Discreteness is not Morita invariant

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Outline

Section 1

Categories of actions

Discrete monoid actions

We may view a monoid M as a one-object category.

The category of presheaves on this category,

$$[M^{\mathrm{op}}, \textbf{Set}] \simeq \textbf{Set}\text{-}M,$$

is precisely the category of *right actions* of M.

Continuous monoid actions

We may equip a monoid M with a topology τ.[1]

We can then consider the full subcategory of $[M^{\mathrm{op}}, \mathbf{Set}]$ on the actions which are continuous with respect to τ (when sets are considered as discrete spaces:

$$[M^{\mathrm{op}}, \mathbf{Set}] \hookleftarrow \mathrm{Cont}(M, \tau) : V.$$

This talk is about categories constructed in this way.

They happen to be examples of Grothendieck toposes.

[1] Don't worry, you could just as well start from a genuine topological monoid!

Morita equivalence

Definition

Two monoids with topologies (M, τ) and (M', τ') are said to be **Morita equivalent** if $\mathrm{Cont}(M, \tau) \simeq \mathrm{Cont}(M', \tau')$.

Clearly, isomorphic monoids are Morita equivalent.

As with other notions of Morita equivalence, this is a way of expressing how much information is lost when we pass from a monoid to its category of right actions.

A property of topological monoids which is recoverable from its category of continuous right actions is called Morita invariant, since it must be shared with all members of its Morita equivalence class.

Morita invariant properties appear at first glance to be rare.

Example

Let M be any monoid. Then $(M, \tau_{\mathrm{indisc}})$ is Morita equivalent to the trivial monoid, where τ_{indisc} is the indiscrete topology.

In other words, *there can be no interesting purely algebraic Morita-invariant property!*

To get around this, we'll first identify canonical representatives of Morita equivalence classes, and then extract interesting invariants of these.

Section 2

Reconstructing topologies

Necessary clopens

Definition

Let M be a monoid equipped with a topology τ, X an M-set. A **necessary clopen** for X is a set of the form,

$$\mathcal{I}_x^p := \{m \in M \mid xm = xp\},$$

where $x \in X$ and $p \in M$.

Lemma

Let M be a monoid equipped with a topology τ and X an M-set. Then X is an (M, τ)-set if and only if all necessary clopens for X are (cl)open in τ.

Note that for each $x \in X$, the subsets $\mathcal{I}_x^p$ partition M, so that if these subsets are all open they are also necessarily closed, hence the name 'necessary clopen'.

Right adjoint to V

Proposition

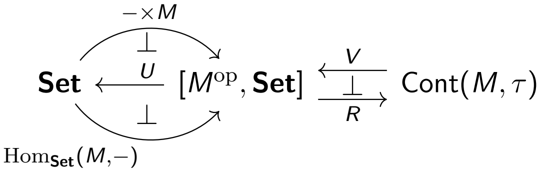
Suppose a monoid M is equipped with a topology τ. Then the forgetful functor $V : \mathrm{Cont}(M, \tau) \to [M^{\mathrm{op}}, \mathbf{Set}]$ is left exact and comonadic; its right adjoint R sends an M-set X to:

$$R(X) := \{x \in X \mid \forall p, q \in M, \mathcal{I}^p_{xq} \in \tau\}.$$

Moreover, if τ makes the multiplication of M left continuous then the expression for $R(X)$ simplifies to:

$$R(X) := \{x \in X \mid \forall p \in M, \mathcal{I}^p_x \in \tau\}.$$

The situation



Theorem

Suppose M is a monoid equipped with a topology τ. Then there is a topology $\tilde{\tau} \subseteq \tau$ generated by clopen sets such that $\mathrm{Cont}(M, \tilde{\tau}) = \mathrm{Cont}(M, \tau)$ as sub-categories of $[M^{\mathrm{op}}, \mathbf{Set}]$. Moreover, $\tilde{\tau}$ is the coarsest topology on M with this property.

Action topology

Definition

We call $\tilde{\tau}$ the (right) **action topology** induced by τ. More generally, we say τ is an **action topology** if $\tau = \tilde{\tau}$.

NB. Since we consider right actions, all of our definitions have an implicit right-handedness, which we shall ignore in the remainder of the talk.

Proposition

The multiplication on M is continuous with respect to the action topology $\tilde{\tau}$ for any starting topology τ.

So we can safely restrict to topological monoids without missing any Morita equivalence classes.

Powder monoids

Another operation we can perform on (M, τ) without changing the Morita-equivalence class is to take its Kolmogorov quotient, since topologically indistinguishable elements of the monoid must act identically.

Definition

We say a topological monoid (M, τ) is a **powder monoid** if τ is a T_0 action topology.

Theorem

Given a monoid with an arbitrary topology (M, τ), there is a canonical powder monoid, which we shall by an abuse of notation denote by $(\tilde{M}, \tilde{\tau})$, such that $\mathrm{Cont}(M, \tau) \simeq \mathrm{Cont}(\tilde{M}, \tilde{\tau})$ and the canonical points of these toposes coincide.

Powder monoids

Powder monoids have nice properties. They are zero-dimensional, since they have a base of clopen sets.

Examples

- Any discrete monoid is a powder monoid.
- Any prodiscrete monoid is a powder monoid. Consider the profinite completion or p-adic completion of the integers or the profinite completion of the natural numbers, say.
- The powder monoid corresponding to the real numbers under addition (with their usual topology) is the trivial monoid.

Section 3

Reconstructing monoids

Best approximation

Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\mathbf{Set} \xrightarrow{\ p\ } \mathcal{E}$$

Theorem

There is a canonical factorization of p through a topos of topological monoid actions.

Best approximation

Suppose that we are given a topos $\mathcal{E}$, i.e. a category with the basic necessary properties to have a chance of being a category of continuous actions of a monoid, and a point of such a topos:

$$\mathbf{Set} \longrightarrow [L^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathcal{E}, \qquad p : \mathbf{Set} \to \mathcal{E}$$

Theorem

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Namely, consider the monoid $L := \mathrm{End}(p^*)^{\mathrm{op}}$, dual to the monoid of natural endomorphisms of p^*.

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$$\mathbf{Set} \longrightarrow [L^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathrm{Cont}(L, \rho) \longrightarrow \mathcal{E} \qquad (p : \mathbf{Set} \to \mathcal{E})$$

Theorem

There is a canonical factorization of p through a topos of topological monoid actions.
Namely, consider the monoid $L := \mathrm{End}(p^*)^{\mathrm{op}}$, dual to the monoid of natural endomorphisms of p^*.
This comes equipped with the coarsest topology making all of the actions of the form $p^*(X)$ continuous.

In particular, the given point expresses $\mathcal{E}$ as a topos of actions of a topological monoid if and only if the comparison morphism $\mathrm{Cont}(L, \rho) \to \mathcal{E}$ is an equivalence.

Definition

We call a topological monoid **complete** if it is isomorphic to the topological monoid of endomorphisms of its canonical point.

Every complete monoid is a powder monoid, but the converse is false.

Example

Consider $\mathbb{Z}$ equipped with the topology where all non-trivial subgroups (and their cosets) are open. This is a powder monoid, but the procedure above produces the profinite completion of the integers.

Complete monoids

Complete monoids are the natural representatives of toposes of actions of topological actions. It therefore makes sense to ask about Morita-invariant properties of complete monoids.

Remark

A complete monoid is completely determined by the corresponding point of its topos of actions. Morita equivalence is thus really a question of the different points of a given topos.

Examples

Example

A complete monoid produces has a topos of actions which is atomic if and only if its group of units is dense. Hence this is a Morita-invariant property.

Example

If a complete monoid has a zero element, then each of its principal actions has a unique fixed point. Since a complete monoid is recoverable as a limit of its continuous principal actions and the fixed points assemble into a cone over this diagram, we find that having a zero element is a Morita invariant property.

Is discreteness Morita invariant?

Section 4

Homomorphisms

Semigroup homomorphisms

Continuous semigroup homomorphisms induce geometric morphisms between toposes of actions of topological monoids.

Remark

We use semigroup homomorphisms rather than monoid homomorphisms because in the discrete case these correspond to functors between the *idempotent completions* of the respective monoids, which are sufficient to induce geometric morphisms.

This works exactly as one would expect: the inverse image functor is restriction along the semigroup homomorphism.

Morita equivalence for discrete monoids

Considering just the discrete case for a moment, any Morita equivalence of discrete monoids is induced by a semigroup homomorphisms.

Proposition

Two discrete monoids M and N are Morita equivalent if and only if there exists an idempotent $e \in M$ and elements $u, v \in M$ with $N \cong eMe$, $uv = 1$, $ue = u$ and $ev = v$.

That is, N must be a subsemigroup of M (and vice versa)!

Unfortunately, continuous homomorphisms aren't enough to determine Morita equivalence for complete monoids.

Continuous homomorphisms are insufficient

Lemma

A continuous monoid homomorphism between complete monoids inducing an equivalence of toposes is an isomorphism. In particular, any non-trivial equivalence must be induced by an inclusion of subsemigroups.

Example

The Schanuel topos is the classifying topos for *infinite, decidable objects*. Equivalently, it is the topos of sheaves for the atomic topology on the category of finite sets and injective functions.

Points of this topos correspond to infinite sets. the monoids of injective endomorphisms of any such set provides a representing topological monoid. But there are no non-identity idempotents of these monoids! In particular, there is no continuous semigroup homomorphism inducing the equivalence between the respective toposes of actions.

Section 5

The titular counterexample

The bicyclic monoid

Consider the bicyclic monoid, which has the following presentation:

$$B := \langle u, v \mid uv = 1 \rangle.$$

Each element of B can be uniquely expressed in the form $v^i u^j$ with $i, j \geq 0$. We equip this with the discrete topology.

Lemma

Points of a presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ correspond to filtered colimits of representable objects in $[\mathcal{C}, \mathbf{Set}]$.

Therefore, consider the point of $[B^{\mathrm{op}}, \mathbf{Set}]$ constructed as the colimit of the following diagram in $[B, \mathbf{Set}]$,

$$B \xrightarrow{\cdot u} B \xrightarrow{\cdot u} B \xrightarrow{\cdot u} \cdots .$$

An explicit description

As a left B-set, we can identify A with the set having elements $\{b^p a^q \mid p \in \mathbb{N},\ q \in \mathbb{Z}\}$, and action determined by:

$$u \cdot b^p a^q = \begin{cases} b^{p-1} a^q & \text{if } p > 0 \\ a^{q+1} & \text{if } p = 0 \end{cases}$$
$$v \cdot b^p a^q = b^{p+1} a^q.$$

There is an epimorphism $A \to B$ sending $b^p a^q$ to $v^p u^q$ if $q \leq 0$ and to v^{p+q} if $q < 0$. This epimorphism splits; the most obvious splitting sends $v^i u^j$ back to $b^i a^j$.

The corresponding complete monoid

Let $L := \mathrm{End}(A)^{\mathrm{op}}$. This monoid can be presented as,

$$L \cong \langle a, a^{-1}, w \mid aa^{-1} = 1 = a^{-1}a,\ wa^k w = wa^k, wa^{-k}w = a^{-k}w\ (k \geq 0)\rangle,$$

where,

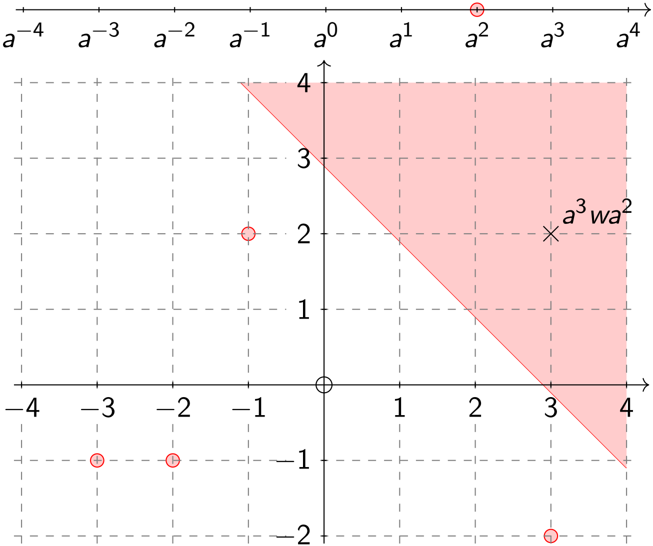
- a acts on A by sending $b^p a^q$ to $b^p a^{q+1}$,
- a^{-1} is its inverse, and
- w is the idempotent endomorphism obtained as the composite $A \to B \to A$ of the morphisms in the retraction described on the last slide.

Note that elements of L can be presented in the form $a^i w a^j$ or as a^k with $i, j, k \in \mathbb{Z}$.

The topology

To compute the topology on L, we use the construction described earlier. To summarize, for any value of i and j, the singleton $\{a^i w a^j\}$ is open. Meanwhile, the basic open neighbourhoods of a^k are infinite and of the form $\{a^k\} \cup \{a^i w a^j \mid i + j \geq k'\}$ for some $k' \in \mathbb{Z}$.

An example of an open set



The conclusion

We have by now seen that $[B^{\mathrm{op}}, \mathbf{Set}] \simeq \mathrm{Cont}(L, \rho)$.

As such, we have shown that *discreteness is not a Morita-invariant property* of complete topological monoids.

On the other hand, observe that (L, ρ) *does* have a dense discrete subsemigroup.

The conclusion

Thanks for listening!

Any questions?