The profinite Schützenberger group defined by a symbolic dynamical system

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Centre for Mathematics
University of Coimbra

Fundação para a Ciência e a Tecnologia


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AN INTRODUCTION TO

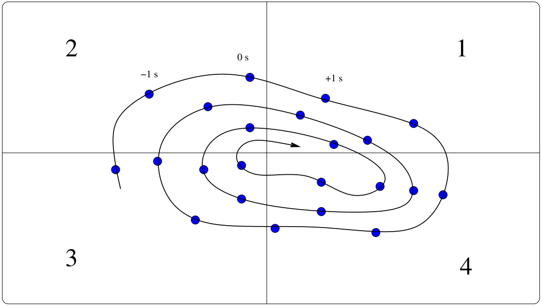
Symbolic Dynamics and Coding

DOUGLAS LIND

and

BRIAN MARCUS

Discretization



$$\ldots 32.2114443332114433214443 \ldots$$

This bi-infinite sequence is an element of $\{1, 2, 3, 4\}^{\mathbb{Z}}$, i.e., a mapping from $\mathbb{Z}$ to {1,2,3,4}.

$$\ldots * * * 32.2114443332114433214443 * \ldots$$

Subshifts

A symbolic dynamical system of $A^{\mathbb{Z}}$, a.k.a. subshift or just shift is a nonempty subset $\mathscr{X}$ of $A^{\mathbb{Z}}$ such that

- $\mathscr{X}$ is topologically closed
- $\sigma(\mathscr{X}) = \mathscr{X}$

$$\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \quad x_i \in A$$

The language of a subshift

$F(\mathscr{X}) = \{u \in A^+ : u = x_i x_{i+1} \dots x_{i+n} \text{ for some } x \in \mathscr{X},\ i \in \mathbb{Z},\ n \geq 0\}$

The elements of $F(\mathscr{X})$ are the blocks of $\mathscr{X}$.

Let $\mathscr{X}$ be the least subshift containing

$x = \cdots 32.211444333211443321443 \cdots$

$F(\mathscr{X}) = \{\dots, 3221, 32, 22, 1, 2, 3, 2114433, 4, 21, 11, \dots\}$

$F(\mathscr{X}) \subseteq F(\mathscr{Y})$ if and only if $\mathscr{X} \subseteq \mathscr{Y}$.

Morphisms between subshifts

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\psi} & \mathscr{Y} \\ \sigma \downarrow & & \downarrow \sigma \\ \mathscr{X} & \xrightarrow{\psi} & \mathscr{Y} \end{array}$$

Isomorphic subshifts are said to be conjugate.

An isomorphism is called a conjugacy.

Sliding block codes

Let $x \in A^{\mathbb{Z}}$. Given a map $g : A^m \to B$, we can code x through g:

- we choose integers $k, l \geq 0$ such that $m = k + l + 1$;
- we make $y_i = g(x_{[i-k,i+l]})$.

$$\ldots x_{i-4}x_{i-3}\,\boxed{x_{i-2}x_{i-1}x_ix_{i+1}}\,x_{i+2}x_{i+3}\ldots$$

$$g \big\downarrow$$

$$\ldots y_{i-2}y_{i-1}\,\boxed{y_i}\,y_{i+1}y_{i+2}\ldots$$

Theorem (Curtis-Hedlund-Lyndon, 1969)

The morphisms between subshifts are precisely the sliding block codes.

Irreducible shifts

Definition

A subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ is irreducible if

$$\mathscr{X} = \overline{\{\sigma^n(x) \mid n \geq 1\}}$$

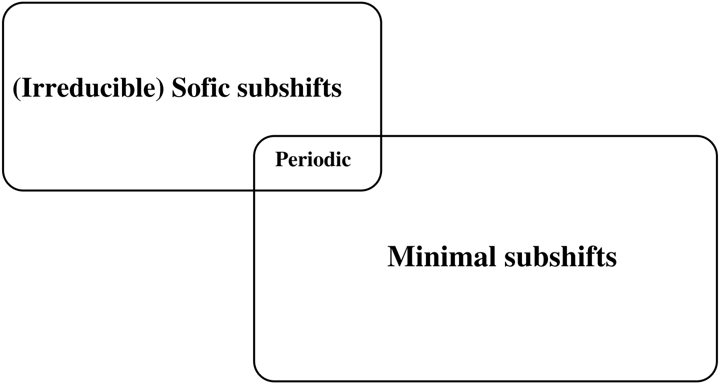
for some $x \in \mathscr{X}$.

$\mathscr{X}$ is irreducible if and only if $F = F(\mathscr{X})$ is recurrent.

Recurrent languages

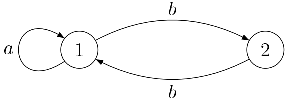
A language F is recurrent when:

- F is factorial
- $u, v \in F \Rightarrow (uwv \in F$ for some $w \in A^+)$


(Irreducible) Sofic subshifts
Periodic
Minimal subshifts

Sofic shifts

A shift $\mathscr{X}$ is *sofic* if $F(\mathscr{X})$ is a rational language, that is, sofic shifts correspond to factorial prolongable languages which are recognized by some finite labeled (oriented) graph.



$$\dots aaabbbbab.baaabbaaa \dots \in \mathscr{X}$$

but

$$\dots aaabbbbab.baa\color{red}{abbba}aa \dots \notin \mathscr{X}$$

Minimal subshifts

Definition

A subshift $\mathscr{X}$ is minimal if

$$\mathscr{Y} \subseteq \mathscr{X} \Rightarrow \mathscr{Y} = \mathscr{X}$$

for every subshift $\mathscr{Y}$. That is, if

$$\mathscr{X} = \overline{\{\sigma^n(x) \mid n \in \mathbb{Z}\}}$$

for every $x \in \mathscr{X}$.

$\mathscr{X}$ is minimal if and only if $F = F(\mathscr{X})$ is uniformly recurrent.

Uniformly recurrent languages

An infinite language F is uniformly recurrent when:

- F is factorial
- $u \in F \Rightarrow u$ is a factor of every word of length $N(u)$ in F

Lecture Notes in Mathematics

N. Pytheas Fogg

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The Fibonacci set and other examples

$$\varphi\colon A^* \to A^* \qquad \varphi(a) = ab \qquad \varphi(b) = a$$

$$\varphi^{n+2}(a) = \varphi^{n+1}(a)\varphi^n(a)$$

$$\varphi^6(a) = abaababaabaababaababa$$

$$F(\varphi) = \{a, b, aa, ab, ba, aab, aba, baa, bab, \ldots\}$$

- $n+1$ words of length n **(uniformly recurrent sets with this property are called Sturmian)**
- φ is an example of a primitive substitution

$$M(\varphi) = \begin{bmatrix} |\varphi(a)|_a & |\varphi(b)|_a \\ |\varphi(a)|_b & |\varphi(b)|_b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

M

- If $\psi\colon B^* \to B^*$ is primitive, then

$$F(\psi) = \{\text{factors of } \psi^n(b) \mid n \geq 1\} \qquad (b \in B)$$

does not depend of the choice of b, is uniformly recurrent, and so it defines a unique minimal subshift $\mathscr{X}(\psi)$

Free profinite monoids

Profinite monoid: inverse limit of finite monoids

Free profinite monoid:

$$\widehat{A^*} = \varprojlim \{A^*/\theta \mid A^*/\theta \text{ is finite}\}$$

- The elements of $\widehat{A^*}$ are called pseudowords
- The ideal structure of the free monoid A^* is very "poor".
- The ideal structure of the free **profinite** monoid $\widehat{A^*}$ is very **"rich"** when $|A| \geq 2$.

A connection introduced by Almeida ≈ 15 years ago

Consider the topological closure $\overline{F(\mathscr{X})}$ in $\widehat{A^*}$.

The set $\overline{F(\mathscr{X})}$ itself is factorial!
This is related with the multiplication on $\widehat{A^*}$ being an open mapping.

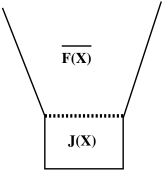
If $\mathscr{X}$ is irreducible, then $\overline{F(\mathscr{X})}$ contains a

$$\mathcal{J}\text{-minimum } \mathcal{J}\text{-class } J(\mathscr{X})$$

containing a

$$\text{maximal (profinite!) subgroup } G(\mathscr{X})$$

called the Schützenberger group of $\mathscr{X}$.



$$\mathscr{X} \subseteq \mathscr{Y} \Leftrightarrow F(\mathscr{X}) \subseteq F(\mathscr{Y}) \Leftrightarrow \overline{F(\mathscr{X})} \subseteq \overline{F(\mathscr{Y})} \Leftrightarrow J(\mathscr{Y}) \leq_{\mathcal{J}} J(\mathscr{X})$$

Corollary

If $|A| \geq 2$, then $\widehat{A^*}$ has an uncountable chain of regular $\mathcal{J}$-classes.

Proof.

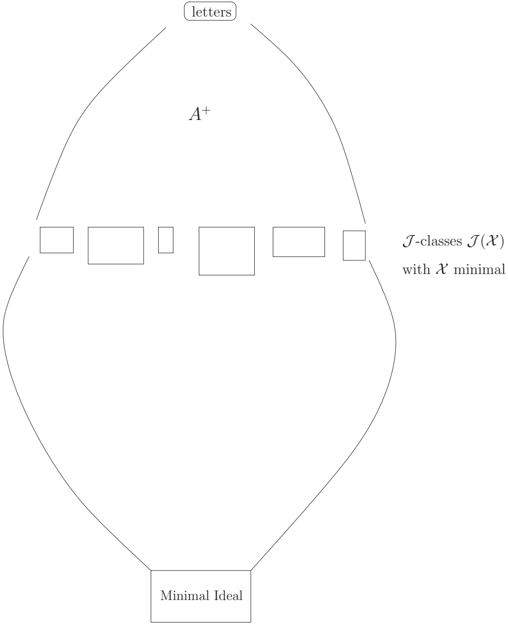
There is an uncountable chain of irreducible subshifts of $A^{\mathbb{Z}}$. □

Corollary

If $|A| \geq 2$, then $\widehat{A^*}$ has an uncountable anti-chain of regular $\mathcal{J}$-classes.

Proof.

There is an uncountable anti-chain of minimal subshifts of $A^{\mathbb{Z}}$. □



letters
A^+
$\mathcal{J}$-classes $\mathcal{J}(\mathcal{X})$
with $\mathcal{X}$ minimal
Minimal Ideal

$G(\mathscr{X})$

Theorem (C, 2006)

$G(\mathscr{X})$ is a conjugacy invariant.

Theorem (C & Steinberg, 2011)

If $\mathscr{X}$ is an irreducible sofic subshift over an alphabet with at least two letters, then $G(\mathscr{X})$ is a free profinite group of rank $\aleph_0$.

Theorem (C & Steinberg, 2020)

$G(\mathscr{X})$ is an invariant of flow equivalence.

Flow equivalence

Suspension flow:

$$(\mathscr{X} \times \mathbb{R})/\sim$$

with

$$(x, r) \sim (\sigma(x), r - 1)$$

$\forall x \in \mathscr{X}$, $r \in \mathbb{R}$.

"Definition"

$\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent whenever there is a well behaved homeomorphism between their suspension flows.

Flow equivalence

Let α be a letter of the alphabet A. Let $B = A \uplus \{\diamond\}$. Define a semigroup homomorphism $\mathcal{E}_\alpha \colon A^+ \to B^+$ by

$$\begin{cases} \mathcal{E}_\alpha(a) = a & \text{if } a \in A \setminus \{\alpha\} \\ \mathcal{E}_\alpha(\alpha) = \alpha\diamond \end{cases}$$

Symbol expansion

The symbol expansion of $\mathscr{X}$, relatively to a letter α of the alphabet of $\mathscr{X}$ is the least shift $\mathscr{X}_\alpha$ such that $F(\mathscr{X}_\alpha)$ contains $\mathcal{E}_\alpha(F(\mathscr{X}))$.

An equivalent definition by Parry & Sullivan (1975)

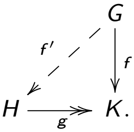
Flow equivalence is the equivalence relation between shifts generated by conjugacy and symbol expansions.

First example (Almeida, 2007) of a non-free maximal subgroup of $\widehat{A^*}$:

$$G(\mathscr{X}(\varphi)): \qquad \varphi(a) = ab, \qquad \varphi(b) = a^3 b.$$

Theorem (Rhodes & Steinberg, 2008)

The closed subgroups of the free profinite monoids are precisely the projective profinite groups.



Theorem (Lubotzky & Kovács, 2001)

If G is a projective profinite group which is finitely generated, then

$$G = \langle X \mid r(x) = x \ (x \in X) \rangle$$

with r an idempotent continuous endomorphism of $\widehat{FG(X)}$, for some finite X

Primitive (proper) substitutions

$\varphi\colon A^* \to A^*$ is proper when there are $a, b \in A$ such that $\varphi(A) \subseteq aA^* \cap A^*b$

e.g. $\varphi(a) = ab$, $\varphi(b) = a^3 b$

Theorem (Hunter, 1983)

If M is a finitely generated profinite monoid, then $\mathrm{End}(M)$ is profinite.

φ^ω: idempotent power of φ in $\mathrm{End}(\widehat{A^*})$

Theorem (Almeida & C, 2013)

If φ is a proper non-periodic primitive substitution, then

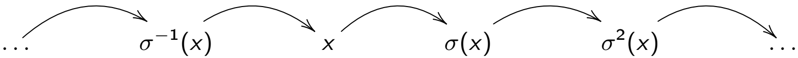
$$G(\mathscr{X}(\varphi)) = \varphi^\omega(\widehat{A^*}) = \langle A \mid \varphi^\omega(a) = a \ (a \in A)\rangle_{\text{Group presentation}}$$

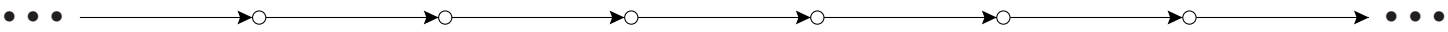
- If φ is invertible on $FG(A)$ (e.g. $\psi(a) = ab$ & $\psi(a) = a^2 b$), then $\psi^\omega(a) = a$ in $\widehat{FG(A)}$ and so $G(\mathscr{X}(\psi))$ is a free profinite group of rank $|A|$.
- $\varphi(a) = ab$, $\varphi(b) = a^3 b$ is not invertible on groups. We can use the presentation to show that $G(\mathscr{X}(\varphi))$ is not a free profinite group.
- For arbitrary primitive substitutions, one can reduce to the proper case, e.g. via a suitable conjugacy (Durand & Host & Skau, 1999)

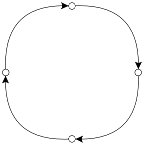
Towards a "geometrical" interpretation

The topological graph $\Sigma(\mathscr{X})$ of a subshift $\mathscr{X}$:

- vertices: the elements of $\mathscr{X}$
- edges: $(x, \sigma(x))$ (the unique edge from x to $\sigma(x)$)







Rauzy graphs

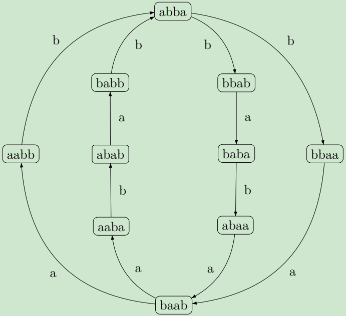
- $\mathcal{X} = \mathcal{X}(\tau)$ the P.T.M. subshift

$$\tau(a) = ab, \quad \tau(b) = ba$$

- Canonical projection

$$\Sigma(\mathcal{X}) \to \Sigma_4(\mathcal{X})$$

Rauzy graph $\Sigma_4(\mathscr{X})$



$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \ldots bbaba.abbaa\ldots & \xrightarrow{a} & \ldots babaa.bbaab\ldots & \xrightarrow{b} & \ldots abaab.baabb\ldots & \xrightarrow{b} & \cdots \\
& \downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_2 & & \\
\cdots \longrightarrow & baab & \xrightarrow{a} & aabb & \xrightarrow{b} & abba & \xrightarrow{b} & \cdots
\end{array}$$

Inverse limits

At the level of graphs

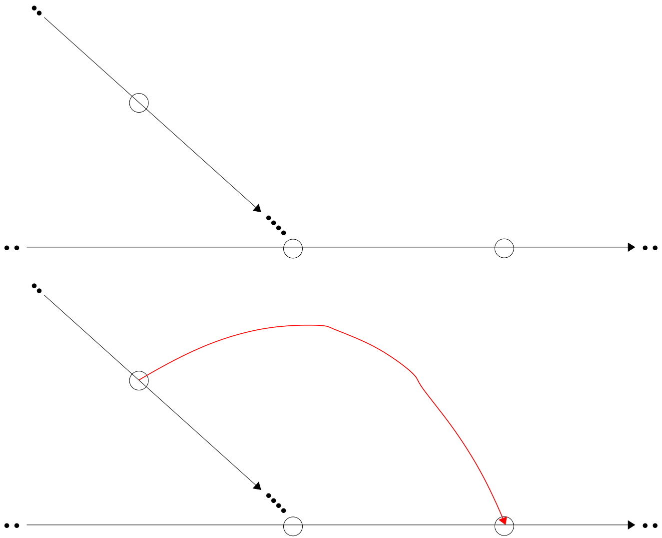
$$\Sigma(\mathscr{X}) = \varprojlim \Sigma_{2n}(\mathscr{X})$$

At the level of free categories

$$\Sigma(\mathscr{X})^* = \varprojlim \Sigma_{2n}(\mathscr{X})^*$$

At the level of free profinite categories

$$\widehat{\Sigma(\mathscr{X})^*} = \varprojlim \widehat{\Sigma_{2n}(\mathscr{X})^*}$$

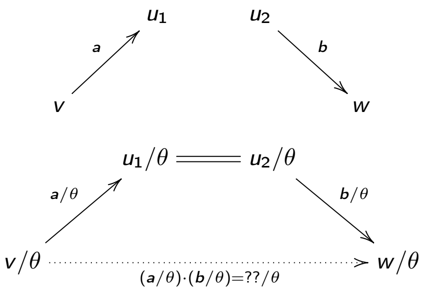


A technical difficulty

Given a (small) category C with a finite number of vertices, its profinite completion is the category:

$$\widehat{C} = \varprojlim \{ C/\theta \mid C/\theta \text{ is a finite category} \}$$

A congruence in a (small) category may identify two arrows with the same source and the same target, but not distinct vertices.



Dealing with the difficulty

- Take $\Gamma = \varprojlim \Gamma_i$, with the Γ_i finite
- Let $\widehat{\Gamma^*}$ be the closed subcategory of $\varprojlim \widehat{\Gamma_i^*}$ generated by Γ.

  Then, $\widehat{\Gamma^*}$ is the free profinite category generated by Γ.
- By definition, the inclusion $\widehat{\Gamma^*} \subseteq \varprojlim \widehat{\Gamma_i^*}$ holds. Is it always an equality?

Theorem (Almeida & C, 2009)

$$\widehat{\Sigma(\mathscr{X})^*} = \varprojlim \widehat{\Sigma_{2n}(\mathscr{X})^*}$$

- The inclusion $\overline{\Sigma(\mathscr{X})^*} \subseteq \widehat{\Sigma(\mathscr{X})^*}$ may be strict:
  - if $\mathcal{X}$ is the even subshift, then

$$\overline{\Sigma(\mathscr{X})^*} \subsetneq \left(\overline{\Sigma(\mathscr{X})^*}\right)^* \subsetneq \overline{\left(\overline{\Sigma(\mathscr{X})^*}\right)^*} = \widehat{\Sigma(\mathscr{X})^*}$$

First "geometrical" interpretation of $G(\mathscr{X})$

Theorem (Almeida & C, 2009)

Suppose that $\mathscr{X}$ is minimal.

1. $\widehat{\Sigma(\mathscr{X})^*} = \overline{\Sigma(\mathscr{X})^*}$
2. *After removing from $\widehat{\Sigma(\mathscr{X})^*}$ the edges of $\Sigma(\mathscr{X})^*$, we obtain a connected profinite groupoid, denoted $\widehat{\Sigma(\mathscr{X})^*}_\infty$*
3. *Let $\mu\colon \Sigma(\mathscr{X}) \to A$ send $(x, \sigma(x))$ to x_0. Its unique extension to a continuous functor*
$$\widehat{\mu}\colon \widehat{\Sigma(\mathscr{X})^*} \to \widehat{A^*}$$
is faithful and its restriction to any local group G_x of $\widehat{\Sigma(\mathscr{X})^*}$ maps G_x isomorphically onto a maximal subgroup of $J(\mathscr{X})$

Corollary

$G(\mathscr{X})$ is a conjugacy invariant.

Proof.

$\mathscr{X} \simeq \mathscr{Y} \Rightarrow \Sigma(\mathscr{X}) \simeq \Sigma(\mathscr{Y}) \Rightarrow \widehat{\Sigma(\mathscr{X})^*} \simeq \widehat{\Sigma(\mathscr{Y})^*} \Rightarrow \widehat{\Sigma(\mathscr{X})^*}_\infty \simeq \widehat{\Sigma(\mathscr{Y})^*}_\infty$ □

Second "geometrical" interpretation of $G(\mathscr{X})$

For $\mathscr{X}$ irreducible, we consider the fundamental groupoid $\Pi_{2n}(\mathscr{X})$ of $\Sigma_{2n}(\mathscr{X})$

$$
\begin{array}{ccc}
\widehat{\Sigma_{2m}(\mathscr{X})} & \xrightarrow{\ \hat{h}_m\ } & \widehat{\Pi_{2m}(\mathscr{X})} \\
\Big\downarrow{\scriptstyle \hat{p}_{m,n}} & & \Big\downarrow{\scriptstyle \hat{q}_{m,n}} \\
\widehat{\Sigma_{2n}(\mathscr{X})} & \xrightarrow[\ \hat{h}_n\ ]{} & \widehat{\Pi_{2n}(\mathscr{X})}
\end{array}
$$

Therefore, we have a continuous onto groupoid homomorphism

$$\hat{h}\colon \widehat{\Sigma(\mathscr{X})}^{*} \to \varprojlim \widehat{\Pi_{2n}(\mathscr{X})}$$

Theorem (Almeida & C, 2016)

If $\mathscr{X}$ is minimal, then $\hat{h}\colon \widehat{\Sigma(\mathscr{X})}^{*}{}_{\infty} \to \varprojlim \widehat{\Pi_{2n}(\mathscr{X})}$ is a continuous isomorphism.

Therefore, $G(\mathscr{X})$ is isomorphic to an inverse limit of the profinite completions of the fundamental groups of $\Sigma_{2n}(\mathscr{X})$.

Application of the (first) geometrical interpretation

- A return word to $u \in F(X)$ is a word $w \in A^* u$ such that $uw \in F(X)$ and u is not an internal factor of uw.

- Every loop of $\widehat{\Sigma(\mathscr{X})}^*$ at vertex x projects onto a loop of $\widehat{\Sigma_{2n}(\mathscr{X})}^*$ labeled by an element of the closed submonoid generated by the conjugate

$$(x_{[-n,-1]})^{-1} \cdot R_n \cdot x_{[-n,-1]}$$

of the set R_n of return words of $x_{[-n,n-1]}$.

- If R_n is always a basis of $FG(A)$, then $G(\mathscr{X})$ is a free profinite group of rank $|A|$.

Theorem (Return theorem — Berthé & De Felice & Dolce & Leroy & Perrin & Reutenauer & Rindone, 2015)

If $\mathscr{X}$ is dendric, then R_n is a basis of $FG(A)$

Dendric: generalize Sturmian subshifts.

Corollary (Almeida & C, 2016)

If $\mathscr{X}$ is dendric, then $G(\mathscr{X})$ is a free profinite group of rank $|A|$

Next slides: first relevant "external" application of $G(F) = G(\mathscr{X})$
($F = F(\mathscr{X})$ will always be uniformly recurrent)

(Joint work with:
Jorge Almeida, Revekka Kyriakoglou & Dominique Perrin)

Maximal bifix codes

A bifix code X of A^* is maximal if

$$X \subseteq Y \text{ and } Y \text{ is bifix} \implies X = Y$$

A bifix code X is F-maximal if $X \subseteq F$ and

$$X \subseteq Y \subseteq F \text{ and } Y \text{ is bifix} \implies X = Y$$

Theorem (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)

Z is maximal bifix $\Rightarrow X = Z \cap F$ is a finite F-maximal bifix code...

Theorem (The five authors of the 2012 paper + Dolce & Leroy; 2015)

... and $X = Z \cap F$ is a basis of a subgroup of index $d(Z)$ of the free group $FG(A)$, if moreover F is dendric.

$d(Z)$: rank of the minimum ideal of the transition monoid $M(Z^*)$ of the minimal automaton of Z^*

F-groups

The group of Z, denoted $G(Z)$:

Schützenberger group of the minimum $\mathcal{J}$-class of $M(Z^*)$.

The F-group of X, denoted $G_F(X)$:

Schützenberger group of the F-minimum $\mathcal{J}$-class of $M(Z^*)$ — the $\mathcal{J}$-class containing the image in $M(X^*)$ of $J(F)$.

Theorem (Berstel & De Felice & Perrin & Reutenauer & Rindone; 2012)

If Z is a group code and F is Sturmian, then

$$G(Z) \simeq G_F(X)$$

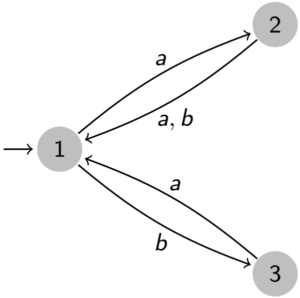
and $d(Z) = d_F(X)$.

$d_F(Z)$: rank of $J_F(X)$

- $Z = \{aa, ab, ba, bb\}$ is a group code of degree 2: $M(Z^*) = G(Z) = \mathbb{Z}/2\mathbb{Z}$

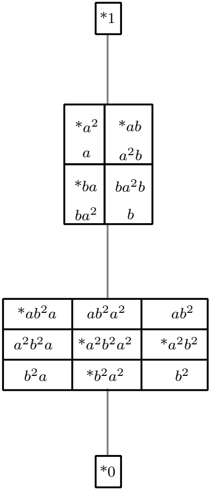
- $F =$ "Fibonacci set"

Minimum automaton of X^*, where $X = Z \cap F = \{aa, ab, ba\}$



$$G(Z) \simeq G_F(X) \simeq \mathbb{Z}/2\mathbb{Z}$$

$M(X^*)$:

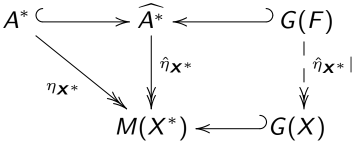


F-charged codes

Definition: X is:

- F-charged if

$$\hat{\eta}_{X^*}(G(F)) = G(X)$$



Theorem (Almeida & C & Kyriakoglou & Perrin; 2020)

Under mild conditions

$$Z \text{ is } F\text{-charged} \Rightarrow G(Z) \simeq G_F(X) \;\&\; d(Z) = d_F(X)$$

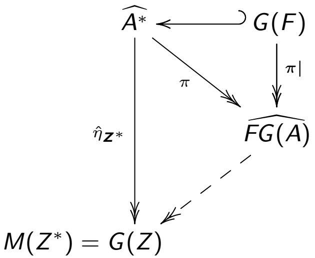
Corollary

$G(Z) \cong G_F(Z \cap F)$ if F is connected and Z is group code

Connected F: generalizes dendric F

Proof:

If F is a connected set, then $\pi|\colon G(F) \to \widehat{FG(A)}$ is onto **(Almeida & C; 2017)**.



Hence Z is F-charged. QED

Notice that this theorem uses no "profinite jargon"!

Three references

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   Symbolic dynamics and semigroup theory
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2. J. Almeida, A. Costa, R. Kyriakoglou & D. Perrin
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   (For applications to the theory of codes)

3. A. Costa & B. Steinberg
   The Karoubi envelope of the mirage of a subshift
   arXiv:2005.07490
   (Most recent research paper)