Maximal left ideals in Banach algebras

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York, Algebra Seminar

20 February 2019

Algebras

Throughout, an algebra is linear and associative and over the complex field $\mathbb{C}$.

A **left ideal** in A is a linear subspace I of A such that $ax \in I$ when $a \in A$ and $x \in I$; a left ideal M is **maximal** if $M \neq A$ and $I = M$ or $I = A$ when I is a left ideal in A with $I \supset M$.

A proper left ideal I in an algebra A is **modular** if there exists $u \in A$ with $a - au \in I \ (a \in A)$.

Let I be a left ideal in an algebra A with such a u. By Zorn's lemma, the family of left ideals J in A with $J \supset I$ and $u \notin J$ has a maximal member, say M. Clearly M is a maximal left ideal in A.

The **radical**, $\operatorname{rad} A$, of A is the intersection of the maximal modular left ideals, with $\operatorname{rad} A = A$ if there are no such (and then A is **radical**). It is an ideal in A. The algebra is **semi-simple** if $\operatorname{rad} A = \{0\}$.

Banach Algebras

These are Banach spaces $(A, \|\cdot\|)$ which are also algebras such that

$$\|ab\| \leq \|a\| \, \|b\| \quad (a, b \in A)\,.$$

Example For a Banach space E, $\mathcal{B}(E)$ is the Banach algebra of all bounded, linear operators on E. $\square$

Example Let S be a semigroup. The point mass at $s \in S$ is δ_s. The Banach space $\ell^{\,1}(S)$ consists of the functions $f : S \to \mathbb{C}$ such that

$$\|f\|_1 = \sum\{|f(s)| : s \in S\} < \infty\,.$$

There is a Banach algebra product $\star$ called convolution such that $\delta_s \star \delta_t = \delta_{st}$ $(s, t \in S)$. Then $(\ell^{\,1}(S), \star)$ is a Banach algebra that is the **semigroup algebra** on S. $\square$

More definitions

Let I be a closed ideal in a Banach algebra A. Then A/I is also a Banach algebra. The algebra $A/\mathrm{rad}\,A$ is a semi-simple Banach algebra.

A maximal left ideal in a Banach algebra is either closed or dense.

A Banach algebra is a **Banach $*$-algebra** if there is an involution $*$ on A such that

$$\|a^*\| = \|a\| \quad (a \in A)\,.$$

For example, C^*-algebras are Banach $*$-algebras.

Matrices

For $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the algebra of $n \times n$ matrices over $\mathbb{C}$. The algebras $\mathbb{M}_n$ are simple, i.e., no proper, non-zero ideals.

Let A be an algebra. Then $\mathbb{M}_n(A)$ is the algebra of all $n \times n$ matrices with coefficients in A. In the case where A is a Banach algebra, $\mathbb{M}_n(A)$ is also a Banach algebra with respect to the norm given by

$$\left\|(a_{i,j})\right\| = \sum_{i,j=1}^{n} \left\|a_{i,j}\right\| \quad ((a_{i,j}) \in \mathbb{M}_n(A)) .$$

Suppose that A is a Banach $*$-algebra. Then $\mathbb{M}_n(A)$ is also a Banach $*$-algebra with respect to the involution given by the transpose map $(a_{i,j}) \mapsto (a_{j,i}^*)$.

Maximal modular left ideals

Let A be a Banach algebra. The following basic result is in all books on Banach algebras.

Theorem Every maximal modular left ideal M in A is closed (and A/M is a simple Banach left A-module).

Proof Take $u \in A$ as in the definition. Assume that there is $a \in M$ with $\|a - u\| < 1$. Then there is $b \in A$ with $a - u + b + b(a - u) = 0$, and so $u = a + ba + (b - bu) \in M$, a contradiction. So M is not dense, and hence it is closed. $\square$

Codimension of maximal modular left ideals in Banach algebras

What is the codimension of such an ideal M?

Suppose that A is commutative. Then A/M is a field, and $A/M = \mathbb{C}$ by Gel'fand–Mazur, so M is the kernel of a continuous character and has codimension 1.

Suppose that A is non-commutative. For example, take $A = \mathcal{B}(E)$ for a Banach space E, and take $x \in E$ with $x \neq 0$. Then

$$M = \{T \in \mathcal{B}(E) : Tx = 0\}$$

is a closed, singly-generated maximal left ideal. When E has dimension $n \in \mathbb{N}$, M has codimension n; when E has infinite dimension, M has infinite codimension.

Detour to Fréchet algebras

A **Fréchet algebra** has a countable series of semi-norms, rather than one norm.

Let A be a commutative, unital Fréchet algebra. Then each closed maximal ideal is the kernel of a continuous character, but it is a formidable open question, called **Michael's problem**, whether all characters on each commutative Fréchet algebra are continuous.

An example

Let $O(\mathbb{C})$ denote the space of entire functions on $\mathbb{C}$, a Fréchet algebra with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$.

Then each maximal ideal M of codimension 1 in $O(\mathbb{C})$ is closed, and there exists $z \in \mathbb{C}$ such that

$$M = M_z := \{f \in O(\mathbb{C}) : f(z) = 0\}\,.$$

Let I be the set of functions $f \in O(\mathbb{C})$ such that $f(n) = 0$ for each sufficiently large $n \in \mathbb{N}$. Clearly I is an ideal in $O(\mathbb{C})$, I is dense in $O(\mathbb{C})$, and I is contained in a maximal ideal, say M. Then M is dense in $O(\mathbb{C})$, but M is not of the form M_z. The quotient A/M is a 'very large field' of infinite dimension. (For large fields, see a book with W. H. Woodin.)

Maximal left ideals that are not modular

Example Let E be an infinite-dimensional Banach space. Then E has a dense subspace F that has codimension 1 in E; it is the kernel of a discontinuous linear functional. The space E is a commutative Banach algebra with respect to the zero product, and F is a maximal (left) ideal in this algebra such that F is not closed, and F is obviously not modular. $\square$

This suggested:

Conjecture Let A be a Banach algebra. Then every maximal left ideal in A is either closed or of codimension 1.

We shall give a counter-example.

Algebraic preliminaries - 1

The following are little calculations.

Here A is any algebra, $A^{[2]} = \{ab : a, b \in A\}$, and $A^2 = \text{lin}\, A^{[2]}$. The algebra A **factors** if $A = A^{[2]}$ and **factors weakly** if $A = A^2$ (not the same).

Fact 1 Suppose that $A^2 \subsetneq A$. Then A contains a maximal left ideal that is an ideal in A and that contains A^2. Each maximal left ideal that contains A^2 has codimension 1 in A.

Just take M to be a subspace of codimension 1 in A such that $A^2 \subset M$. $\square$

Algebraic preliminaries - 2

Fact 2 Suppose that M is a maximal left ideal in A and $b \in A$, and set $J_b = \{a \in A : ab \in M\}$. Then either $J_b = A$ or J_b is a maximal modular left ideal in A.

Either $Ab \subset M$, and hence $J_b = A$, or A/M is a simple left A-module, and $J_b = (b + M)^\perp$ is a maximal modular left ideal. □

Fact 3 A has no maximal left ideals iff A is a radical algebra and $A^2 = A$.

Suppose that A has no maximal left ideals. Then A is radical, and $A^2 = A$ by Fact 1.

For the converse, assume that M is a maximal left ideal, and take $b \in A$. By Fact 2, $J_b = A$, and so $Ab \subset M$, whence $A^2 \subset M \neq A$, a contradiction. □

A simple, radical algebra

A simple, radical algebra was constructed by Paul Cohn in 1967. Since a simple algebra A is such that $A^2 = A$, it follows from Fact 3 that this algebra has no maximal left or maximal right ideal. However, it does have a maximal ideal, namely $\{0\}$.

A **topologically simple** Banach algebra A is one in which the only closed ideals are $\{0\}$ and A. Is there a commutative, radical Banach algebra that is topologically simple?

Maybe this is the hardest question in Banach algebra theory.

An example

A Banach algebra A with a bounded approximate identity is such that $A = A^{[2]}$; this follows from **Cohen's factorization theorem**

Let $\mathcal{V}$ be the **Volterra algebra**. This is the Banach space $L^1([0,1])$ with truncated convolution multiplication:

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s \quad (t \in [0,1])$$

for $f, g \in \mathcal{V}$. This is a radical Banach algebra with a BAI, and so $\mathcal{V}^{[2]} = \mathcal{V}$. Thus there are no maximal ideals in $\mathcal{V}$ (and so the conjecture holds vacuously for $\mathcal{V}$).

Some positive results

Theorem Let A be a Banach algebra with maximal left ideal M. Suppose that $A^2 \not\subset M$ and M is also a right ideal. Then M is closed.

Proof Set $J_A = \{a \in A : aA \subset M\}$. By Fact 2, J_A is a closed left ideal. Since $A^2 \not\subset M$, it is not true that $J_A = A$. Since M is a right ideal, $M \subset J_A$. So $M = J_A$ is closed. □

Corollary Let A be a commutative Banach algebra with a maximal ideal M. Then M has codimension 1. Either $A/M = \mathbb{C}$ and M is closed, or $A^2 \subset M$. □

Thus the conjecture holds in the commutative case.

Null sequences factoring

Let A be a Banach algebra. A null sequence (a_n) **factors** if there is a null sequence (b_n) in A and $a \in A$ with $a_n = b_n a$ $(n \in \mathbb{N})$. This holds when A has a BAI (but is more general).

Theorem Let A be a Banach algebra in which null sequences factor. Then every maximal left ideal M in A is closed.

Proof Take $a \in A$ and (a_n) in M with $a_n \to a$. There is a null sequence (b_n) and $b \in A$ with $a - a_n = b_n b$ and $a = b_0 b$. Again set

$$J = J_b = \{x \in A : xb \in M\}.$$

By Fact 2, J is closed. Now we have $(b_0 - b_n)b = a_n \in M$, so $b_0 = \lim(b_0 - b_n) \in J$, whence $a \in M$. So M is closed. $\square$

Applications

Corollary Every maximal left ideal in each C^*-algebra is closed. $\square$

Let E be a Banach space. Then $\mathcal{A}(E)$ and $\mathcal{K}(E)$ are the Banach algebras of approximable and compact operators, respectively. Suppose that E has certain approximation properties. Then null sequences in $\mathcal{A}(E)$ and $\mathcal{K}(E)$ factor, and so every maximal left ideal is closed.

What are they? Are they all modular? What happens if E does not have the 'certain approximation properties'?

An example - algebraic preliminary

Definition Let A be an algebra with a character φ. Then M_φ is the kernel of φ and

$$J_\varphi = \text{lin}\,\{ab - \varphi(a)b : a, b \in A\}\,.$$

Then J_φ is a right ideal and $M_\varphi A \subset J_\varphi \subset M_\varphi$.

Suppose that there is an idempotent u in $A \setminus M_\varphi$. Then

$$J_\varphi = M_\varphi^2 + M_\varphi u + (1 - u)M_\varphi\,.$$

Fact Take a non-zero linear functional λ on A with $\lambda \mid J_\varphi = 0$, and set $M = \ker\lambda$. Then M is a maximal left ideal in A of codimension 1 and $A^2 \not\subset M$.

This is easily checked.

A Banach algebra

Theorem Let A be a Banach algebra with a character φ, and suppose that J_φ is not closed. Then there is a dense maximal left ideal M of codimension 1 in A with $A^2 \not\subset M$.

Proof Take a linear functional λ with $\lambda \mid J_\varphi = 0$ and $\lambda \mid \overline{J_\varphi} \neq 0$, and set $M = \ker \lambda$. $\square$

A starting point

We suppose that we have a Banach algebra $(I, \|\cdot\|_I)$ with $I^2 \subsetneq \overline{I^2} = I$, and we take $B = I^\sharp$ to be the unitization of I, so that B is a unital Banach algebra, with identity e_B, say, and I is a maximal ideal in B.

A construction

From our starting point, consider the Banach algebra $\mathfrak{B} = \mathbb{M}_2(B)$, so that $\mathfrak{B}$ is also a unital Banach algebra. Set $\mathfrak{I} = \mathbb{M}_2(I)$. Then $\mathfrak{I}$ is a closed ideal in $\mathfrak{B}$ (of codimension 4).

Consider the elements

$$P = \begin{pmatrix} e_B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & e_B \end{pmatrix}$$

in $\mathfrak{B}$. Then $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, and $P + Q$ is the identity of $\mathfrak{B}$.

Next, consider the subset $\mathfrak{A} = \mathfrak{I} + \mathbb{C}P$ in $\mathfrak{B}$. Symbolically, $\mathfrak{A}$ has the form

$$\mathfrak{A} = \begin{pmatrix} B & I \\ I & I \end{pmatrix}.$$

Then $\mathfrak{A}$ is a closed subalgebra of $\mathfrak{B}$, and $\mathfrak{I}$ is a maximal ideal in $\mathfrak{A}$ of codimenison 1; the quotient map $\varphi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ is a character on $\mathfrak{A}$.

A construction, continued

We define M_φ and J_φ (in relation to $\mathfrak{A}$ and the character φ) as above. Then $\mathfrak{I} = M_\varphi$ and

$$J_\varphi = \mathfrak{I}^2 + \mathfrak{I}P + Q\mathfrak{I} \subset P\mathfrak{I}^2Q + P\mathfrak{I}P + Q\mathfrak{I} \subset \mathfrak{I}\,,$$

and so $\mathfrak{I}^2 \subset J_\varphi \subset \mathfrak{I} = M_\varphi$. Also

$$\mathfrak{I} = (P+Q)\mathfrak{I}(P+Q) = P\mathfrak{I}P + P\mathfrak{I}Q + Q\mathfrak{I}\,.$$

We *claim* that $\mathfrak{I}^2$ is dense in M_φ. Indeed, given $\varepsilon > 0$ and $x \in I$, there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n, v_1, \dots, v_n \in I$ with $\left\| x - \sum_{i=1}^n u_i v_i \right\|_I < \varepsilon$. It follows that

$$\left\| \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^n \begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i & 0 \\ 0 & 0 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} x - \sum_{i=1}^n u_i v_i & 0 \\ 0 & 0 \end{pmatrix} \right\| < \varepsilon\,,$$

with similar calculations in the other positions. The claim follows. Hence $\overline{J_\varphi} = M_\varphi$.

A construction, continued further

We also *claim* that $J_\varphi \neq M_\varphi$. Assume towards a contradiction that $J_\varphi = M_\varphi$. Then

$$\mathfrak{I} = P\mathfrak{I}P + P\mathfrak{I}Q + Q\mathfrak{I} = P\mathfrak{I}^2Q + P\mathfrak{I}P + Q\mathfrak{I}\,.$$

Since $\mathfrak{I} = P\mathfrak{I}P \oplus P\mathfrak{I}Q \oplus Q\mathfrak{I}$, this implies that $P\mathfrak{I}Q = P\mathfrak{I}^2Q$. However, take $x \in I \setminus I^2$, and consider the element

$$\mathbf{x} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}\,.$$

Since $P\mathbf{x}Q = \mathbf{x}$, we see that $\mathbf{x} \in P\mathfrak{I}Q$. But every element of $P\mathfrak{I}^2Q$ has the form

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix},$$

where $u \in I^2$, and so $\mathbf{x} \notin P\mathfrak{I}^2Q$, the required contradiction. Thus the claim holds.

So far we have:

Theorem The Banach algebra $\mathfrak{A}$ contains a dense maximal left ideal $\mathfrak{M}$ with $\mathfrak{A}^2 \not\subset \mathfrak{M}$ such that $\mathfrak{M}$ has codimension 1 in $\mathfrak{A}$. □

Another algebraic calculation

Proposition Let A be an algebra containing a maximal left ideal M of codimension 1 such that $A^2 \not\subset M$, and take $n \in \mathbb{N}$. Then the matrices $(a_{i,j})$ in $\mathbb{M}_n(A)$ such that $a_{i,1} \in M \ (i \in \mathbb{N}_n)$ form a maximal left ideal in $\mathbb{M}_n(A)$ of codimension n.

Proof The matrices that we are considering have the form

$$\mathcal{M} = \begin{pmatrix} M & A & \dots & A \\ M & A & \dots & A \\ \dots & \dots & \dots & \dots \\ M & A & \dots & A \end{pmatrix} .$$

It is clear that $\mathcal{M}$ is a left ideal of codimension n in $\mathbb{M}_n(A)$. Consider a left ideal $\mathcal{J}$ in $\mathbb{M}_n(A)$ with $\mathcal{J} \supsetneq \mathcal{M}$. Since $A^2 \not\subset M$, there exist $a, b \in A$ with $ab \notin M$, and so $b \notin M$ and this implies that $\mathbb{C}ab + M = \mathbb{C}b + M = A$. A little multiplication shows that $\mathcal{J} = \mathbb{M}_n(A)$, and so $\mathcal{M}$ is maximal. $\square$

Conclusion

We combine the above results to exhibit our main example (assuming that we can reach the starting point).

Theorem Let $n \in \mathbb{N}$. Then there is a Banach algebra $\mathcal{A}$ with a dense maximal left ideal $\mathcal{M}$ with codimension n in $\mathcal{A}$. We can arrange that $\mathcal{A}$ be semi-simple and a Banach $*$-algebra. $\square$

Challenge Modify the above to find a Banach algebra with a dense maximal left ideal of infinite codimension. Maybe a semigroup algebra of the form $\ell^1(S)$?

An equivalence

The existence of such a Banach algebra is equivalent to the existence of a Banach algebra A that has a discontinuous left A-module homomorphism into an infinite-dimensional, simple Banach left A-module, an 'automatic continuity' question.

See a book of mine on 'automatic continuity'.

A small modification

Replace $\mathfrak{A}$ and $\mathfrak{I}$ by

$$\mathfrak{A} = \begin{pmatrix} B & I \\ B & I \end{pmatrix} \quad \text{and} \quad \mathfrak{I} = \begin{pmatrix} I & I \\ B & I \end{pmatrix},$$

respectively. Then nearly the same calculation works, and the bonus is that we get $\mathfrak{A}^2 = \mathfrak{A}$, and hence $\mathcal{A}^2 = \mathcal{A}$, so that $\mathcal{A}$ factors weakly. Indeed, take

$$\mathbf{x} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \mathfrak{A}\,,$$

where $x_{1,1}, x_{2,1} \in B$ and $x_{1,2}, x_{2,2} \in I$. Then

$$\mathbf{x} = P\mathbf{x} + \begin{pmatrix} 0 & 0 \\ e_B & 0 \end{pmatrix} \begin{pmatrix} x_{2,1} & x_{2,2} \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}^2\,.$$

However I do not know the answer to the following:

Let A be a Banach algebra that factors. Is it true that every maximal left ideal in A is closed?

Commutative starting points

Recall that we require Banach algebras I such that I^2 is dense in I and $I^2 \subsetneq I$.

1) Let $I = (\ell^p, \|\cdot\|_p)$, where $1 \leq p < \infty$, taken with the coordinatewise product, so that I is a commutative, semi-simple Banach algebra. The final algebra $\mathcal{A}$ is semi-simple.

2) Take R to be the commutative Banach algebra $C([0,1])$ with the above truncated convolution multiplication. Here R has an approximate identity, so $R^{[2]}$ is dense in R, but $R^2 \subsetneq R$. This example is radical. So the final algebra $\mathcal{A}$ has a large radical.

Non-commutative starting points

3) Let H be an infinite-dimensional Hilbert space, and take I to be the non-commutative Banach algebra of all Hilbert–Schmidt operators on H, with the standard norm on I. Then $I^2 = I^{[2]}$ is the space of trace-class operators. Here I is a semi-simple algebra and a Banach $*$-algebra, and we can show that he corresponding algebra $\mathcal{A}$ has the same properties. $\square$

4) Let E be an infinite-dimensional Banach space, and let $I = \mathcal{N}(E)$, the nuclear operators on E, so that I is a non-commutative Banach algebra with respect to the nuclear norm. Then $I^{[2]}$ is dense in I and I^2 has infinite codimension in I. $\square$

Finitely-generated maximal left ideals

A left ideal I in a unital algebra A is **finitely-generated** if there exist $a_1, \ldots, a_n \in A$ such that $I = Aa_1 + Aa_2 + \cdots + Aa_n$.

Theorem, Sinclair-Tullo, 1974 Let A be a unital Banach algebra. Suppose that all closed left ideals are finitely-generated. Then A is finite dimensional. $\square$

Conjecture, D-Zelazko, 2012 Let A be a unital Banach algebra. Suppose that all **maximal** left ideals are finitely-generated. Then A is finite dimensional.

Theorem True when A is commutative, and for various other examples. $\square$

Theorem, D-Kania-Kochanek-Koszmider-Laustsen, 2013 Consider $\mathcal{B}(E)$. Then the conjecture holds for very many different classes of Banach spaces E. No counter-example is known. $\square$

Semigroup algebras

Theorem, Jared White, 2017 Consider $\ell^1(S)$ for a monoid S, or its weighted version $\ell^1(S,\omega)$. Then the conjecture holds for many different classes of semigroup S, including all groups. $\square$

For a semigroup algebra $\ell^1(S)$, set

$$\ell_0^1(S) = \left\{ f : \sum_{s\in S} f(s) = 0 \right\}$$

Then $\ell_0^1(S)$ is a maximal ideal, called the **augmentation ideal**.

Theorem, Jared White, 2017 Let S be a monoid. Then $\ell_0^1(S)$ is finitely generated (as a left ideal) iff S is 'pseudo-finite'. $\square$

Could infinite, pseudo-finite semigroups give counters to the DZ conjecture? One needs **all** maximal left ideals to be finitely generated.

Counter-examples?

White There are rather trivial infinite, pseudo-finite semigroups. But these do not give counter-examples to the main conjecture.

Example, VG, et al There is a non-trivial infinite, pseudo-finite semigroup. □

Question Does this give a counter-example to the DZ conjecture? What are the maximal left ideals for this example?