

COMMUTATIVE ORDERS REVISITED

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ABSTRACT. This article studies commutative orders, that is, commutative semigroups having a semigroup of quotients. In a commutative order S , the square-cancellable elements $\mathcal{S}(S)$ constitute a well-behaved separable sub-semigroup. Indeed, $\mathcal{S}(S)$ is also an order and has a maximum semigroup of quotients R , which is Clifford. We present a new characterisation of commutative orders in terms of semilattice decompositions of $\mathcal{S}(S)$ and families of ideals of S . We investigate the role of tensor products in constructing quotients, and show that all semigroups of quotients of S are homomorphic images of the tensor product $R \otimes_{\mathcal{S}(S)} S$. By introducing the notions of generalised order and semigroup of generalised quotients, we show that if S has a semigroup of generalised quotients, then it has a greatest one. For this we determine those semilattice congruences on $\mathcal{S}(S)$ that are restrictions of congruences on S .

1. INTRODUCTION

Commutative semigroups, in spite of possessing a well-developed theory, remain far from being fully understood. For a relatively recent general presentation, see [4]. Our aim here is to study a commutative semigroup S by dividing it into two parts. Namely, $S = \mathcal{S}(S) \cup T$ where $\mathcal{S}(S)$ is the subsemigroup of square-cancellable elements of S , and $T = S \setminus \mathcal{S}(S)$. Our tool is that of *quotients*: for the convenience of the reader we immediately recall the main relevant notions, beginning with that of square-cancellability.

An element $a \in S$ is *square-cancellable* if for all $x, y \in S^1$ we have that $xa^2 = ya^2$ implies $xa = ya$ and also $a^2x = a^2y$ implies $ax = ay$. It is clear that being square-cancellable is a necessary condition for an element to lie in a subgroup of an oversemigroup. Let S be a subsemigroup of a semigroup Q . Then S is a *left order* in Q and Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^\#b$ where $a \in \mathcal{S}(S), b \in S$ and $a^\#$ is the inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q . *Right orders* and *semigroups of right*

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quotients are defined dually. If S is both a left order and a right order in Q , then S is an *order* in Q and Q is a *semigroup of quotients* of S . We remark that if a commutative semigroup is a left order in Q , then Q is commutative [1, Theorem 3.1] so that S is an order in Q . A given commutative order S may have more than one semigroup of quotients. The semigroups of quotients of S are pre-ordered by the relation $Q \geq P$ if and only if there exists an onto homomorphism $\phi : Q \rightarrow P$ which restricts to the identity on S . Such a ϕ is referred to as an *S -homomorphism*; the classes of the associated equivalence relation are the *S -isohomomorphism classes of orders*, giving us a partially ordered set $\mathcal{Q}(S)$. In the best case, $\mathcal{Q}(S)$ contains maximum and minimum elements.

Our rationale is as follows. Let S be a commutative semigroup. The set $\mathcal{S}(S)$ is a subsemigroup of S and, if S is an order, then $\mathcal{S}(S)$ is also. In this case $\mathcal{S}(S)$ is a commutative separative semigroup and thus has a well-understood structure. Namely, $\mathcal{S}(S)$ is a semilattice of commutative cancellative semigroups and as such possesses a semigroup of quotients that is a semilattice of commutative groups, that is, a commutative Clifford semigroup [5]. Moreover, every semigroup of quotients of $\mathcal{S}(S)$ is a commutative Clifford semigroup and $\mathcal{Q}(\mathcal{S}(S))$ forms a complete lattice [1]. The subset T consists of what may be thought of as ‘bad’ elements, including any nilpotents. We aim to understand these elements in terms of their relation to elements of $\mathcal{S}(S)$, in the case S is an order.

Unfortunately, not all commutative semigroups are orders (not even all those in which every element is square-cancellable, see Example 2.5 below). Easdown and the second author [1] gave a description of those that are, in terms of compatible pre-orders, using a direct construction. They also give examples of commutative orders having, respectively, a maximum but no minimum and a minimum but no maximum semigroup of quotients. By using an entirely different approach we re-establish the description of orders given in [1] and give a deeper analysis of $\mathcal{Q}(S)$ for commutative orders S . We do so by using the decomposition $S = \mathcal{S}(S) \cup T$ mentioned above.

In Section 2 we give the necessary preliminaries, and recap our knowledge of commutative orders, summarising and clarifying existing results. We present the description of commutative orders S in terms of compatible pre-orders from [1], and then proceed in Section 3 to give a new characterisation via semilattice decompositions of $\mathcal{S}(S)$ and families of ideals of S .

Section 4 introduces a notion of a *generalised* quotient semigroup, which will be of use in our final results. We observe that a semigroup of generalised quotients of a commutative semigroup S is always commutative. If S is commutative and either S is a monoid or $S = \mathcal{S}(S)$, then the notions of quotient semigroup and generalised quotient semigroup coincide.

We proceed as follows in Section 5. Let S be a commutative order, so that $\mathcal{S}(S)$ is also an order. In particular, if S is an order in Q , then $\mathcal{S}(S)$

is an order in $\mathcal{H}(Q)$, the commutative Clifford semigroup of the group \mathcal{H} -classes of Q . Note that S may also be an order in another semigroup Q' (so that $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q')$) such that $\mathcal{H}(Q) \cong \mathcal{H}(Q')$, without Q being isomorphic to Q' (see [1, Example 7.4]). Put $R = \mathcal{H}(Q)$. We construct the tensor product $R \otimes_{\mathcal{S}(S)} S$ and show that R embeds into $R \otimes_{\mathcal{S}(S)} S$ and that Q is a morphic image of $R \otimes_{\mathcal{S}(S)} S$. We thus obtain *all* quotient semigroups of S that induce the same quotient semigroup R of $\mathcal{S}(S)$ as morphic images of the fixed semigroup $R \otimes_{\mathcal{S}(S)} S$. Moreover, we can recover the characterisation of commutative orders given in [1]. A further consequence is that every semigroup of quotients of S is the image of $M \otimes_{\mathcal{S}(S)} S$, where M is the maximum semigroup of quotients of $\mathcal{S}(S)$.

Now let S be any commutative semigroup such that $\mathcal{S}(S)$ is an order in a semigroup R . Again in Section 5, we give a necessary and sufficient condition for R to embed into $R \otimes_{\mathcal{S}(S)} S$, namely, that if ρ is the semilattice congruence on $\mathcal{S}(S)$ induced by that of R , then $\bar{\rho}|_{\mathcal{S}(S)} = \rho^1$, where $\bar{\rho}$ is the congruence on S generated by ρ . This is easily seen to be a necessary condition for S to be an order in some Q such that $\mathcal{H}(Q) = R$. Our first aim in Section 6 is to find a straightforward condition in terms of elements of S for $\bar{\rho}|_{\mathcal{S}(S)} = \rho$. If a congruence ρ on $\mathcal{S}(S)$ satisfies this property, then one further condition on ρ tells us when S has a semigroup of generalised quotients. Our final results show that if S has a generalised semigroup of quotients (for example, if S is an order), then it has a maximum one.

2. PRELIMINARIES

We recall that a *pre-order* (or *quasi-order*) \preceq on a set S is a reflexive, transitive relation. From a pre-order \preceq we can define an equivalence relation \equiv_{\preceq} by

$$a \equiv_{\preceq} b \text{ if and only if } a \preceq b \preceq a.$$

If S is a semigroup, then we say that a pre-order \preceq is *compatible* if for any $a, b, c \in S$, we have that if $a \preceq b$, then $ac \preceq bc$ and $ca \preceq cb$. If \preceq is a compatible pre-order, $a \preceq b$ and $c \preceq d$, then it is clear by transitivity that $ac \preceq bd$ and, in this case, the associated equivalence relation is a congruence.

Lemma 2.1. *Let κ be a relation on a semigroup S .*

(i) *The smallest compatible pre-order $\bar{\kappa}$ containing κ is given by the rule that for any $a, b \in S$, $a \bar{\kappa} b$ if and only if $a = b$ or there exists a sequence*

$$a = c_1 u_1 d_1, c_1 v_1 d_1 = c_2 u_2 v_2, \dots, c_n v_n d_n = b,$$

where for $1 \leq i \leq n$ we have that $(u_i, v_i) \in \kappa$ and $c_i, d_i \in S^1$.

¹A word on notation: for any relation μ on a set X , we denote by $\mu|_Y$ the restriction of μ to a subset Y of X .

(ii) The smallest congruence $\bar{\kappa}$ containing κ is given by the rule that for any $a, b \in S$, $a\bar{\kappa}b$ if and only if $a = b$ or there exists a sequence

$$a = c_1u_1d_1, c_1v_1d_1 = c_2u_2v_2, \dots, c_nv_nd_n = b,$$

where for $1 \leq i \leq n$ we have that $(u_i, v_i) \in \kappa$ or $(v_i, u_i) \in \kappa$ and $c_i, d_i \in S^1$.

Where the notation $\bar{\kappa}$ is not convenient, we may use the more standard $\langle \kappa \rangle$.

Let Q be a commutative semigroup. Clearly, Green's relations $\mathcal{H}, \mathcal{L}, \mathcal{R}$ and \mathcal{J} all coincide on Q and we will denote this relation, which is a congruence, by \mathcal{H} . Moreover, \mathcal{H} is the equivalence associated with the (compatible) pre-order $\leq_{\mathcal{H}}$, where for $a, b \in Q$ we have $a \leq_{\mathcal{H}} b$ if and only if $a = bq$ for some $q \in Q^1$. Where there is danger of ambiguity we will denote \mathcal{H} and $\leq_{\mathcal{H}}$ on Q by \mathcal{H}^Q and $\leq_{\mathcal{H}^Q}$, respectively, with corresponding notation for \mathcal{H} -classes.

The following result is folklore. Its straightforward proof runs as that of the corresponding statement for \mathcal{H} , which can be found in, for instance, [6, Proposition II.4.5].

Lemma 2.2. *Let T be a regular subsemigroup of a commutative semigroup Q . Then*

$$\leq_{\mathcal{H}^T} = \leq_{\mathcal{H}^Q|_T}.$$

We now explain the concept of square-cancellability. Let S be a semigroup. The relation $\leq_{\mathcal{R}^*}$ is defined on S by the rule that for $a, b \in S$ we have $a \leq_{\mathcal{R}^*} b$ in S if and only if $a \leq_{\mathcal{R}} b$ in some oversemigroup of S . It is well known, and easy to see from the right regular representation of S in \mathcal{T}_{S^1} , that $a \leq_{\mathcal{R}^*} b$ if and only if for all $x, y \in S^1$ we have that $xb = yb$ implies $xa = ya$. Clearly, $\leq_{\mathcal{R}^*}$ is a pre-order; we denote the associated equivalence relation by \mathcal{R}^* .

The relations $\leq_{\mathcal{L}^*}$ and \mathcal{L}^* are defined dually and we let $\leq_{\mathcal{H}^*}$ and \mathcal{H}^* be the intersections $\leq_{\mathcal{R}^*} \cap \leq_{\mathcal{L}^*}$ and $\mathcal{R}^* \cap \mathcal{L}^*$, respectively. It is clear from their second characterisations that if S is commutative then

$$\leq_{\mathcal{R}^*} = \leq_{\mathcal{L}^*} = \leq_{\mathcal{H}^*} \quad \text{and} \quad \mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$$

and moreover $\leq_{\mathcal{H}^*}$ is compatible, so that \mathcal{H}^* is a congruence. From the definition given in the Introduction, $a \in S$ is square-cancellable if $a \mathcal{H}^* a^2$. We have already observed that being square-cancellable is a necessary condition for an element of S to lie in a subgroup of an oversemigroup.

We denote by $\mathcal{H}(S)$ the set of elements of S lying in group \mathcal{H} -classes, and by $\mathcal{S}(S)$ the set of square-cancellable elements of S . Recall from the definition that if S is an order in Q , then $\mathcal{S}(S) = \mathcal{H}(Q) \cap S$. The next lemma builds on the preceding remarks.

Lemma 2.3. [1] *Let T be a commutative semigroup. Then:*

- (i) \mathcal{H}^* is a congruence on T and $\mathcal{S}(T)$ is empty or is a subsemigroup of T ;
- (ii) \mathcal{H} is a congruence on T and $\mathcal{H}(T)$ is empty or is a subsemigroup and moreover a semilattice of the group \mathcal{H} -classes of T ;
- (iii) for all $a, b \in \mathcal{H}(T)$, $(ab)^{\#} = a^{\#}b^{\#} = b^{\#}a^{\#}$.

Further, if S is an order in a commutative semigroup Q , then $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q)$.

Recall that a subset X of a commutative semigroup S is *separative* if for all $x, y \in X$ with $x^2 = xy = y^2$, we have $x = y$. Since any Clifford semigroup is separative, and separability is clearly inherited by subsemigroups, the following lemma is clear.

Lemma 2.4. *Let S be a commutative subsemigroup of Q such that $\mathcal{S}(S) = \mathcal{H}(Q) \cap S$. Then $\mathcal{S}(S)$ is separative.*

The following is an example of a commutative semigroup $S = \mathcal{S}(S)$ which is not separative, hence, in view of Result 2.6, not an order.

Example 2.5. (Ruškuc) Let S be the semigroup defined by the presentation

$$S = \langle a, b \mid a^2 = ab = ba = b^2 \rangle.$$

It is readily seen that $S = \{a^i : i \in \mathbb{N}\} \cup \{b\}$ and that b and all powers of a are distinct. Hence S is commutative, but not separative.

Every element of S^1 has length $|a^i| = i$, $|b| = 1$, $|1| = 0$ and clearly $|uv| = |u| + |v|$ for all $u, v \in S^1$. Let $c \in S$ and $x, y \in S^1$ with $xc^2 = yc^2$. Then $|x| = |y|$ so that either $x = y = 1$ or $xc = a^{|xc|} = a^{|yc|} = yc$. Thus every element of S is square-cancellable.

On the positive side we have the following, which draws together relevant results from [1, 3, 5] and [4]. First, we recall that an order S in a commutative semigroup Q is said to be *straight* if every element of Q can be written in the form $q = a^\sharp b$ where $a \in \mathcal{S}(S)$, $b \in S$, and $a \mathcal{H} b$ in Q .

Result 2.6. *The following conditions are equivalent for a semigroup S :*

- (i) S is commutative and separative;
- (ii) S is a semilattice of commutative, cancellative semigroups;
- (iii) S is an order in a commutative Clifford semigroup;
- (iv) S is a subsemigroup of a commutative Clifford semigroup;
- (v) S is a commutative order such that $S = \mathcal{S}(S)$;
- (vi) S is a commutative straight order in some semigroup of quotients;
- (vii) S is a commutative order which is straight in each of its semigroups of quotients;

(viii) S is commutative, $S = \mathcal{S}(S)$ and the \mathcal{H}^* -classes of S are cancellative.

If any (all) of the above conditions hold, then S has a semigroup of quotients Q such that $\leq_{\mathcal{H}\phi|_S} = \leq_{\mathcal{H}^*}$.

Proof. The equivalence of (i) to (iv) comes from [3, Corollary 6.1] (cf. [7, Theorem II.6.6] and that of (v), (vi), (vii) was noted in the beginning of Section 7 in [1]. Corollary 4.4 of [1] shows that (v) and (viii) are equivalent. Clearly, (iii) implies (v), so we need only show that (v) implies (iii). If $S = \mathcal{S}(S)$ is a commutative order in Q , then by [1, Theorem 3.1], Q is commutative. Let $q = a^\sharp b$ where $a, b \in S$. Then $q \mathcal{H} ab \in Q$ and ab lies in a subgroup of Q . Hence Q is a union of groups, and so Clifford. \square

Example 2.5 shows that in Condition (v) of Result 2.6 the requirement that S is an order cannot be omitted.

Our next result is taken from [3, Theorem 3.1] and [1, Corollary 4.4, Proposition 5.3]. Here \mathcal{N} denotes the least semilattice congruence on a semigroup S .

Result 2.7. *Let $S = \mathcal{S}(S)$ be a commutative order. Then:*

(i) *S is an order in a semilattice Y of groups $G_\alpha, \alpha \in Y$, if and only if S is a semilattice Y of cancellative semigroups $S_\alpha, \alpha \in Y$;*

(ii) *S is an order in Q where $\leq_{\mathcal{H}^*} \downarrow_S = \leq_{\mathcal{H}^*}$, so that \mathcal{H}^* is a semilattice congruence on S with cancellative classes;*

(iii) *$\mathcal{Q}(S)$ forms a complete lattice, isomorphic to the dual of the interval $[\mathcal{N}, \mathcal{H}^*]$ in the lattice of congruences of S . The isomorphism is given by*

$$\rho \longleftrightarrow \mathcal{H}^Q|_S,$$

where Q is a representative of its equivalence class.

Part (iii) of the above is achieved from the following.

Result 2.8. [1, Theorem 5.1] *Let S be a commutative semigroup and an order in semigroups Q_1 and Q_2 . The following conditions are equivalent:*

(i) $Q_2 \leq Q_1$;

(ii) for all $a, b \in S$,

$$a \leq_{\mathcal{H}} b \text{ in } Q_1 \text{ implies that } a \leq_{\mathcal{H}} b \text{ in } Q_2;$$

(iii) for all $a, b \in S$,

$$a \mathcal{H} b \text{ in } Q_1 \text{ implies that } a \mathcal{H} b \text{ in } Q_2.$$

We would like to say that every commutative order S has a maximum and a minimum semigroup of quotients. Unfortunately, this is not the case [1, Section 7]. One of our reasons in introducing ‘generalised’ semigroups of quotients is that in Section 6 we show that for an arbitrary commutative order S , we can find a semigroup Q that is a ‘generalised’ semigroup of quotients of S , and is such that every semigroup of quotients of S is an image of Q in a natural way.

The existence and behaviour of quotient semigroups of a commutative S is closely tied to that of pre-orders on S , as is already apparent from the last claim of Result 2.6. Let \preceq be a compatible pre-order on S . We recall from [1] some conditions on \preceq that are crucial in determining quotient semigroups of S . We supplement this list with a related condition that will be required later.

(A) For all $b, c \in S$, we have $bc \preceq b$.

(B) For all $b, c \in S$ and $a \in \mathcal{S}(S)$, if

$$b \preceq a, c \preceq a, \text{ and } ab = ac,$$

then

$$b = c \preceq ab.$$

Conditions (A) and (B) restricted to $\mathcal{S}(S)$ clearly imply that the semigroup $\mathcal{S}(S)$ is separative.

(C) For all $b \in S$ there exists $x \in \mathcal{S}(S)$ with $b \preceq x$.

(C') For all $b, c \in S$, $b \preceq c$ implies that $bx = cy$ for some $x \in \mathcal{S}(S), y \in S$ with $b \preceq x$.

(C'') For all $b, c \in S$, $b \preceq c$ implies that $bx = cy$ for some $x, y \in S^1$ such that if $x \in S$, then $x \in \mathcal{S}(S)$ and $b \preceq x$.

The motivation for introducing conditions of the kind above is made clear by the next result.

Theorem 2.9. [1, Theorem 4.3] *Let S be a commutative semigroup and let \preceq be a relation on S . Then S is an order in a semigroup Q such that $\leq_{\mathcal{H}Q|_S} = \preceq$ if and only if \preceq is a compatible pre-order on S satisfying Conditions (A), (B) and (C').*

With the above result in mind we introduce some terminology. We say that a compatible pre-order \preceq on a commutative semigroup S is a *quotient pre-order* or *q-pre-order* if it satisfies Conditions (A), (B) and (C'). We normally denote the associated congruence \equiv_{\preceq} on S by \mathcal{H}' . The restriction of a q-pre-order and its associated congruence to $\mathcal{S}(S)$ will be normally denoted by \leq and \mathcal{H}'' and we will refer to these as being *induced* by \preceq and \mathcal{H}' . Before continuing we make some technical observations concerning Conditions (A) and (B).

Lemma 2.10. *Let \preceq be a compatible pre-order on a commutative semigroup S satisfying (A) and (B), and let \equiv_{\preceq} be the associated congruence. Let $a \in \mathcal{S}(S)$ and $b \in S$. Then the following conditions are equivalent:*

- (i) $b \preceq a$;
- (ii) $ba \equiv_{\preceq} b$;
- (iii) $ca \equiv_{\preceq} b$ for some $c \in S$.

Proof. (i) \Rightarrow (ii) We remark that by (A), $ba \preceq b$. If $b \preceq a$, then with $b = c$ in (B), we have $b \preceq ba$ so that $b \equiv_{\preceq} ba$ as required.

(ii) \Rightarrow (iii) Clear.

(iii) \Rightarrow (i) We have $b \equiv_{\preceq} ca \preceq a$ by (A), so that $b \preceq a$. □

Lemma 2.11. *Let \preceq be a compatible pre-order on a commutative semigroup S satisfying (A) and (B) and let \equiv_{\preceq} be the associated congruence. Let $u \in S$. If there exist $a_1, \dots, a_n \in \mathcal{S}(S)$ and $v_1, \dots, v_n \in S$ with $u \equiv_{\preceq} a_i v_i, 1 \leq i \leq n$, then $u \preceq a_1 \dots a_n$.*

Proof. We proceed by induction. Clearly the result is true if $n = 1$, by Lemma 2.10.

Suppose now that $n > 1$ and the result is true for $n-1$. Then $u \preceq a_1 \dots a_{n-1}$. By Lemma 2.10,

$$u \equiv_{\preceq} v_n a_n \equiv_{\preceq} u a_n \preceq a_1 \dots a_{n-1} a_n,$$

so that the result follows by induction. □

Our aim in Section 4 is to show how Theorem 2.9 can be obtained with a rather different construction to that in [1]. In fact, we need only the characterisation of orders in commutative Clifford semigroups and a particular use of

tensor products, which forms the major construction of this paper, to produce all semigroups of quotients of a given commutative order.

Let T and U be semigroups. We say that T is a U -semigroup if there is a homomorphism $\phi : U \rightarrow T$. The extension of ϕ to $U^1 \rightarrow T^1$ defines an action of U^1 on T given by $ut = (u\phi)t$. Throughout this article, U is a subsemigroup of T and ϕ is inclusion. If V is also a U -semigroup then we can form the *tensor product* $T \otimes_{U^1} V$, which for convenience we abbreviate as $T \otimes_U V$. Specifically, this is the set $T \times V$ factored by the equivalence relation \mathcal{T} generated by

$$\{((tu), v), (t, uv) : t \in T, u \in U^1, v \in V\}.$$

We write $t \otimes v$ for the \mathcal{T} -equivalence class of (t, v) . Note that for elements $(p, s), (p', s') \in T \times V$ we have that $p \otimes s = p' \otimes s'$ if and only if there exists a system of equations

$$(1) \quad \begin{array}{rcl} & & s = s_1 b_1 \\ ps_1 & = & a_2 t_1 \quad t_1 b_1 = s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 \\ & \vdots & \vdots \\ a_{m-1} s_{m-1} & = & a_m t_{m-1} \quad t_{m-1} b_{m-1} = s_m b_m \\ a_m s_m & = & p' t_m \quad t_m b_m = s' \end{array}$$

for some $s_1, t_1, \dots, s_m, t_m \in U^1, a_2, \dots, a_m \in T$ and $b_1, \dots, b_m \in V$.

The tensor product $T \otimes_U V$ comes with a tensor map $\tau : T \times V \rightarrow T \otimes_U V$ given by $(p, s)\tau = p \otimes s$. The map τ is *balanced*, that is, $(pu, s)\tau = (p, us)\tau$ for all $(p, s) \in T \times V$ and $u \in U^1$. Conversely, it is clear that every balanced mapping $\phi : T \times V \rightarrow X$ factors uniquely through τ , that is, there is a mapping $\psi : T \otimes_U V \rightarrow X$ that is unique with respect to $\tau\psi = \phi$.

If T and V are commutative semigroups, then so is the direct product $T \times V$ and $T \otimes_U V$, and in the above, τ, ϕ and ψ are homomorphisms.

Lemma 2.12. *Let T, V be commutative U -semigroups. Then $T \otimes_U V$ is a commutative semigroup under*

$$(p \otimes s)(q \otimes t) = pq \otimes st.$$

Clearly $\tau : T \times V \rightarrow T \otimes_U V$ is a homomorphism. If X is a commutative semigroup and $\phi : T \times V \rightarrow X$ is a balanced homomorphism, then the unique map $\psi : T \otimes_U V \rightarrow X$ such that $\tau\psi = \phi$ is a homomorphism.

Proof. It is clear that the set of generators of \mathcal{T} is compatible, hence so is \mathcal{T} , giving that \mathcal{T} is a congruence. Thus $T \otimes_U V$ is a commutative semigroup as in the statement, and τ is the natural homomorphism. Given that ϕ is a homomorphism, it follows from standard algebraic arguments that so also is ψ . \square

Lemma 2.13. *Suppose that T_1, V_1, T_2, V_2 are commutative U -semigroups, and there are U -homomorphisms $\phi : T_1 \rightarrow T_2$ and $\psi : V_1 \rightarrow V_2$, that is, for all $u \in U, t_i \in T_i, (ut_1)\phi = u(t_1\phi)$ and $(ut_2)\psi = u(t_2\psi)$. Then $\phi \otimes \psi : T_1 \otimes_U V_1 \rightarrow$*

$T_2 \otimes_U V_2$ given by $(p \otimes s)(\phi \otimes \psi) = p\phi \otimes s\psi$ is a homomorphism. Further, if ϕ and ψ are onto, then so is $\phi \otimes \psi$.

Proof. The map $T_1 \times V_1 \rightarrow T_2 \otimes_U V_2$ given by $(p, s) \mapsto p\phi \otimes s\psi$ is a balanced homomorphism. Now call upon Lemma 2.12. \square

Example 2.14. We briefly consider the special case of a commutative cancellative semigroup S . Certainly $\mathcal{H}^* = S \times S$, $S = \mathcal{S}(S)$ and S is an order (in, for example, a group). From Result 2.7 we know that $\mathcal{Q}(S)$ is a lattice and is isomorphic to the dual of the interval $[\mathcal{N}, S \times S]$ in the lattice of congruences on S , and hence therefore to the dual of the lattice of semilattice congruences on S .

Put $Y = S/\mathcal{N}$, so that $\mathcal{Q}(S)$ is therefore isomorphic to the dual of the lattice of congruences on Y . Moreover, S has a greatest semigroup of quotients Q , where Q is a semilattice Y of groups G_α , $\alpha \in Y$, and S is a semilattice Y of orders S_α in G_α . Let e_α denote the identity of G_α , $\alpha \in Y$. By definition of the ordering on $\mathcal{Q}(S)$, it is clear that $\mathcal{Q}(S)$ corresponds to the set of congruences on Q that restrict to the identity relation ι on S . We now show directly that these are exactly the congruences generated by sets of the form

$$C = \{(e_{\alpha_i}, e_{\beta_i}) : i \in I\}.$$

Proof. Let τ be the congruence on Q generated by a set C as above. We may assume that C is symmetric. If $u, v \in S$ and $u \tau v$, then $u = v$ or there is a sequence

$$u = e_{\alpha_{i_1}} q_1, e_{\beta_{i_1}} q_1 = e_{\alpha_{i_2}} q_2, \dots, e_{\beta_{i_n}} q_n = v$$

where $n \in \mathbb{N}$, $q_1, \dots, q_n \in Q$ and $i_1, \dots, i_n \in I$. With $e = e_{\alpha_{i_1}} e_{\beta_{i_1}} \dots e_{\alpha_{i_n}} e_{\beta_{i_n}}$ we have $eu = ev$. If $e \in G_\gamma$, then choosing $c \in S_\gamma$ we have $cu = cv$ so that as S is cancellative, $u = v$. Thus $\tau \upharpoonright_S = \iota$.

Conversely, let κ be a congruence on Q that restricts to the identity on S . Suppose that $a^\#b \kappa c^\#d$, where $a, b \in S_\alpha$ and $c, d \in S_\beta$. Then $cb \kappa ad$ so that $cb = ad$ by assumption. Moreover, $e_\alpha = (a^\#b)^\#(a^\#b) \kappa (a^\#b)^\#(c^\#d) = (bc)^\#ad = e_{\alpha\beta}$ and similarly, $e_\beta \kappa e_{\alpha\beta}$. Also from $cb = ad$ we have $e_{\alpha\beta}cb = e_{\alpha\beta}ad$ so that $e_{\alpha\beta}a^\#b = e_{\alpha\beta}c^\#d$. Put $\kappa_{\alpha,\beta} = \langle (e_\alpha, e_{\alpha\beta}), (e_\beta, e_{\alpha\beta}) \rangle$. Then

$$a^\#b = a^\#be_\alpha \kappa_{\alpha,\beta} a^\#be_{\alpha\beta} = c^\#de_{\alpha\beta} \kappa_{\alpha,\beta} c^\#de_\beta = c^\#d.$$

The result follows. \square

We end this section with an example which demonstrates the complexities that can arise for commutative orders, even when $S = \mathcal{S}(S)$.

Example 2.15. Consider the multiplicative semigroup of natural numbers \mathbb{N} and denote by \mathbb{P} the set of prime numbers. Since \mathbb{N} is cancellative, we have $\mathbb{N} = \mathcal{S}(\mathbb{N})$. Let Θ be the smallest semilattice congruence on \mathbb{N} , then the Θ -classes are in a 1-1 correspondence with the finite subsets of \mathbb{P} and \mathbb{N}/Θ is the free semilattice monoid (free semilattice with an identity adjoined) F_ω on countably many generators. Each Θ -class is uniquely determined by the smallest square-free number n_Θ in it, and thus by the set X of prime factors of

n_Θ ; we may therefore write $n_\Theta = n_X$. For each non-empty finite subset X of \mathbb{P} , let \mathbb{N}_X be the Θ -class containing n_X , and let G_X be the group of quotients of \mathbb{N}_X , with identity element e_X . Clearly, \mathbb{N}_X is a free semigroup and G_X a free abelian group of rank $|X|$.

Let $\mathcal{P}_f(\mathbb{P})$ denote the set of finite subsets of \mathbb{P} and let $Q = \bigcup_{X \in \mathcal{P}_f(\mathbb{P})} G_X$, where we take the union to be disjoint. The multiplication in Q works as follows. For $q_1, q_2 \in Q$ with $q_1 \in G_{X_1}, q_2 \in G_{X_2}$ we have $q_1 = a^\sharp b, q_2 = c^\sharp d$ for some $a, b \in \mathbb{N}_{X_1}, c, d \in \mathbb{N}_{X_2}$. Then $ac, bd \in \mathbb{N}_{X_1 \cup X_2}$, and we put

$$q_1 q_2 = (ac)^\sharp bd \in G_{X_1 \cup X_2}.$$

Clearly, Q is the greatest element of $\mathcal{Q}(\mathbb{N})$.

Since \mathbb{N} is cancellative, \mathcal{H}^* is universal. It follows that the lattice $\mathcal{Q}(\mathbb{N})$ is isomorphic to the dual of the lattice of semilattice congruences on \mathbb{N} and hence to the dual of the lattice of congruences on F_ω . The latter is a vast lattice known to satisfy no lattice identity [2].

3. CHARACTERISATION BY IDEALS

The aim of this section is to give a new description of commutative orders, in terms of ideal decompositions.

Theorem 3.1. *Let S be a commutative semigroup. Then S is an order in a semigroup Q such that for each $e \in E = E(Q)$ we have*

$$C_e = \mathcal{S}(S) \cap H_e^Q \text{ and } I_e = S \cap eQ$$

if and only if S has a set $\{C_e, I_e : e \in E\}$ of subsets such that:

- (1) $\mathcal{S}(S)$ is a semilattice E of subsemigroups $C_e, e \in E$ and $S = \bigcup_{e \in E} I_e$; and for any $e, f \in E$
- (2) I_e is an ideal of S with $C_e \subseteq I_e$ and $I_e \cap I_f = I_{ef}$;
- (3) if $C_e \cap I_f \neq \emptyset$, then $C_e \subseteq I_f$ and $e \leq f$ in E ;
- (4) if $x \in I_e, a \in C_e$ and $xa \in I_f$, then $x \in I_f$;
- (5) if $a \in C_e$ and $x, y \in I_e$ with $ax = ay$, then $x = y$.

Proof. Suppose that S is an order in a semigroup Q and put $E = E(Q)$. For each $e \in E$ define

$$C_e = \mathcal{S}(S) \cap H_e^Q \text{ and } I_e = S \cap eQ.$$

It is straightforward to see that (1)–(5) hold, but to this end we note that for $e, f \in E$ we have

$$I_e \cap I_f = \{s \in S : es = s = fs\} = \{s \in S : efs = s\} = I_{ef}.$$

If $x \in I_e$ and $a \in C_e$, then $ex = x$ and $x \mathcal{H} ax$, so that if $ax \in I_f$, then $x \in I_f$. Moreover, if also $y \in I_e$ and $ax = ay$, then $x = ex = a^\sharp ax = a^\sharp ay = ey = y$.

Now suppose that S has a set of subsets $\{C_e, I_e : e \in E\}$ such that Conditions (1)–(5) hold.

Define a relation \preceq on S by the rule that

$$x \preceq y \Leftrightarrow \exists e \in E, a \in C_e, z \in S \text{ with } x \in I_e \text{ and } ax = zy.$$

Note immediately that if $x \in I_e$ and $a \in C_e$ we have $x \preceq a$ and $x \preceq x$. Moreover, it is clear that (C') holds.

Suppose now that $x, y, z \in S$ with $x \preceq y \preceq z$. Then $\exists e, f \in E$ with $x \in I_e, y \in I_f$ and $a \in C_e, b \in C_f$ with

$$ax = dy, by = hz$$

for some $d, h \in S$. It follows that

$$bax = bdy = dhz$$

and $ba \in C_{ef}$ by (1). From $ax = dy \in I_f$ and (4), we have $x \in I_f$ so that $x \in I_e \cap I_f = I_{ef}$. Hence $x \preceq z$ and \preceq is a pre-order which is clearly compatible with multiplication.

Let $x, y \in S$, say $xy \in I_e$; choosing $a \in C_e$ the triviality $a(xy) = (ax)y$ gives that $xy \preceq y$ so that (A) holds. For (B), we note that if $x \preceq a$ where $x \in I_e$ and $a \in C_f$, then $bx = ay$ for some $b \in C_e$ and $y \in S$. Then $bx \in I_f$ so that by (4), $x \in I_f$. Now $a^2x = a(ax)$ tells us that $x \preceq ax$. The remaining part of (B) follows immediately from (5).

From Theorem 2.9 we have that S is an order in a commutative semigroup Q such that $\leq_{\mathcal{H}Q|_S} = \preceq$.

Let $a, b \in \mathcal{S}(S)$ with $a \in C_e$ and $b \in C_f$. If $e \leq f$ then as $I_e = I_{ef} \subseteq I_f$ we have $a \preceq b$. On the other hand, if $a \preceq b$, then it follows as above that $a \in I_f$ so that $e \leq f$ by (3). We may therefore assume that E is the semilattice of idempotents of Q and for each $e \in E$ we have $C_e = S \cap H_e^Q$. Choose $a \in C_e$. Then for any $x \in S$ we have that

$$x \in eQ \Leftrightarrow x \leq_{\mathcal{H}Q} e \Leftrightarrow x \leq_{\mathcal{H}Q} a \Leftrightarrow x \preceq a.$$

Now if $x \preceq a$ then we have seen that $x \in I_e$. On the other hand, if $x \in I_e$ then we noted earlier that $x \preceq a$. It follows that $eQ \cap S = I_e$. □

4. GENERALISED QUOTIENTS

For later purposes we introduce and make some comments concerning a generalisation of the notion of order. For a subset X of a semigroup Q , we denote by $\langle X \rangle$ the subsemigroup of Q generated by X .

Definition 4.1. Let S be a subsemigroup of a semigroup Q . Then Q is a *generalised quotient semigroup* of S and S is a *generalised order* in Q if every square-cancellable element of S lies in a subgroup of Q and

$$Q = \langle S \cup \{a^\sharp : a \in \mathcal{S}(S)\} \rangle.$$

It is clear that semigroups of quotients are generalised quotient semigroups. We will see that the notions almost coincide when S is commutative. The methods in the lemma below are similar to those in [1, Theorem 3.1], but we give a proof for completeness.

Lemma 4.2. *If S is a commutative subsemigroup of Q and Q is generated by $S \cup \{a^\sharp : a \in S \cap \mathcal{H}(Q)\}$, then Q is commutative.*

Proof. Let $a, b \in S$ and suppose that $a^\#$ exists. Then

$$a^\#b = (a^\#)^2ab = (a^\#)^2ba = (a^\#)^2ba^3(a^\#)^2 = (a^\#)^2a^3b(a^\#)^2 = ab(a^\#)^2 = ba(a^\#)^2 = ba^\#.$$

If in addition we have that $c \in S$ and $c^\#$ exists, then a similar calculation, making use of the above, gives that

$$\begin{aligned} a^\#c^\# &= (a^\#)^2ac^\# = (a^\#)^2c^\#a = (a^\#)^2c^\#a^3(a^\#)^2 = \\ &= (a^\#)^2a^3c^\#(a^\#)^2 = ac^\#(a^\#)^2 = c^\#a(a^\#)^2 = c^\#a^\#. \end{aligned}$$

□

The following corollary now follows easily from Lemmas 2.3 and 4.2.

Corollary 4.3. *Let S be commutative and a subsemigroup of Q such that every square-cancellable element of S lies in a subgroup of Q . Then Q is a semigroup of generalised quotients of S if and only if for any $q \in Q$, either $q \in S$ or $q = a^\#b$ for some $a, b \in S$.*

If S is commutative, and is a monoid or $S = \mathcal{S}(S)$, then we get nothing new by moving to *generalised* quotients.

Lemma 4.4. *Let S be a commutative monoid. Then*

- (i) *S is a generalised order in Q if and only if S is an order in Q ;*
- (ii) *if S is an order in Q , then Q is a monoid.*

Proof. (i) If S is a generalised order in Q and $s \in S$, then $s = 1^\#s$.

(ii) Let $a^\#b \in Q$, where $a, b \in S$. Then $(a^\#b)1 = a^\#b1 = a^\#b$ and $1(a^\#b) = 1a(a^\#)^2b = a(a^\#)^2b = a^\#b$.

□

Lemma 4.5. *Let S be commutative with $S = \mathcal{S}(S)$. Then S is a generalised order in Q if and only if S is an order in Q ;*

Proof. Suppose that S is a generalised order in Q . If $s \in S$, then as $s \in \mathcal{S}(S) \subseteq \mathcal{H}(Q)$ we have $s = s^\#s^2$. □

In the commutative case, we can answer the question of whether a semigroup Q of (generalised) quotients of S is a semigroup of (generalised) quotients of itself.

Proposition 4.6. *Let S be a commutative (generalised) order in Q . Then Q is a (generalised) order in Q .*

Proof. Clearly we need only show that $\mathcal{S}(Q) \subseteq \mathcal{H}(Q)$. Let $q \in \mathcal{S}(Q)$. If $q \in S$ then clearly $q \in \mathcal{H}(Q)$. Otherwise, $q = a^\#b$ for some $a \in \mathcal{S}(S)$ and $b \in S$. Then $q = (a^2)^\#ab$, so that we can assume $b \leq_{\mathcal{H}} a$ and so $b\mathcal{H}a^\#b$ in Q . Consequently, $b^2\mathcal{H}(a^\#b)^2$ in Q and so as $\mathcal{H} \subseteq \mathcal{H}^*$ we have that $b\mathcal{H}^*b^2$ in Q . Certainly then $b\mathcal{H}^*b^2$ in S , so that b lies in a subgroup of Q . Using Lemma 2.3 we conclude $q = a^\#b \in \mathcal{H}(Q)$. □

5. A CONSTRUCTION

Let S be a commutative semigroup and let \preceq be a q-pre-order, that is, a compatible pre-order satisfying Conditions (A), (B) and (C'). By Theorem 2.9, S is an order in a commutative semigroup Q such that $\leq_{\mathcal{H}^Q}|_S = \preceq$ and hence from Lemma 2.3, $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q)$. It is this latter fact that we need, that can be shown independently of the main construction of [1].

Lemma 5.1. *Let S be a commutative semigroup and let \preceq be a q-pre-order on S . Putting*

$$\leq = \preceq|_{\mathcal{S}(S)}$$

we have that $\mathcal{S}(S)$ is an order in a Clifford semigroup R such that

$$\leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} = \leq.$$

Moreover, with $\mathcal{H}' = \equiv_{\preceq}$ and $\mathcal{H}'' = \equiv_{\leq}$ being the equivalence relations associated with \preceq and \leq , respectively, we have that

$$\mathcal{H}'' = \mathcal{H}'|_{\mathcal{S}(S)} = \mathcal{H}^R|_{\mathcal{S}(S)}.$$

Proof. It is clear from the definitions that $\mathcal{H}'' = \mathcal{H}'|_{\mathcal{S}(S)}$. Suppose that $a \in \mathcal{S}(S)$; by Lemma 2.10 we have that $a\mathcal{H}''a^2$ so that \mathcal{H}'' is a semilattice congruence on $\mathcal{S}(S)$. Writing H''_u for the \mathcal{H}'' -class of $u \in \mathcal{S}(S)$, and again using Lemma 2.10, we have that for any $a, b \in \mathcal{S}(S)$,

$$\begin{aligned} a \leq b &\Leftrightarrow a \preceq b \\ &\Leftrightarrow ab\mathcal{H}'a \\ &\Leftrightarrow ab\mathcal{H}''a \\ &\Leftrightarrow H''_a \leq H''_b \text{ in the semilattice } \mathcal{S}(S)/\mathcal{H}'' . \end{aligned}$$

Consider an \mathcal{H}'' -class H'' . Clearly (B) gives that H'' is cancellative and it is right reversible, as it is commutative. By Result 2.7, we have that $\mathcal{S}(S)$ is an order in R , where R is a semilattice $\mathcal{S}(S)/\mathcal{H}''$ of commutative groups. It follows that $\mathcal{H}'' = \mathcal{H}^R|_{\mathcal{S}(S)}$. Moreover, for any $a, b \in \mathcal{S}(S)$ we have

$$a \leq_{\mathcal{H}^R} b \Leftrightarrow ab\mathcal{H}^Rb \Leftrightarrow ab\mathcal{H}''a \Leftrightarrow a \leq b,$$

so that $\leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} = \leq$. □

The following lemma is clear, and certainly applies to the foregoing relations \mathcal{H}'' and \mathcal{H}' .

Lemma 5.2. *Let S be a commutative semigroup and let ρ be a congruence on $\mathcal{S}(S)$. Let $\bar{\rho}$ be the congruence on S generated by ρ . If ρ is the restriction to $\mathcal{S}(S)$ of a congruence on S , then $\bar{\rho}|_{\mathcal{S}(S)} = \rho$.*

Suppose now that S is a commutative semigroup and $\mathcal{S}(S)$ is an order in a (commutative Clifford) semigroup R . We are *not* assuming here that S is an order. From (vii) of Result 2.6 we have that $\mathcal{S}(S)$ is straight in R , that is, if $q \in R$ then $q = x\#y$ where $x, y \in \mathcal{S}(S)$ and $x\mathcal{H}^Ry$. We let

$$\leq = \leq_{\mathcal{H}^R}|_{\mathcal{S}(S)} \text{ and } \rho = \equiv_{\leq} = \mathcal{H}^R|_{\mathcal{S}(S)}.$$

From Lemma 2.12, $Q = R \otimes_{\mathcal{S}(S)} S$ is a commutative semigroup in which $(p \otimes s)(q \otimes t) = pq \otimes st$.

The next lemma is phrased in such a way that we can maximise its implications.

Lemma 5.3. *Let S, R and Q be as above and let $\bar{\rho}$ be the congruence on S generated by ρ . Suppose that*

$$p^\#q \otimes s = x^\#y \otimes t$$

where $p, q, x, y \in \mathcal{S}(S)$, $s, t \in S$, $p \rho q$ and $x \rho y$. Then

- (i) $qs \bar{\rho} yt$;
- (ii) if $\rho \subseteq \mathcal{H}''$, where \mathcal{H}'' is a congruence on $\mathcal{S}(S)$ induced by a q -pre-order on S , then

$$qs \mathcal{H}' yt \text{ and } xqs = pyt;$$

- (iii) if $\rho = \bar{\rho}|_{\mathcal{S}(S)}$, and $s, t \in \mathcal{S}(S)$, then

$$qs \rho yt \text{ and } xqs = pyt.$$

Proof. (i) We have a system of equalities

$$(2) \quad \begin{array}{rcc} & & s = s_1 b_1 \\ p^\#q s_1 & = & a_2 t_1 \quad t_1 b_1 = s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 \\ & \vdots & \vdots \\ a_{m-1} s_{m-1} & = & a_m t_{m-1} \quad t_{m-1} b_{m-1} = s_m b_m \\ a_m s_m & = & x^\#y t_m \quad t_m b_m = t \end{array}$$

for some $s_1, t_1, \dots, s_m, t_m \in \mathcal{S}(S)^1$, $a_2, \dots, a_m \in R$ and $b_1, \dots, b_m \in S$.

Since $\mathcal{S}(S)$ is a straight left order in R , we have that $a_i = c_i^\# d_i$ for some $c_i, d_i \in \mathcal{S}(S)$ with $c_i \rho d_i$, $2 \leq i \leq m$. Let $w = pc_2 \dots c_m x \in \mathcal{S}(S)$. Then, multiplying each of the equations in the left hand column of (2) by w , we have

$$\begin{array}{rcc} c_2 \dots c_m x q s_1 & = & p c_3 \dots c_m x d_2 t_1 \quad \text{as } p^\#p q = q \text{ and } c_2^\# c_2 d_2 = d_2 \\ p c_3 \dots c_m x d_2 s_2 & = & p c_2 c_4 \dots c_m x d_3 t_2 \quad \text{as } c_2^\# c_2 d_2 = d_2 \text{ and } c_3^\# c_3 d_3 = d_3 \\ & \vdots & \\ p c_2 \dots c_{m-1} x d_m s_m & = & p c_2 \dots c_m y t_m \quad \text{as } c_m^\# c_m d_m = d_m \text{ and } x^\# x y = y. \end{array}$$

This gives us that

$$\begin{array}{l} c_2 \dots c_m x q s \\ = c_2 \dots c_m x q s_1 b_1 \\ = p c_3 \dots c_m x d_2 t_1 b_1 \\ = p c_3 \dots c_m x d_2 s_2 b_2 \\ = p c_2 c_4 \dots c_m x d_3 t_2 b_2 \\ \vdots \\ = p c_2 \dots c_{m-1} x d_m t_{m-1} b_{m-1} \\ = p c_2 \dots c_{m-1} x d_m s_m b_m \\ = p c_2 \dots c_m y t_m b_m \\ = p c_2 \dots c_m y t. \end{array}$$

Again using our list of equalities (2), we have that

$$qs_1 \rho c_2 t_1, c_2 s_2 \rho c_3 t_2, \dots, c_m s_m \rho y t_m,$$

so that

$$qs = qs_1 b_1 \bar{\rho} c_2 t_1 b_1 = c_2 s_2 b_2 \bar{\rho} c_3 t_2 b_2 \bar{\rho} \dots \bar{\rho} c_m t_{m-1} b_{m-1} = c_m s_m b_m \bar{\rho} y t_m b_m = yt.$$

(ii) Suppose now that $\rho \subseteq \mathcal{H}''$, where \mathcal{H}'' is a congruence on $\mathcal{S}(S)$ induced by a q-pre-order \preceq on S . Then $\rho \subseteq \bar{\rho} \subseteq \langle \mathcal{H}'' \rangle \subseteq \mathcal{H}'$.

From (i) we certainly have $qs \mathcal{H}' yt$. Hence $pqxs \mathcal{H}' pxyt$ and so $xqs \mathcal{H}' pyt$. From Lemma 2.11 we have that $qs \preceq c_2 \dots c_m \in \mathcal{S}(S)$. Now from $c_2 \dots c_m xqs = pc_2 \dots c_m yt$, Conditions (A) and (B) give that $xqs = pyt$.

(iii) Suppose now that $\rho = \bar{\rho}|_{\mathcal{S}(S)}$, and $s, t \in \mathcal{S}(S)$. We certainly have that $qs \rho yt$. We have observed that for any $i \in \{2, \dots, m\}$, we have that $qs \bar{\rho} c_i w$ for some $w \in S$. Now $c_i \rho c_i^2$, so that

$$c_i qs \bar{\rho} c_i^2 w \bar{\rho} c_i w \bar{\rho} qs,$$

and $qs \rho c_2 \dots c_m qs$. With a familiar argument we see that $xqs \rho pyt$ and so from $c_2 \dots c_m xqs = pc_2 \dots c_m yt$ and the fact that $\mathcal{S}(S)$ is an order in the Clifford semigroup R , we deduce that $xqs = pyt$. \square

Lemma 5.4. *With notation as above, the map $\theta : R \rightarrow Q$ given by*

$$(a^\#b)\theta = a^\# \otimes b$$

where $a, b \in \mathcal{S}(S)$ and $a \rho b$, is a well-defined homomorphism.

Further, θ is an embedding if and only if $\bar{\rho}|_{\mathcal{S}(S)} = \rho$.

Proof. Suppose that $a^\#b = c^\#d$ where $a, b, c, d \in \mathcal{S}(S)$, $a \rho b$ and $c \rho d$. Then a, b, c, d are all ρ -related and lie in the same subgroup of R . We then calculate that $cb = ad$ and

$$a^\# \otimes b = a^\# c^\# c \otimes b = a^\# c^\# \otimes cb = a^\# c^\# \otimes ad = a^\# c^\# a \otimes d = c^\# \otimes d,$$

so that θ is well defined.

To see that θ is a homomorphism, again let $a^\#b, c^\#d \in R$ where $a, b, c, d \in \mathcal{S}(S)$, $a \rho b$ and $c \rho d$, so that $ac \rho bd$. Using the fact that in R we have $(uv)^\# = u^\#v^\#$, we see that

$$((a^\#b)(c^\#d))\theta = ((ac)^\#bd)\theta = (ac)^\# \otimes bd = a^\#c^\# \otimes bd = (a^\# \otimes b)(c^\# \otimes d) = (a^\#b)\theta(c^\#d)\theta,$$

so that θ is a homomorphism.

We remark that for any $u^\#v \in R$, where $u, v \in \mathcal{S}(S)$, we have that

$$\begin{aligned} (u^\#v)\theta &= ((u^2v)^\#uv^2)\theta = (u^2v)^\# \otimes uv^2 = (u^2)^\#v^\#v^2 \otimes u = \\ &= (u^2)^\#v \otimes u = (u^2)^\# \otimes vu = (u^2)^\#u \otimes v = u^\# \otimes v. \end{aligned}$$

Suppose now that $\bar{\rho}|_{\mathcal{S}(S)} = \rho$. Again let $a^\#b, c^\#d \in R$ where $a, b, c, d \in \mathcal{S}(S)$, $a \rho b$ and $c \rho d$ and suppose that $(a^\#b)\theta = (c^\#d)\theta$. Then $a^\# \otimes b = c^\# \otimes d$ and, re-writing to fit in with the notation of Lemma 5.3, we have that $(a^2)^\#a \otimes b = (c^2)^\#c \otimes d$. From (iii) of Lemma 5.3, we have that $ab \rho cd$ and $c^2ab = a^2cd$.

Since a, b, c, d all lie in the same subgroup of R , we see that $a^\sharp b = c^\sharp d$ so that θ is an embedding.

Finally, let us assume that θ is an embedding and $u, v \in \mathcal{S}(S)$ are such that $u \bar{\rho} v$. If $u = v$, then certainly $u \rho v$. Otherwise, there exists $n \in \mathbb{N}$ and elements $c_1, \dots, c_n \in S^1$ and $(x_1, y_1), \dots, (x_n, y_n) \in \rho$ such that

$$u = x_1 c_1, y_1 c_1 = x_2 c_2, \dots, y_n c_n = v.$$

We have

$$\begin{aligned} u\theta &= (u^\sharp u^2)\theta \\ &= u^\sharp \otimes u^2 \\ &= u^\sharp \otimes x_1 c_1 u \\ &= u^\sharp x_1 \otimes c_1 u \\ &= u^\sharp x_1 y_1^\sharp y_1 \otimes c_1 u \\ &= u^\sharp x_1 y_1^\sharp \otimes y_1 c_1 u \\ &= u^\sharp x_1 y_1^\sharp \otimes x_2 c_2 u \\ &\quad \vdots \\ &= u^\sharp x_1 y_1^\sharp \dots x_n y_n^\sharp \otimes y_n c_n u \\ &= u^\sharp x_1 y_1^\sharp \dots x_n y_n^\sharp \otimes v u \\ &= (u y_1 \dots y_n)^\sharp \otimes x_1 \dots x_n v u \\ &= ((u y_1 \dots y_n)^\sharp x_1 \dots x_n v u)\theta. \end{aligned}$$

Since θ is an embedding, we deduce that

$$u = (u y_1 \dots y_n)^\sharp x_1 \dots x_n v u$$

and hence that $u \leq v$. Together with the dual we have that $u \rho v$ as required. \square

We now apply Lemmas 5.2 and 5.4.

Corollary 5.5. *With notation as above, suppose that ρ is induced by a q -pre-order on S . Then R embeds into Q .*

In view of Lemma 2.13 the following is clear.

Corollary 5.6. *Let S be a commutative semigroup and let R_1 and R_2 be semigroups of quotients of $\mathcal{S}(S)$, and suppose there is an S -homomorphism from R_1 to R_2 . Then $p^\sharp q \otimes s \mapsto p^* q \otimes s$ is a homomorphism from $R_1 \otimes_{\mathcal{S}(S)} S$ onto $R_2 \otimes_{\mathcal{S}(S)} S$, where for clarity we write the inverse of $p \in \mathcal{S}(S)$ in R_2 as p^* .*

We will now suppose that our commutative semigroup S is an order, which is a stronger statement than saying that $\mathcal{S}(S)$ is an order. Again, our next result is phrased in such a way that we maximise its usage.

Theorem 5.7. *Let S be a commutative order in a semigroup W , such that W induces \preceq and \mathcal{H}' on S and \leq and \mathcal{H}'' on $\mathcal{S}(S)$. Let ρ be any semilattice congruence on $\mathcal{S}(S)$ such that $\rho \subseteq \mathcal{H}''$, and let R be a semigroup of quotients of $\mathcal{S}(S)$ inducing ρ . Then $\psi : R \otimes_{\mathcal{S}(S)} S \rightarrow W$ given by $(a^\sharp b \otimes s)\psi = a^* b s$, where*

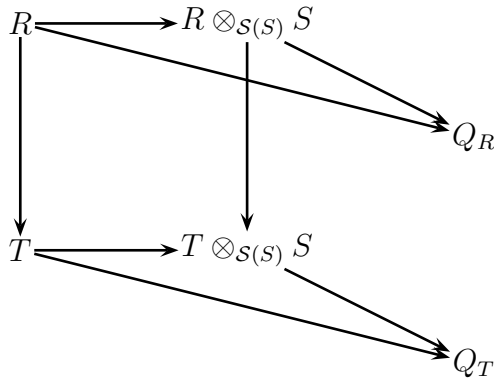
$a, b \in \mathcal{S}(S)$, $a \rho b$ and a^* denotes the group inverse of a in W , is a well-defined onto homomorphism.

Proof. From Result 2.8, there is an S -homomorphism from R to $\mathcal{H}(W)$, which must be given by $a^\#b \mapsto a^*b$. Now from Corollary 5.6, we have that there is a homomorphism from $R \otimes_{\mathcal{S}(S)} S \rightarrow \mathcal{H}(W) \otimes_{\mathcal{S}(S)} S$ given by $a^\#b \otimes s \mapsto a^*b \otimes s$. Clearly the map from $\mathcal{H}(W) \otimes_{\mathcal{S}(S)} S$ to W given by $a^*b \otimes s \mapsto a^*bs$ is an onto homomorphism. \square

Corollary 5.8. *Let S be a commutative order, let ρ be the smallest semilattice congruence on $\mathcal{S}(S)$, and let R be a semigroup of quotients of $\mathcal{S}(S)$ inducing ρ . Then every semigroup of quotients of S is a morphic image of $Q = R \otimes_{\mathcal{S}(S)} S$ under $(a^\#b \otimes s)\psi = a^*bs$. Moreover, if $\theta : R \rightarrow Q$ is given by $(a^\#b)\theta = a^\# \otimes b$, then $(a^\#b)\theta\psi = a^*b$.*

Corollary 5.9. *Let S be a commutative order in W and let \preceq be the q -pre-order on S induced by W . Let $R = \mathcal{H}(W)$ be a semigroup of quotients of $\mathcal{S}(S)$ corresponding to \mathcal{H}'' . Then R embeds into $R \otimes_{\mathcal{S}(S)} S$ under $(a^\#b)\theta = a^\# \otimes b$, $\psi : R \otimes_{\mathcal{S}(S)} S \mapsto W$ given by $(a^\#b \otimes s)\psi = a^*bs$ is an onto homomorphism, where $a, b \in \mathcal{S}(S)$ with $a \mathcal{H}'' b$ and $s \in S$. Further, $(a^\#b)\theta\psi = a^*b$ and $\theta\psi$ is the identity map on R .*

The diagram below represents the relation between the various semigroups constructed. Here, S is a commutative order and R, T are semigroups of quotients of $\mathcal{S}(S)$ with R being the greatest such. The semigroups Q_R and Q_T are any quotient semigroups of S such that $\mathcal{H}(Q_R)$ ($\mathcal{H}(Q_T)$) are isomorphic to R and T , respectively. In general we cannot deduce that Q_T is an image of Q_R , since this depends upon the pre-orders induced by Q_R and Q_T on the whole of S .



We would like to say in Corollary 5.9 above that $W \cong R \otimes_{\mathcal{S}(S)} S$. However, this is not true, owing to the fact that \mathcal{H}'' on $\mathcal{H}(W)$ may be induced by *different* q -pre-orders on S and hence by *different* semigroups of quotients. We

now show how to recover each such W by factoring $R \otimes_{\mathcal{S}(S)} S$. We hence recover the constructive part of Theorem 2.9.

To get the widest applications, we again proceed in the most general way. Let \preceq be a q-pre-order on S , with associated equivalence relation \mathcal{H}' , and let \leq and \mathcal{H}'' be the restrictions of \preceq and \mathcal{H}' to $\mathcal{S}(S)$. Let R be a semigroup of quotients of $\mathcal{S}(S)$, and let \leq_ρ and ρ be the restrictions of $\leq_{\mathcal{H}}$ and \mathcal{H} in R to $\mathcal{S}(S)$. Suppose that $\leq_\rho \subseteq \leq$, so that $\rho \subseteq \mathcal{H}''$.

Let $Q = R \otimes_{\mathcal{S}(S)} S$ and put

$$\overline{Q} = Q / \overline{K}$$

where \overline{K} is the congruence generated by

$$K = \{(uu^\# \otimes s, vv^\# \otimes s) : u, v \in \mathcal{S}(S), s \preceq u, v\}.$$

By the standard construction of a semigroup congruence from a symmetric set of generators, \overline{K} is the reflexive transitive closure of

$$\overline{K} = \{((uu^\# \otimes s)\alpha, (vv^\# \otimes s)\alpha) : (uu^\# \otimes s, vv^\# \otimes s) \in K, \alpha \in Q^1\}.$$

Denoting the \overline{K} -equivalence class of $p \otimes s \in Q$ by $[p \otimes s]$, let $\theta : S \rightarrow \overline{Q}$ be given by

$$s\theta = [uu^\# \otimes s] \text{ where } s \preceq u \in \mathcal{S}(S).$$

It is easy to see that Condition (C) may be deduced from (C'), so that $s\theta$ is defined for any $s \in S$. By definition of \overline{K} , it is clear that θ is well defined. We proceed via a series of lemmas.

Lemma 5.10. *If $[u^\#x \otimes s] = [v^\#y \otimes t]$ where $u, x, v, y \in \mathcal{S}(S), u \rho x, v \rho y$ and $s, t \in S$, then $xs \mathcal{H}' yt$.*

Proof. If

$$u^\#x \otimes s = v^\#y \otimes t,$$

then using Lemma 5.3, we have $xs \rho yt$, so that $xs \mathcal{H}' yt$ as required.

Suppose now that

$$u^\#x \otimes s = \alpha(pp^\# \otimes r), \alpha(qq^\# \otimes r) = v^\#y \otimes t,$$

where $(pp^\# \otimes r, qq^\# \otimes s) \in K$ and $\alpha \in Q^1$. We have either $\alpha = 1$ or $\alpha = h^\#k \otimes z$ for some $h, k \in \mathcal{S}(S)$ with $h \rho k$ and $z \in S$. For $\alpha = 1$, let $h = k = z = 1$ in S^1 . Then, by definition of multiplication in Q , we have in either case that

$$u^\#x \otimes s = h^\#kpp^\# \otimes zr, h^\#kqq^\# \otimes zr = v^\#y \otimes t.$$

Making use of Lemmas 2.10 and 5.3, we have

$$xs \mathcal{H}' kpzr \mathcal{H}' kqr \mathcal{H}' kqzr \mathcal{H}' yt.$$

The result now follows by transitivity. \square

Lemma 5.11. *The function θ is an embedding of S in Q .*

Proof. Suppose first that $s\theta = t\theta$, that is,

$$[uu^\# \otimes s] = [vv^\# \otimes t]$$

for some $u, v \in \mathcal{S}(S)$ with $s \preceq u$ and $t \preceq v$.

By Lemmas 2.10 and 5.10, we have

$$s \mathcal{H}' us \mathcal{H}' vt \mathcal{H}' t.$$

If $u^\#u \otimes s = v^\#v \otimes t$, then from Lemma 5.3,

$$uvs = uvt$$

so that as $s, t \preceq uv$, Condition (B) gives that $s = t$.

Otherwise, since $\overline{\overline{K}}$ is the reflexive transitive closure of \overline{K} , there exist $n \in \mathbb{N}$ and for $1 \leq i \leq n$,

$$\alpha_i \in Q^1, p_i^\#p_i \otimes r_i, q_i^\#q_i \otimes r_i \in Q$$

where $p_i, q_i \in \mathcal{S}(S), r_i \in S$, with $r_i \preceq p_i, q_i$ such that

$$\begin{aligned} u^\#x \otimes s &= \alpha_1(p_1^\#p_1 \otimes r_1) \\ \alpha_1(q_1^\#q_1 \otimes r_1) &= \alpha_2(p_2^\#p_2 \otimes r_1) \\ &\vdots \\ \alpha_{n-1}(q_{n-1}^\#q_{n-1} \otimes r_{n-1}) &= \alpha_n(p_n^\#p_n \otimes r_n) \\ \alpha_n(q_n^\#q_n \otimes r_n) &= v^\#y \otimes t. \end{aligned}$$

For $1 \leq i \leq n$, either $\alpha_i = 1$ or $\alpha_i = x_i^\#y_i \otimes z_i$ for some $x_i, y_i \in \mathcal{S}(S)$ with $x_i \rho y_i$ and $z_i \in S$. For $\alpha_i = 1$, let $x_i = y_i = z_i = 1$ in S^1 . Then by definition of multiplication in Q , we have

$$\begin{aligned} u^\#x \otimes s &= (p_1x_1)^\#p_1y_1 \otimes z_1r_1 \\ (q_1x_1)^\#q_1y_1 \otimes z_1r_1 &= (p_2x_2)^\#p_2y_2 \otimes z_2r_2 \\ &\vdots \\ (q_{n-1}x_{n-1})^\#q_{n-1}y_{n-1} \otimes z_{n-1}r_{n-1} &= (p_nx_n)^\#p_ny_n \otimes z_nr_n \\ (q_nx_n)^\#q_ny_n \otimes z_nr_n &= v^\#y \otimes t. \end{aligned} \tag{3}$$

Making use of Lemmas 2.10 and 5.3 (or Lemma 5.10), we have

$$xs \mathcal{H}' p_1y_1z_1r_1 \mathcal{H}' y_1z_1r_1 \mathcal{H}' q_1y_1z_1r_1 \mathcal{H}' p_2y_2z_2r_2 \mathcal{H}' \dots \mathcal{H}' y_nz_nr_n \mathcal{H}' q_ny_nz_nr_n \mathcal{H}' yt.$$

From (3), we also have

$$\begin{aligned} p_1x_1us &= up_1y_1z_1r_1 \\ p_2x_2q_1y_1z_1r_1 &= q_1x_1p_2y_2z_2r_2 \\ &\vdots \\ p_nx_nq_{n-1}y_{n-1}z_{n-1}r_{n-1} &= q_{n-1}x_{n-1}p_ny_nz_nr_n \\ vq_ny_nz_nr_n &= q_nx_nv t. \end{aligned} \tag{4}$$

In view of Lemma 2.11, we can cancel $u, p_1, q_1, \dots, p_n, q_n$ and v from the equalities (4) to obtain

$$\begin{aligned} x_1 s &= y_1 z_1 r_1 \\ x_2 y_1 z_1 r_1 &= x_1 y_2 z_2 r_2 \\ &\vdots \\ x_n y_{n-1} z_{n-1} r_{n-1} &= x_{n-1} y_n z_n r_n \\ y_n z_n r_n &= x_n t. \end{aligned}$$

We deduce that

$$\begin{aligned} x_1 \dots x_n s &= y_1 x_2 \dots x_n z_1 r_1 = x_1 y_2 x_3 \dots x_n z_2 r_2 = \dots \\ &= x_1 \dots x_{n-1} y_n z_n r_n = x_1 \dots x_n t. \end{aligned}$$

If $x_1 \dots x_n = 1$, clearly $s = t$. Otherwise, $x_1 \dots x_n \in \mathcal{S}(S)$, and using Lemma 2.11, $s \mathcal{H}' t \preceq x_1 \dots x_n$, yielding $s = t$. Thus θ is an injection.

To see that θ is an embedding, notice that if $a\theta = [c^\# c \otimes a]$ and $b\theta = [d^\# d \otimes b]$, where $a \preceq c$ and $b \preceq d$, then $ab \preceq cd$, so that

$$a\theta b\theta = [c^\# c \otimes a][d^\# d \otimes b] = [(c^\# c \otimes a)(d^\# d \otimes b)] = [(cd)^\# cd \otimes ab] = (ab)\theta.$$

□

Lemma 5.12. *Let*

$$[x^\# y \otimes s], [u^\# v \otimes t] \in \overline{Q},$$

where $x, y, u, v \in \mathcal{S}(S)$, $x \rho y, u \rho v, s, t \in S$. Then $[x^\# y \otimes s] \leq_{\mathcal{H}} [u^\# v \otimes t]$ if and only if $ys \preceq vt$.

Proof. If $[x^\# y \otimes s] \leq_{\mathcal{H}} [u^\# v \otimes t]$, then either $[x^\# y \otimes s] = [u^\# v \otimes t]$ or there exists $[a^\# b \otimes c] \in \overline{Q}$, where $a, b \in \mathcal{S}(S)$, $c \in S$ and $a \rho b$, such that

$$[x^\# y \otimes s] = [a^\# b \otimes c][u^\# v \otimes t] = [(ua)^\# bv \otimes ct].$$

In the first case, Lemma 5.10 gives directly that $ys \mathcal{H}' vt$ and in the second, we deduce that $ys \mathcal{H}' bvct \preceq vt$.

Conversely, suppose that $ys \preceq vt$. By (C'), there exist $a \in \mathcal{S}(S)$, $b \in S$ with $ys \preceq a$, such that $ysa = vtb$. We then calculate that

$$\begin{aligned} [x^\# a^\# \otimes ub][u^\# v \otimes t] &= [x^\# a^\# u^\# v \otimes ubt] = [x^\# a^\# u^\# uv \otimes bt] = [x^\# a^\# v \otimes bt] = \\ [x^\# a^\# \otimes vtb] &= [x^\# a^\# \otimes ays] = [x^\# a^\# a \otimes ys] = [(x^\#)^2 a a^\# \otimes xys] = [(x^\#)^2 \otimes x][a a^\# \otimes ys] = \\ &[(x^\#)^2 \otimes x][x x^\# \otimes ys] = [(x^\#)^3 x \otimes xys] = [x^\# y \otimes s], \end{aligned}$$

since $(a a^\# \otimes ys, x x^\# \otimes ys) \in K$. We therefore deduce that $[x^\# y \otimes s] \leq_{\mathcal{H}} [u^\# v \otimes t]$ as required. □

The following corollary is now straightforward:

Corollary 5.13. *For any $s, t \in S$,*

$$s\theta \leq_{\mathcal{H}} t\theta \text{ in } \overline{Q} \text{ if and only if } s \preceq t.$$

Lemma 5.14. *The semigroup \overline{Q} is a semigroup of quotients of $S\theta$.*

Proof. If $a \in \mathcal{S}(S)$, then an easy calculation gives that $a\theta = [aa^\# \otimes a]$ lies in a subgroup of \overline{Q} with identity $[a^\# \otimes a]$, such that $[a^\#a \otimes a]^\# = [(a^\#)^2 \otimes a]$.

Suppose now that $[p^\#q \otimes s] \in \overline{Q}$, where $p, q \in \mathcal{S}(S)$, $s \in S$ and $p \rho q$. Then $(p\theta)^\#(qs)\theta = [(p^2)^\# \otimes p][q^\#q \otimes qs] = [(p^2)^\#q^\#q \otimes pqs] = [(p^2)^\#q^\#qpq \otimes s] = [p^\#q \otimes s]$.

□

Theorem 5.15. *Let \preceq be a q -pre-order on a commutative semigroup S . Then S is an order in the semigroup \overline{Q} inducing \preceq .*

Effectively, what we have achieved in the preceding argument is to determine the kernel of ψ in Theorem 5.7. It is worth making specific one further consequence.

Corollary 5.16. *Let S be a commutative semigroup and let ρ be a congruence on $\mathcal{S}(S)$ induced by a q -pre-order on S . Let R be the corresponding semigroup of quotients of $\mathcal{S}(S)$. Then for any q -pre-order \preceq inducing ρ we have that*

$$W \cong R \otimes_{\mathcal{S}(S)} S / \langle \{(u^\#u \otimes s, v^\#v \otimes s) : s \in S, u, v \in \mathcal{S}(S) \text{ } s \preceq u, v\} \rangle.$$

6. EXTENSION OF SEMILATTICE CONGRUENCES ON $\mathcal{S}(S)$

Let S be a commutative semigroup. In view of Lemmas 5.2 and 5.4, we wish to determine under which conditions do we have that a semilattice congruence ρ on $\mathcal{S}(S)$ with associated preorder \leq is such that $\overline{\rho}|_{\mathcal{S}(S)} = \rho$. Further, if this holds, when is it the case that ρ and \leq are induced by \mathcal{H}' and \preceq , where \preceq is a q -pre-order on S with associated congruence \mathcal{H}' ? The first question we answer completely. As for the second, we get a full answer in the case where S is a monoid and show how to understand the result in the case that S is not.

Lemma 6.1. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated compatible pre-order \leq . Then for any $c, d \in S$ with $c \overline{\leq} d$, where $\overline{\leq}$ is the smallest compatible pre-order on S containing \leq , we have*

- (i) $c = d$ or
- (ii) $c \overline{\rho} xd$ and $yc = xd$ for some $x, y \in \mathcal{S}(S)$ with $x \leq y$.

Proof. Let $c \overline{\leq} d$. In view of Lemma 2.1, $c = d$ or there exist $n \in \mathbb{N}$, and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $x_i, y_i \in \mathcal{S}(S)$ with $x_i \leq y_i$, such that

$$c = x_1c_1, y_1c_1 = x_2c_2, \dots, y_nc_n = d.$$

Suppose the latter holds. Notice that for $1 \leq i \leq n$ we have that $x_iy_i \rho x_i$. Then

$$c = x_1c_1 \overline{\rho} x_1y_1c_1 = x_1x_2c_2 \overline{\rho} \dots \overline{\rho} x_1 \dots x_nc_n \overline{\rho} x_1 \dots x_ny_nc_n = x_1 \dots x_nd,$$

so that $c \overline{\rho} xd$ where $x = x_1 \dots x_n \in \mathcal{S}(S)$.

Let $y = y_1 \dots y_n$, so that $y \in \mathcal{S}(S)$ and $x \leq y$. We have

$$\begin{aligned} yc &= y_1 \dots y_nc = y_1 \dots y_nx_1c_1 = x_1x_2y_2 \dots y_nc_2 \\ &= \dots = x_1 \dots x_{n-1}y_{n-1}y_nc_{n-1} = x_1 \dots x_ny_nc_n = xd. \end{aligned}$$

□

We will return later to the \leq -sequence connecting c to d .

Lemma 6.2. *Let S be a commutative semigroup and let ρ be a semilattice congruence on S with associated compatible pre-order \leq . Then*

- (i) $\bar{\rho}$ is the congruence $\equiv_{\bar{\leq}}$ associated with $\bar{\leq}$;
- (ii) $\bar{\rho}|_{\mathcal{S}(S)} = \rho$ if and only if $\bar{\leq}|_{\mathcal{S}(S)} = \leq$.

Proof. (i) Suppose that $c \bar{\rho} d$. Then either $c = d$ (so that clearly $c \equiv_{\bar{\leq}} d$) or there exists a ρ -sequence connecting c to d . As this sequence and its reverse are certainly \leq -sequences, we see that $c \equiv_{\bar{\leq}} d$.

Conversely, suppose that $c \equiv_{\bar{\leq}} d$. Then either $c = d$ (so that $c \bar{\rho} d$), or by Lemma 6.1 we have that

$$c \bar{\rho} u d \text{ and } d \bar{\rho} v c$$

for some $u, v \in \mathcal{S}(S)$. Then $c \bar{\rho} u v c$ so that

$$d \bar{\rho} v c \bar{\rho} u v^2 c \bar{\rho} u v c \bar{\rho} c.$$

(ii) (\Leftarrow) Let $a, b \in \mathcal{S}(S)$ and suppose that $a \bar{\rho} b$. By (i), $a \bar{\leq} b \bar{\leq} a$ so that by assumption $a \leq b \leq a$ and so $a \rho b$.

(\Rightarrow) Let $a, b \in \mathcal{S}(S)$ and suppose that $a \bar{\leq} b$. By Lemma 6.1 we have that either $a = b$ (so that $a \leq b$) or $a \bar{\rho} u b$ for some $u \in \mathcal{S}(S)$. By assumption, $a \rho u b$, so that $a \leq b$ as required. □

We can now give the first result promised at the beginning of this section.

Proposition 6.3. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated compatible pre-order \leq . Then $\bar{\rho}|_{\mathcal{S}(S)} = \rho$ if and only if Condition (R) holds.*

(R) *For all $a, b \in \mathcal{S}(S)$ and $c \in S$ with $a = bc$, we have that $a \leq b$.*

Proof. Suppose that $\bar{\rho}|_{\mathcal{S}(S)} = \rho$. Let $a, b \in \mathcal{S}(S)$ and $c \in S$ be such that $a = bc$. Then

$$a = bc \bar{\rho} b^2 c = ba,$$

so that $a \rho ab$ and consequently, $a \leq b$.

Conversely, suppose that (R) holds. Let $u, v \in \mathcal{S}(S)$ be such that $u \bar{\leq} v$. As in Lemma 6.1, either $u = v$ (and so $u \leq v$), or there exist $n \in \mathbb{N}$, and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $x_i, y_i \in \mathcal{S}(S)$ with $x_i \leq y_i$, such that

$$u = x_1 c_1, y_1 c_1 = x_2 c_2, \dots, y_n c_n = v.$$

From $u = x_1 c_1$ our given condition tells us that $u \leq x_1 \leq y_1$ and

$$u \rho y_1 u = y_1 x_1 c_1.$$

Suppose that for some i with $1 \leq i < n$ we have that $u \leq x_j \leq y_j$ for all $1 \leq j \leq i$ and $y_1 \dots y_i u = x_1 \dots x_i y_i c_i$. Then

$$u \rho y_1 \dots y_i u = x_1 \dots x_i x_{i+1} c_{i+1}$$

and again using our given condition we find that $u \leq x_{i+1} \leq y_{i+1}$ and further, $y_1 \dots y_{i+1} u = x_1 \dots x_{i+1} y_{i+1} c_{i+1}$.

By finite induction we obtain that

$$u \rho y_1 \dots y_n u = x_1 \dots x_n y_n c_n = x_1 \dots x_n v.$$

Hence $u \leq v$ as required. The result now follows using Lemma 6.2 (ii). \square

We recall that a necessary condition for a semilattice congruence ρ on $\mathcal{S}(S)$ to be induced by \mathcal{H}' , where \mathcal{H}' is \equiv_{\leq} for a q-pre-order on S , is that we have $\overline{\bar{\rho}}|_{\mathcal{S}(S)} = \rho$. We have now determined when the latter condition holds. If it does, what further conditions do we need in order that ρ be induced by \mathcal{H}' ? Surprisingly, at least in the case where S is a monoid, only one. First, we examine how to find a compatible pre-order on S containing \leq and satisfying Condition (A).

Lemma 6.4. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated pre-order \leq . We define*

$$A = \{(bc, b) : b, c \in S\}$$

and let $\overline{\leq}_A$ be the compatible pre-order on S generated by $\leq \cup A$. Then

- (i) for any $c, d \in S$, $c \overline{\leq}_A d$ if and only if $c \leq wd$ for some $w \in S^1$;
- (ii) $\overline{\leq}|_{\mathcal{S}(S)} = \leq$ if and only if $\overline{\leq}_A|_{\mathcal{S}(S)} = \leq$.

Proof. We remark that certainly $\overline{\leq} \subseteq \overline{\leq}_A$.

- (i) Suppose that $c \leq wd$ for some $w \in S^1$. Then using the definition of A ,

$$c \overline{\leq}_A wd \overline{\leq}_A d.$$

Conversely, suppose that $c \overline{\leq}_A d$. Then either $c = d$ (so that $c \leq d1$) or there exist $n \in \mathbb{N}$ and for $1 \leq i \leq n$, elements $c_i \in S^1$ and $(x_i, y_i) \in \leq \cup A$, such that

$$c = x_1 c_1, y_1 c_1 = x_2 c_2, \dots, y_n c_n = d.$$

If every $(x_i, y_i) \in \leq$, then $c \leq d = d1$. Otherwise, let

$$i_1 < i_2 < \dots < i_m$$

be those integers in $\{1, \dots, n\}$ such that $(x_{i_j}, y_{i_j}) \in A$ for $1 \leq j \leq m$; write $(x_{i_j}, y_{i_j}) = (h_{i_j} k_{i_j}, k_{i_j})$.

We calculate:

$$\begin{aligned} c \overline{\leq} y_{i_1-1} c_{i_1-1} &= x_{i_1} c_{i_1} = h_{i_1} k_{i_1} c_{i_1} = h_{i_1} y_{i_1} c_{i_1} = h_{i_1} x_{i_1+1} c_{i_1+1} \overline{\leq} \\ &h_{i_1} y_{i_2-1} c_{i_2-1} = h_{i_1} x_{i_2} c_{i_2} = h_{i_1} h_{i_2} k_{i_2} c_{i_2} = h_{i_1} h_{i_2} y_{i_2} c_{i_2} \overline{\leq} \dots \overline{\leq} \\ &h_{i_1} h_{i_2} \dots h_{i_m} y_{i_m} c_{i_m} \overline{\leq} h_{i_1} h_{i_2} \dots h_{i_m} y_n c_n = h_{i_1} h_{i_2} \dots h_{i_m} d, \end{aligned}$$

so that $c \overline{\leq} dw$ for $w = h_{i_1} h_{i_2} \dots h_{i_m}$.

(ii) Suppose that $\overline{\leq}|_{\mathcal{S}(S)} = \leq$. Let $a, b \in \mathcal{S}(S)$ with $a \overline{\leq}_A b$. Then by (i), $a \overline{\leq} bw$ for some $w \in S^1$ and so by Lemma 6.1 we have that $a = bw$ or $a \overline{\bar{\rho}} bw$ for some $x \in \mathcal{S}(S)$. In either case therefore we have that $a \overline{\bar{\rho}} yb$ for some $y \in S^1$. Then it is easy to see that $ab \overline{\bar{\rho}} a$ so that as $\overline{\bar{\rho}} = \equiv_{\leq}$, our assumption gives that $ab \rho a$ and so $a \leq b$.

Conversely, if $\overline{\leq}_A|_{\mathcal{S}(S)} = \leq$ then as

$$\leq \subseteq \overline{\leq} \subseteq \overline{\leq}_A$$

is is clear that $\overline{\leq}|_{\mathcal{S}(S)} = \leq$. \square

Under the assumption that $\overline{\rho}|_{\mathcal{S}(S)} = \rho$, our relation $\overline{\leq}_A$ automatically satisfies many of the conditions required to be a q-pre-order.

Lemma 6.5. *Suppose that $\overline{\rho}|_{\mathcal{S}(S)} = \rho$. Then*

- (i) $\overline{\equiv}_{\overline{\leq}_A}|_{\mathcal{S}(S)} = \rho$;
- (ii) $\overline{\leq}_A$ satisfies Conditions (A) and (C'');
- (iii) for any $a \in \mathcal{S}(S)$ and $b \in S$,

$$b \overline{\leq}_A a \Leftrightarrow ab \overline{\equiv}_{\overline{\leq}_A} b.$$

Proof. (i) By Lemmas 6.2 and 6.4 we have that $\overline{\leq}_A|_{\mathcal{S}(S)} = \leq$, so that as ρ is $\overline{\equiv}_{\leq}$, the result is clear.

(ii) Clearly (A) holds by construction of $\overline{\leq}_A$. Suppose that $b, c \in S$ with $b \overline{\leq}_A c$. From Lemma 6.4 we have that $b \overline{\leq} cw$ for some $w \in S^1$. From Lemma 6.1, we have that either $b = cw$, or $b \overline{\rho} xcw$ and $yb = xcw$ for some $x, y \in \mathcal{S}(S)$ with $x \leq y$. Since $\overline{\rho} \subseteq \overline{\equiv}_{\overline{\leq}_A}$, in the latter case we have that

$$b \overline{\equiv}_{\overline{\leq}_A} xwc \overline{\leq}_A x \overline{\leq}_A y$$

so that $b \overline{\leq}_A y$.

(iii) Let $b \in S$ and $a \in \mathcal{S}(S)$. By definition we have that $ab \overline{\leq}_A b$.

If $ab \overline{\equiv}_{\overline{\leq}_A} b$, then $b \overline{\leq}_A ab \overline{\leq}_A a$, by definition of $\overline{\leq}_A$.

On the other hand, if $b \overline{\leq}_A a$, then $b \overline{\leq} wa$ for some $w \in S^1$. From Lemma 6.1, we have that $b \overline{\rho} va$ for some $v \in S^1$. We deduce that $ba \overline{\rho} b$ so that certainly $b \overline{\equiv}_{\overline{\leq}_A} ab$. \square

To get the smoothest final conclusions we make use of generalised orders. The motivation is as follows. In trying to construct a semigroup of quotients of S , this may be prevented by there being elements of S that are not less than any square-cancellable element in any suitable pre-order. If S is an order in Q , then for any $s \in S$ there must be an $a \in \mathcal{S}(S)$ such that $s \leq_{\mathcal{H}Q} a$, simply because we must be able to write s as a quotient $a^\#b$, where $a, b \in S$. To artificially make a choice of pairs (s, a) to add to our relation $\overline{\leq}_A$ may destroy the nice properties of that relation.

We say that a compatible pre-order on a commutative semigroup S is a *generalised quotient pre-order* or *gq-pre-order* if it satisfies Conditions (A), (B) and (C'').

Lemma 6.6. *Let S be a commutative and such that S is monoid or $S = \mathcal{S}(S)$. Then a pre-order \preceq on S is a gq-pre-order if and only if it is a q-pre-order.*

Proof. The result is clear if S is a monoid. If $S = \mathcal{S}(S)$, then just notice that (C') holds for any pre-order. \square

For any pre-order on S we denote by \preceq^1 the relation $\preceq \cup \{(s, 1) : s \in S^1\}$ on S^1 . Notice that if S is a monoid and \preceq a q-pre-order, then for any $s \in S = S^1$ we have that $s = s1 \preceq 1$, by Condition (A), so that $\preceq = \preceq^1$. We remark that from the definition of \mathcal{H}^* it follows that $\mathcal{S}(S^1) = \mathcal{S}(S) \cup \{1\}$.

Proposition 6.7. *Let S be a commutative semigroup which is not a monoid. Then the following conditions are equivalent:*

- (i) S is a generalised order in a semigroup Q such that $\leq_{\mathcal{H}Q}|_S = \preceq$;
- (ii) \preceq is a gq -pre-order on S ;
- (iii) \preceq^1 is a q -pre-order on S^1 ;
- (iv) S^1 is an order in a monoid P such that $P \setminus \{1\}$ is a semigroup Q and $\leq_{\mathcal{H}P}|_{S^1} = \preceq^1$.

Proof. (ii) \Rightarrow (iii) Suppose that (ii) holds. It is clear that \preceq^1 is a pre-order.

Let $u, v, w \in S^1$ with $u \preceq^1 v$. If $w = 1$, then clearly $uw \preceq^1 vw$; we suppose therefore that $w \neq 1$. If $u, v \in S$ then again it is clear that $uw \preceq^1 vw$. If $u = 1$, then from the definition of \preceq^1 , we have that $v = 1$ so that certainly $uw \preceq^1 vw$. If $u \neq 1$ and $v = 1$, then $uw \preceq w = vw$, by Condition (A) for \preceq . Thus \preceq^1 is compatible.

For any $b, c \in S$ we know that $bc \preceq c$, so that $bc \preceq^1 c$. Clearly $11 \preceq^1 1$. Also, $b1 = b \preceq^1 b$ and $b1 \preceq^1 1$, so that \preceq^1 satisfies Condition (A).

We remarked above that $\mathcal{S}(S^1) = \mathcal{S}(S) \cup \{1\}$. It is now easy to see that Condition (B) holds for \preceq^1 .

Suppose now that $b, c \in S$ and $b \preceq^1 c$. Then $b \preceq c$ so by (C''), we have that $bu = vc$ for some $u, v \in S^1$, such that if $u \in S$, then $u \in \mathcal{S}(S)$ and $b \preceq u$. But if $u = 1$, then certainly $u \in \mathcal{S}(S^1)$ and we have $b \preceq^1 u$. We also have that for any $s \in S^1$, $s \preceq^1 1$, and $s1 = 1s$. Hence \preceq^1 satisfies Condition (C').

(iii) \Rightarrow (ii) This is clear.

(iii) \Rightarrow (iv) From Theorem 2.9, S^1 is an order in a semigroup P such that $\leq_{\mathcal{H}P}|_{S^1} = \preceq^1$. From Lemma 4.4, we have that P is a monoid with identity 1; we claim that $P = Q^1$ for some proper subsemigroup Q of P . To see this, observe that if $1 = (a^\#b)(c^\#d)$ where $a, b, c, d \in S^1$, then we must have $1 \mathcal{H}^P ac \mathcal{H}^P bd$ so that in S^1 we must have that $1 \preceq^1 ac$ and $1 \preceq^1 bd$. This tells us that $a = b = c = d = 1$. Consequently, $P = Q^1$ where $Q = P \setminus \{1\}$ is a semigroup containing S as a subsemigroup.

(iv) \Rightarrow (iii) This is immediate from Theorem 2.9 or Theorem 5.15.

(iv) \Rightarrow (i) If (iv) holds, then it is clear that S is a generalised order in Q , since if $q \in Q$, then $q = a^\#b$ where a, b cannot both be 1. If $b = 1$, then $q = (a^\#)^2a$; otherwise, if $a = 1$, then $q = b \in S$. For $a, b \in S$ we have that

$$a \leq_{\mathcal{H}Q} b \Leftrightarrow a = bq \text{ for some } q \in Q^1 \Leftrightarrow a \leq_{\mathcal{H}Q^1} b \Leftrightarrow a \preceq^1 b \Leftrightarrow a \preceq b,$$

so that Q induces \preceq on S .

(i) \Rightarrow (iv) This is clear from the definitions. □

Theorem 6.8. *Let S be a commutative semigroup and let ρ be a semilattice congruence on $\mathcal{S}(S)$ with associated pre-order \leq . Then S is a generalised order in a semigroup Q inducing \leq if and only if Conditions (R) and (B') hold.*

(B') for all $b, c \in S$ and $a \in \mathcal{S}(S)$ with $b \overline{\leq} au, c \overline{\leq} av$ for some $u, v \in S^1$, if $ab = ac$, then $b = c$.

Proof. Let S be a generalised order in Q such that $\leq = \preceq|_{\mathcal{S}(S)}$ where $\preceq = \leq_{\mathcal{H}Q}|_S$. By Lemma 5.2 we have that $\bar{\rho}|_{\mathcal{S}(S)} = \rho$ and so by Proposition 6.3, we have that (R) holds.

Suppose now that $b, c \in S$ and $a \in \mathcal{S}(S)$ with $b \overline{\leq} au, c \overline{\leq} av$ for some $u, v \in S^1$ and $ab = ac$. Then as $\overline{\leq} \subseteq \preceq$, and \preceq is a gq-pre-order, we have that $b \preceq au \preceq a$ and similarly, $c \preceq a$. As \preceq satisfies (B), we deduce that $b = c$ so that (B') holds.

Conversely, suppose that (R) and (B') hold. By Proposition 6.3 we have that $\bar{\rho}|_{\mathcal{S}(S)} = \rho$ and so Lemmas 6.2 and 6.4 give $\overline{\leq}_A|_{\mathcal{S}(S)} = \leq$. Moreover, by Lemma 6.5, $\overline{\leq}_A$ is a compatible pre-order satisfying Conditions (A) and (C''), and is such that if $b \overline{\leq}_A a \in \mathcal{S}(S)$, then $b \equiv_{\overline{\leq}_A} ab$. If $b \in S$ and $a \in \mathcal{S}(S)$ with $b \overline{\leq}_A a$, then again Lemma 6.4 $b \overline{\leq} au$ for some $u \in S^1$. Condition (B) for $\equiv_{\overline{\leq}_A}$ now follows from (B'). Thus $\overline{\leq}_A$ is a gq-pre-order, so by Proposition 6.7, S is a generalised order in Q inducing $\overline{\leq}_A$ on S and hence \leq on $\mathcal{S}(S)$. \square

The preceding theorem gives rise to the following:

Question Which commutative semigroups S have the property that every semilattice congruence on $\mathcal{S}(S)$ lifts to a congruence on S induced by a semi-group of quotients?

We are now able to present a series of corollaries that throw some light on the existence and structure of the set of quotients of a commutative order. First, we must extend Result 2.8 to generalised orders. If Q_1 and Q_2 are semigroups of generalised quotients of a commutative semigroup S , then as for quotient semigroups we write $Q_1 \geq Q_2$ if there is a homomorphism from Q_1 to Q_2 fixing the elements of S .

Proposition 6.9. *Let S be a commutative semigroup and a generalised order in semigroups Q_1 and Q_2 . The following conditions are equivalent:*

(i) $Q_2 \leq Q_1$;

(ii) for all $a, b \in S$,

$$a \leq_{\mathcal{H}} b \text{ in } Q_1 \text{ implies that } a \leq_{\mathcal{H}} b \text{ in } Q_2;$$

(iii) for all $a, b \in S$,

$$a \mathcal{H} b \text{ in } Q_1 \text{ implies that } a \mathcal{H} b \text{ in } Q_2.$$

Proof. It is only necessary to prove (iii) implies (i). To do so, let us temporarily denote by T^* a semigroup T with an identity adjoined *whether or not T is a monoid*. Clearly S^* is an order in Q_i^* for $i = 1, 2$. If (iii) holds then certainly for all $a, b \in S^*$,

$$a \mathcal{H} b \text{ in } Q_1^* \text{ implies that } a \mathcal{H} b \text{ in } Q_2^*$$

so that by Result 2.8 we have that there exists a homomorphism $\theta : Q_1^* \rightarrow Q_2^*$ fixing the elements of S^* . Note that if $x \in Q_1$, then $x \in S$ or $x = u^\#v$ for some $u, v \in S$, so that in either case $x\theta \in Q_2$. Hence θ restricts to a homomorphism from Q_1 to Q_2 fixing the elements of S , as required. \square

Corollary 6.10. *Let S be a commutative generalised order. Then S has a greatest generalised semigroup of quotients.*

Proof. Suppose that S is a generalised order: fix a semigroup Q of generalised quotients and let \preceq be the induced pre-order on S . Let I index the set of semilattice congruences on $\mathcal{S}(S)$ induced by a semigroup of generalised quotients. For any $\rho_i, i \in I$, we have that (R) holds, so that if the associated pre-order on $\mathcal{S}(S)$ is denoted by \leq_i , then $a = bc$ ($a, b \in \mathcal{S}(S), c \in S$) implies that $a \leq_i b$. Let $\rho = \bigcap_{i \in I} \rho_i$ and let \leq be the associated pre-order. Clearly (R) holds for ρ . By Lemma 6.5, $\overline{\leq_A}$ satisfies (A) and (C'') and for $b \overline{\leq_A} a$ where $a \in \mathcal{S}(S)$, we have that $b \equiv_{\overline{\leq_A}} ab$. If $a, b \in \mathcal{S}(S)$ and $a \leq b$, then $ab \rho a$, so that in particular, $ab \tau a$ where τ is induced by \preceq ; thus $a \preceq b$. Suppose now that $ab = ac$ where $b, c \in S, a \in \mathcal{S}(S)$ and $b, c \overline{\leq_A} a$. Since $\leq \subseteq \preceq$ and \preceq satisfies (A), we must have that $\leq \cup A \subseteq \preceq$ and so $\overline{\leq_A} \subseteq \preceq$. It follows that $b = c$ by Condition (B) for \preceq . Thus $\overline{\leq_A}$ is a gq-pre-order and clearly is the smallest such. The result now follows from Proposition 6.9. \square

Corollary 6.11. *Let S be a commutative semigroup and let \leq and ρ be a pre-order and its associated congruence on $\mathcal{S}(S)$ induced by a generalised semigroup of quotients. Then there is a greatest semigroup of generalised quotients Q^ρ inducing \leq and ρ on $\mathcal{S}(S)$.*

Proof. Using Lemma 6.5 it is easy to check that $\overline{\leq_A}$ is the least gq-pre-order inducing \leq and ρ on $\mathcal{S}(S)$. Denote the corresponding semigroup of generalised quotients by Q^ρ . \square

Our final result is now straightforward, using Propositions 6.7 and 6.9.

Corollary 6.12. *Let S be a commutative semigroup and let ρ_i for $i = 1, 2$ be semilattice congruences on $\mathcal{S}(S)$ induced by generalised semigroups of quotients Q_1 and Q_2 . Then $\rho_1 \subseteq \rho_2$ if and only if $Q^{\rho_1} \geq Q^{\rho_2}$.*

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