark: we have the inclusions $\mathcal{L}^* \subseteq \widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{D}}$, $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{D}}$ and $\mathcal{L}^*, \mathcal{R}^* \subseteq \mathcal{D}^* \subseteq \widetilde{\mathcal{D}}$.

Our goal: Express $\widetilde{\mathcal{L}}$ and $\mathcal{L}^*$ in terms of characteristics that are easy to grasp combinatorially.

# $N$-tableaux

## Definition

- A *semi-standard Young tableau* on $\mathcal{A}_n$ is a staircase shaped diagram where each box is filled with a letter from $\mathcal{A}_n$ such that entries in rows are weakly increasing (from left to right) and entries in column are strictly increasing (from bottom to top).
- An *$N$-tableau* on $\mathcal{A}_n$ is a semi-standard Young tableau in which each row is strictly increasing and contained in the row below it.

*Example:* $T_1 = \begin{array}{|c|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-3} 3 & 5 & 5 \\ \hline 1 & 2 & 3 & 3 & 4 \\ \hline \end{array}$ and $T_2 = \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} 2 & 5 & 6 \\ \hline 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$ are both semi-standard Young tableaux, but only $T_2$ is an $N$-tableau.

## Fact

The stylic monoid $\mathrm{styl}_n$ is the monoid of $N$-tableaux on $\mathcal{A}_n$ under the operation of insertion via Schensted-like algorithms.

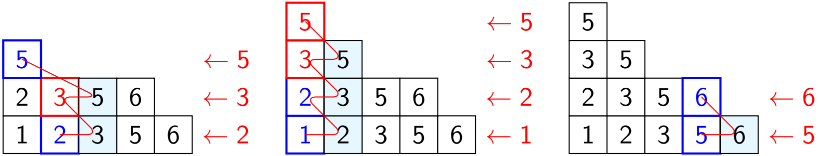
# The right $N$-insertion algorithm

Consider the tableau $T =$

| | | | | |
|---|---|---|---|---|
| 5 | | | | |
| 2 | 5 | 6 | | |
| 1 | 2 | 3 | 5 | 6 |

for insertion of $\boxed{w = 215632}$

In order to insert the word $w$ into $T$, for each letter $a$ of $w$ reading from left to right, do as follows:

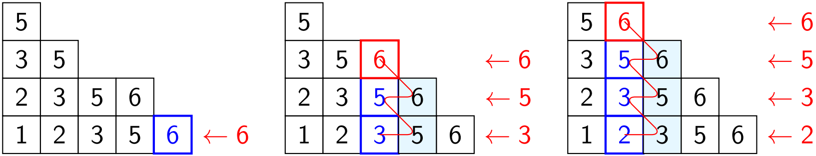



1. insert $a$ into the bottom row,
2. if the element inserted is not the largest element of that row, the smallest element larger than it is then said to be *bumped*,
3. insert the bumped element into the row above,
4. repeat steps 2. and 3. until we no longer have a bumped element.

# The right $N$-insertion algorithm

Consider the tableau $T = \begin{array}{|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 2 & 5 & 6 \\ \cline{1-3} 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$ for insertion of $\boxed{w = 215632}$

In order to insert the word $w$ into $T$, for each letter $a$ of $w$ reading from left to right, do as follows:




1. insert $a$ into the bottom row,
2. if the element inserted is not the largest element of that row, the smallest element larger than it is then said to be *bumped*,
3. insert the bumped element into the row above,
4. repeat steps 2. and 3. until we no longer have a bumped element.

# The bijection between words and $N$-tableaux

The *$N$-tableau associated to $w$* and denoted by $N(w)$ is obtained by (right-)insertion of the letters of $w$ into an empty tableau.

Conversely, given an $N$-tableau $T$ on $\mathcal{A}_n$, the *row word of $T$* is the word $RW(T)$ obtained by concatenating the consecutive reading of the rows of $T$ starting from the topmost one.

Notation: we write $T \leftarrow u$ to denote the insertion of a word $u \in \mathcal{A}_n^*$ into a tableau $T$ on $\mathcal{A}_n$, and $\mathrm{hgt}(w)$ for the height of $N(w)$, that is, the number of rows of $N(w)$.

## Key fact

For all $N$-tableaux $T$ on $\mathcal{A}_n$ and all words $w \in \mathcal{A}_n^*$, we have that $N(RW(T)) = T$ and $w \equiv RW(N(w))$. Hence, in $\mathrm{styl}_n$ the multiplication of $N$-tableaux $T_1 T_2$ is defined as $T_1 \leftarrow RW(T_2)$ which means that for $u \in \mathcal{A}_n^*$, $N(wu) = N(w) \leftarrow u$.

Note: Idempotents in $\mathrm{styl}_n$ correspond to $N$-tableaux with height equal to their support, that is to tableaux in which each row has all the elements of the one below except from its minimum element.

# The left $N$-insertion algorithm

Consider the tableau $T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} 2 & 5 & 6 \\ \hline 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$ for insertion of $\boxed{w = 215632}$

In order to insert the word $w$ into $T$, for each letter $a$ of $w$ reading from right to left, do as follows:

$$2 \rightarrow \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} \boxed{2} & 5 & 6 \\ \hline 1 & 2 & 3 & 5 & 6 \\ \hline \end{array} \qquad 3 \rightarrow \begin{array}{|c|c|c|c|c|} \cline{1-2} 3 & \cancel{5} \\ \cline{1-4} 2 & 3 & 5 & 6 \\ \hline 1 & 2 & 3 & 5 & 6 \\ \hline \end{array} \qquad 6 \rightarrow \begin{array}{|c|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 3 & 6 \\ \cline{1-4} 2 & 3 & 5 & 6 \\ \hline 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$$

1. check if $a$ is present in the first column and if yes, do nothing and stop there, otherwise, insert it into the first column while keeping it ordered,
2. insert $a$ in all the rows below where it is missing,
3. in rows where we have inserted $a$, look at the smallest value larger than $a$ in this row
4. if the same highlighted value is present in consecutive rows, delete it from the higher rows.

# The left $N$-insertion algorithm

Consider the tableau $T = \begin{array}{|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 2 & 5 & 6 \\ \cline{1-3} 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$ for insertion of $\boxed{w = 215632}$

In order to insert the word $w$ into $T$, for each letter $a$ of $w$ reading from right to left, do as follows:

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline \boxed{5} & \cancel{6} \\ \cline{1-2} 3 & \boxed{5} & 6 \\ \cline{1-3} 2 & 3 & 5 & 6 \\ \cline{1-4} 1 & 2 & 3 & 5 & 6 \\ \hline \end{array} \qquad 1 \rightarrow \begin{array}{|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 3 & 5 & 6 \\ \cline{1-3} 2 & 3 & 5 & 6 \\ \cline{1-4} \boxed{1} & 2 & 3 & 5 & 6 \\ \hline \end{array} \qquad 2 \rightarrow \begin{array}{|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 3 & 5 & 6 \\ \cline{1-3} \boxed{2} & 3 & 5 & 6 \\ \cline{1-4} 1 & 2 & 3 & 5 & 6 \\ \hline \end{array}$$

1. check if $a$ is present in the first column and if yes, do nothing and stop there, otherwise, insert it into the first column while keeping it ordered,
2. insert $a$ in all the rows below where it is missing,
3. in rows where we have inserted $a$, look at the smallest value larger than $a$ in this row
4. if the same highlighted value is present in consecutive rows, delete it from the higher rows.

# The $\tau$ and $\beta$ maps

Notation: For $x \in \mathcal{A}_n$ and $B \subseteq \mathcal{A}_n$, denote by $x_B^{\uparrow}$ the smallest letter of $B$ strictly greater than $x$, or $\varepsilon$ when no such letter exists. Similarly, $x_B^{\downarrow}$ denotes the largest letter of $B$ strictly smaller than $x$ when such a letter exists and $\varepsilon$ otherwise. We also write $R_j^w$ for the $j$-th row of $N(w)$.

## Definition

Let $a \in \mathcal{A}_n$ and $1 \leq r \leq \mathrm{hgt}(w) + 1$. The insertion tracking maps $\tau^w$ and $\beta^w$ from $(\mathcal{A}_n^* \cup \{\varepsilon\}) \times \{1, \ldots, \mathrm{hgt}(w) + 1\}$ to $\mathcal{A}_n^* \cup \{\varepsilon\}$ are defined recursively as follows:

- $\tau^w(a, 1) = \beta^w(a, 1) = a$,
- $\tau^w(\varepsilon, r) = \beta^w(\varepsilon, r) = \varepsilon$,
- $\tau^w(a, r) = \tau^w(a, r-1)^{\uparrow}_{R^w_{r-1}}$ and $\beta^w(a, r) = \beta^w(a^{\downarrow}_{R^w_{r-1}}, r-1)$,

where $R_s^w$ denotes the $s$-th row of $N(w)$.

## Example

For $w = 2653879 \in \mathcal{A}_9^*$, we have $\tau(3,3) = 6$, $\tau(6,3) = 8$, $\tau(9,2) = \tau(9,3) = \varepsilon$, while $\beta(6,3) = 3$, $\beta(9,3) = 7$ and $\beta(3,3) = \beta(5,3) = \varepsilon$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 6 | | | | | | |
| 5 | 6 | 8 | | | | |
| 2 | 3 | 5 | 6 | 7 | 8 | 9 |

# Some properties and relationships of $\beta$ and $\tau$

## Lemma

*Let $a \in \mathcal{A}_n$, and $1 < r \leq \operatorname{hgt}(w)$. Then:*

1. $\tau^w(a,r) \in \operatorname{supp}(w) \cup \{\varepsilon\}$ *with* $\tau^w(a,r) \neq \varepsilon$ *if and only if* $\tau^w(a,j) \neq \varepsilon$ *and* $\tau^w(a,j) < \max(R_j^w)$ *for all* $1 \leq j < r$;
2. $\beta^w(a,r) \in \operatorname{supp}(w) \cup \{\varepsilon\}$ *with* $\beta^w(a,r) \neq \varepsilon$ *if and only if* $a > \min(R_{r-1}^w)$, *that is, if and only if* $a^{\downarrow}_{R_{r-1}^w} \neq \varepsilon$;
3. *the first* $r-1$ *rows of the two tableaux* $N(w) \leftarrow \beta^w(a,r)$ *and* $N(w)$ *are equal. Consequently,* $\tau^w(\beta^w(a,r),s) \in R_s^w$ *for all* $1 \leq s \leq r-1$;
4. *if* $\tau^w\left(\beta^w(a,r),r\right) \neq \varepsilon$, *then* $\tau^w\left(\beta^w(a,r),r\right) \geq a$ *with equality if and only if* $a \in R_{r-1}^w$;
5. *if* $\beta^w(\tau^w(a,r),r) \neq \varepsilon$, *then* $\beta^w(\tau^w(a,r),r) \leq a$ *with equality if and only if* $\tau^w(a,j) \in R_j^w$, *for* $1 \leq j \leq r-1$;
6. *if* $\beta^w(a,r) \neq \varepsilon$ *then* $\beta^w(a,r-1) \neq \varepsilon$ *and* $\beta^w(a,r) < \beta^w(a,r-1)$;
7. *if* $\tau^w(a,r) \neq \varepsilon$, *then* $\tau^w(b,r) \neq \varepsilon$ *for all* $b < a$ *and* $\tau^w(b,r) \leq \tau^w(a,r)$, *with equality if and only if there exists* $s < r$ *with* $\tau^w(a,s) \notin R_s^w$ *and* $\tau^w(b,s)^{\uparrow}_{R_s} = \tau^w(a,s)^{\uparrow}_{R_s}$.

# Fix of a word

## Definition

The *fix* of $w$ in $\mathrm{styl}_n$ is the set $\mathrm{Fix}(w) \subseteq \mathcal{A}_n$ defined by

$$\mathrm{Fix}(w) = \{\beta(\max(R_r^w), r) \mid 1 \leq r \leq \mathrm{hgt}(w)\}.$$

Notice in particular that $|\mathrm{Fix}(w)| = \mathrm{hgt}(w)$.

## Lemma

*Let $a \in \mathcal{A}_n$. Then the following are equivalent:*

1. $a \in \mathrm{Fix}(w)$;
2. $N(wa) = N(w)$ *(or similarly* $wa \equiv w$*)*;
3. *there exists* $1 \leq r \leq \mathrm{hgt}(w)$ *such that* $\tau^w(a, r) = \max(R_r^w)$ *and* $\tau^w(a, j) \in R_j^w$ *for all* $1 \leq j \leq r - 1$.

*Moreover, if* $a \in \mathrm{Fix}(w)$ *then* $\mathrm{Fix}(wa) = \mathrm{Fix}(w)$ *and otherwise*

$$\mathrm{Fix}(wa) = \begin{cases} \mathrm{Fix}(w) \cup \{a\} & \text{if } a < \min(\mathrm{Fix}(w)), \\ \left(\mathrm{Fix}(w) \setminus \{a^{\downarrow}_{\mathrm{Fix}(w)}\}\right) \cup \{a\} & \text{if } a > \min(\mathrm{Fix}(w)). \end{cases}$$

# Common local identities: $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$

Recall: $U \, \widetilde{\mathcal{R}} \, V$ if and only if $(EU = U \Leftrightarrow EV = V \quad \forall E = E^2)$ with $U, V, E \in \mathrm{styl}_n$, and dually for $\widetilde{\mathcal{L}}$.

## Proposition

*Let $u, v \in \mathcal{A}_n^*$. Then:*

- $[u] \, \widetilde{\mathcal{R}} \, [v]$ *if and only if $N(u)$ and $N(v)$ have the same first column.*
- $[u] \, \widetilde{\mathcal{L}} \, [v]$ *if and only if* $\mathrm{Fix}(u) = \mathrm{Fix}(v)$.

## Proof.

- In the left insertion, $a \to N(u) = N(u)$ if and only if $a$ belongs to the first column of $N(u)$. Thus for an idempotent $E$, we have that $RW(E) \to N(u) = N(u)$ if and only if all the letters of $RW(E)$ belong to the first column of $N(u)$.
- Similarly, for the right insertion, $N(u) \leftarrow RW(E) = N(u)$ if and only if all letters of $RW(E)$ are in $\mathrm{Fix}(u)$. □

# The $\widetilde{\mathcal{D}}$-relation

Recall: $\widetilde{\mathcal{D}} = \widetilde{\mathcal{R}} \vee \widetilde{\mathcal{L}}$

## Proposition

*For $u, v \in \mathcal{A}_n^*$ we have $[u]\,\widetilde{\mathcal{D}}\,[v]$ if and only if $\mathrm{hgt}(u) = \mathrm{hgt}(v)$.*

## Proof.

Since $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ preserve height, it is clear that $\mathrm{hgt}(u) = \mathrm{hgt}(v)$ is a necessary condition.
On the other hand, suppose that $\mathrm{hgt}(u) = k$, then we can show that $[u]\,\widetilde{\mathcal{D}}\,E$ where $E = N(k(k-1)\dots 1)$. To do so, we construct a sequence from $T_1 = T_1' = 1 \to N(u)$ by letting $T_i$ have the same rows of $T_{i-1}'$ except in row $i$ where $i$ is added to its first column, and $T_i'$ is build from $T_i$ by adding $i$ to all the rows below row $i$. With this we then have:

$$N(u)\,\widetilde{\mathcal{L}}\,T_1\,\widetilde{\mathcal{R}}\,T_2\,\widetilde{\mathcal{L}}\,T_2' \cdots \widetilde{\mathcal{R}}\,T_i\,\widetilde{\mathcal{L}}\,T_i' \cdots \widetilde{\mathcal{R}}\,T_k,$$

and since by construction we have that the first column of $T_k$ consists of all values from 1 to $k$, it follows that $T_k\,\widetilde{\mathcal{R}}\,E$. Therefore $N(u)\,\widetilde{\mathcal{D}}\,E$. $\square$

# Gaps of a word

## Definition

- For $1 \leq i \leq \mathrm{hgt}(w)$, $a \in \mathcal{A}_n$ is a *gap in* $R_i^w$ if $a \notin R_i^w$ and $a > \min(R_i^w)$. The set of all gaps in $R_i^w$ is denoted by $G_i^w$.
- The gap sequence of $a$ in $w$ is the strictly decreasing sequence $\gamma_{[a]}^w = \{a - i + 1 \mid 1 \leq i \leq \mathrm{hgt}(w) \text{ and } a \in G_i^w\}$ and the set of non-empty gap sequences in $w$ is denoted by $\Gamma(w)$.

Example: For $w = 2653879$, we have $G_1^w = \{4\}$, $G_2^w = \{7, 9\}$ and $G_3^w = \{7, 8, 9\}$, so that $\Gamma(w) = \left\{\gamma_{[4]}^w, \gamma_{[7]}^w, \gamma_{[8]}^w, \gamma_{[9]}^w\right\}$ where $\gamma_{[4]}^w = \{4\}$, $\gamma_{[7]}^w = \{6, 5\}$, $\gamma_{[8]}^w = \{6\}$ and $\gamma_{[9]}^w = \{8, 7\}$.

## Lemma

*For all* $\gamma_{[a]}^w, \gamma_{[b]}^w \in \Gamma(w)$ *we have* $\min(\gamma_{[a]}^w) \neq \min(\gamma_{[b]}^w)$ *and furthermore,* $\min(\gamma_{[a]}^w) < \min(\gamma_{[b]}^w)$ *if and only if* $a < b$.

Notation: For $x = \min(\gamma_{[a]}^w)$, we write $\gamma_x^w = \gamma_{[a]}^w$, and then for $u, v \in \mathcal{A}_n^*$ we say that $\Gamma(u) = \Gamma(v)$ if and only if $\{\gamma_x^u \in \Gamma(u)\} = \{\gamma_x^v \in \Gamma(v)\}$.

# Link between gap sequences and $\beta$ map

Using the gap sequences of $w$, we can give an explicit formula for the map $\beta^w$ as follows:

## Lemma

*Let $a \in \mathcal{A}_n$, $1 \leq r \leq \mathrm{hgt}(w) + 1$ and $\Gamma'_{[a]}(w) = \left\{ \gamma^w_{[b]} \in \Gamma(w) \mid b < a \right\}$. We now define the set $\mathcal{S}^w_{a,r} \subseteq \mathcal{P}(\Gamma'_{[a]}(w))$ as follows:*

$$\mathcal{S}^w_{a,r} = \Big\{ \left\{ \gamma^w_{[b_1]}, \ldots, \gamma^w_{[b_\ell]} \right\} \subseteq \Gamma'_{[a]}(w) \mid b_\ell < \cdots < b_1 < a \text{ and } a - r + 1 - (i-1) \in \gamma^w_{[b_i]} \text{ for each } 1 \leq i \leq \ell \Big\}.$$

*Then $\mathcal{S}^w_{a,r}$ either has a unique maximal element under inclusion, which we denote by $S^w_{a,r}$, or $\mathcal{S}^w_{a,r} = \emptyset$, in which case we let $S^w_{a,r} = \emptyset$. Moreover, if $\beta^w(a,r) \neq \varepsilon$, we then have that $\beta^w(a,r) = a - r + 1 - |S^w_{a,r}|$.*

# The importance of gap sequences

Recall: $U\,\mathcal{L}^*\,V$ if and only if $\big(UC = UD \Leftrightarrow VC = VD \quad \forall C, D \in \mathrm{styl}_n\big)$.

Gap sequences seem to be the appropriate characteristic to consider when trying to describe $\mathcal{L}^*$ because of its consequences:

## Lemma

*Let $u, v \in \mathcal{A}_n^*$. Suppose that $\mathrm{hgt}(u) = \mathrm{hgt}(v)$ and $\Gamma(u) = \Gamma(v)$. Then:*

- $\mathrm{Fix}(u) = \mathrm{Fix}(v)$, *so that* $[u]\,\widetilde{\mathcal{L}}\,[v]$;
- $\Gamma(uz) = \Gamma(vz)$ *for all* $z \in \mathcal{A}_n^*$;
- *for all* $\gamma_x^u \in \Gamma(u)$ *and* $p \in \gamma_x^u$, *we have that the moorings and anchor of* $p$ *in* $\gamma_x^u$ *and* $\gamma_x^v$ *are equal.*

# Moorings

## Definition

Let $\gamma^w_{[a]} \in \Gamma(w)$ and $1 \leq r \leq \mathrm{hgt}(w)$. For $p \in \gamma^w_{[a]}$, we define:

- the *first mooring* of $p$ in $\gamma^w_{[a]}$, as $\mu_1(\gamma^w_{[a]}, p) = \beta^w(a, a-p+1)$;
- the *second mooring* of $p$ in $\gamma^w_{[a]}$, as $\mu_2(\gamma^w_{[a]}, p) = \beta^w(a, a-p+2)$; and
- the *anchor* of $p$ in $\gamma^w_{[a]}$, as $\chi(\gamma^w_{[a]}, p) = \beta^w(a, a-p)$ if $p \neq a$, and $\varepsilon$ otherwise.

Some useful facts about the moorings are the following:

- when $p = \max(\gamma^w_{[a]})$, the first mooring corresponds to the value we need to insert so that $a$ appears one row higher than before, the second mooring is the value required to bump $a$ on the row above after that, while the anchor is the value bumping $a$ to the highest row it currently sits in;
- $\mu_2(\gamma^w_{[a]}, p) < \mu_1(\gamma^w_{[a]}, p) < \chi(\gamma^w_{[a]}, p)$ for all $p \in \gamma^w_{[a]}$;
- for $b \in \mathcal{A}_n$, the gap sequence $\gamma^{wb}_{[a]}$ of $a$ in $wb$ is uniquely determined by the initial value of $\gamma^w_{[a]}$ and the value of $b$ relatively to its moorings and anchor (elongating or shortening the gap sequence by one, or keeping it the same).

# Double bumping

## Definition

Let $t, y \in \mathcal{A}_n$. We say that the (right-)insertion of $ty$ into $w$ does a *double bumping* if there exists $1 \leq r \leq \mathrm{hgt}(w)$ such that:

- $\tau^w(t, r) = \tau^w(y, r)$ and
- $\tau^w(t, j) \in R_j^w$ for all $1 \leq j < r$.

Example: For $w = 2653879$, taking $t = 5$ and $y = 6$, we have that $ty$ does a double bumping in $w$ for $r = 3$, since $\tau^w(5, 3) = 8 = \tau^w(6, 3)$ as well as $\tau(5, 1) = 5 \in R_1^w$ and $\tau(5, 2) = 6 \in R_2^w$.

## Lemma

*Let $t, y \in \mathcal{A}_n$. Then the insertion of $ty$ into $w$ does a double bumping if and only if there exists $\gamma_{[a]}^w \in \Gamma(w)$ such that $\mu_2(\gamma_{[a]}^w, p) = t$ and $\mu_1(\gamma_{[a]}^w, p) \leq y < \chi(\gamma_{[a]}^w, p)$ where $p = \max(\gamma_{[a]}^w)$.*
*Moreover, we then have that $wty \equiv wy$, that is:*

$$N(w) \leftarrow ty = N(w) \leftarrow y.$$

# Description of $\mathcal{L}^*$

Recall: $U \,\mathcal{L}^*\, V$ if and only if $\big(UC = UD \Leftrightarrow VC = VD \quad \forall C, D \in \mathrm{styl}_n\big)$.

## Theorem

*Let $u, v \in \mathcal{A}_n^*$ be such that $\mathrm{hgt}(u) = \mathrm{hgt}(v)$.*
*Then $[u] \,\mathcal{L}^*\, [v]$ if and only if $\Gamma(u) = \Gamma(v)$.*

What do we actually need to do to prove the theorem:

$\boxed{\Rightarrow:}$ For two words $u, v \in \mathcal{A}_n^*$ with $\Gamma(u) \neq \Gamma(v)$, we need to find $w_1, w_2 \in \mathcal{A}_n^*$ such that $N(uw_1) = N(uw_2)$ but $N(vw_1) \neq N(vw_2)$.

$\boxed{\Leftarrow:}$ For two words $u, v \in \mathcal{A}_n^*$ with $\Gamma(u) = \Gamma(v)$, and $w_1, w_2 \in \mathcal{A}_n^*$ such that $N(uw_1) = N(uw_2)$, we need to show that this forces $N(vw_1) = N(vw_2)$. This is like looking at all the diamond shape paths in the right ideal graph of $u$ and $v$.

# Sketch of $\mathcal{L}^*$ proof: the necessary condition

## Lemma

*Let $u, v \in \mathcal{A}_n^*$ with $\mathrm{hgt}(u) = \mathrm{hgt}(v)$ and $\mathrm{Fix}(u) = \mathrm{Fix}(v)$. If $\Gamma(u) \neq \Gamma(v)$ then $[u]$ and $[v]$ are not $\mathcal{L}^*$-related.*

The main idea is now that we want to create $w_1$ and $w_2$ such that $w_1 = w'ty$ and $w_2 = w'y$ with $ty$ doing a double bumping on $uw'$ but not on $vw'$.

Steps are as follows:

1. look at the first gap sequence that differs (when ordered by increasing minimum value), say $\gamma_x^u = \{p_1 < \cdots < p_r\}$ and $\gamma_x^v$ and wlog assume that $\gamma_x^u$ is the longest of the two;
2. take $w' = \varepsilon$ and $i = 1$;
3. set $t = \mu_2(\gamma_x^u, p_i)$, $y = \mu_1(\gamma_x^u, p_i)$;
4. if $ty$ does a double bumping on $uw'$ but not on $vw'$ we are done;
5. otherwise, set $w'$ to $w'y$, increment $i$ and go back to step 3.

Example: For $u = RW(N(u)) = 9493459$ and $v = 2653879$, the first non-common sequence are $\gamma_6^u = \{7, 6\}$ and $\gamma_6^v = \{6\}$. Then $w_1 = 57$ and $w_2 = 7$ separate $[u]$ and $[v]$.

# Sketch of $\mathcal{L}^*$ proof: the sufficient condition

**Lemma**

*Let $u, v \in \mathcal{A}_n^*$ with $\mathrm{hgt}(u) = \mathrm{hgt}(v)$ and $\mathrm{Fix}(u) = \mathrm{Fix}(v)$. If $\Gamma(u) = \Gamma(v)$ then $[u]$ and $[v]$ are $\mathcal{L}^*$-related.*

Take $w_1, w_2 \in \mathcal{A}_n^*$ such that $N(uw_1) = N(uw_2)$. Then we have that

$$\Gamma(vw_1) = \Gamma(uw_1) = \Gamma(uw_2) = \Gamma(vw_2).$$

We now need to prove the following:

- show that the gap sequences in $vw_1$ and $vw_2$ are associated with the same values;
- show that in fact $vw_1$ and $vw_2$ have exactly the same gaps, which mean that the only difference between them must be related to consecutive elements on the lower end of each row;
- show that the minimum of each row of $N(vw_1)$ and $N(vw_2)$ are actually the same.

# What about $\mathcal{R}^*$?

## Definition

The *Schützenberger involution* on $\mathrm{styl}_n$ is the anti-automorphism $\theta$ defined by $\bar{a} = a\theta = n - a + 1$ for all $a \in \mathcal{A}_n$.

So if $w = x_1 \ldots x_n \in \mathcal{A}_n^*$, we have that $\overline{w} = (x_n\theta) \cdots (x_1\theta)$.

## Proposition

*Let $u, v \in \mathcal{A}_n^*$. Then $[u]\,\mathcal{R}^*\,[v]$ if and only if $\Gamma(\bar{u}) = \Gamma(\bar{v})$.*

But we don't get a nice combinatorical description. So the question is: what do the following tableaux have in common?

$$\begin{array}{ccccc} 6 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{array} \qquad \begin{array}{ccccc} 6 \\ 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{array} \qquad \begin{array}{ccccc} 6 \\ 3 & 6 \\ 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\begin{array}{cccc} 6 \\ 3 & 4 & 6 \\ 2 & 3 & 4 & 6 \end{array} \qquad \begin{array}{cccc} 6 \\ 3 & 6 \\ 2 & 3 & 4 & 6 \end{array} \qquad \begin{array}{ccc} 6 \\ 3 & 6 \\ 2 & 3 & 6 \end{array}$$