# A MUNN TYPE REPRESENTATION FOR A CLASS OF $E$-SEMIADEQUATE SEMIGROUPS

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ABSTRACT. Munn's construction of a fundamental inverse semigroup $T_E$ from a semilattice $E$ provides an important tool in the study of inverse semigroups. We present here a semigroup $F_E$ that plays for a class of $E$-semiadequate semigroups the role that $T_E$ plays for inverse semigroups. Every inverse semigroup with semilattice of idempotents $E$ is $E$-semiadequate. There are however many interesting $E$-semiadequate semigroups that are not inverse; we consider various such examples arising from Schützenberger products.

## 1. INTRODUCTION

One of the significant early approaches to the structure theory of inverse semigroups was via *fundamental* inverse semigroups, that is, inverse semigroups having no non-trivial idempotent separating congruences. Munn [M] showed how an important fundamental inverse semigroup $T_E$ could be constructed from any semilattice $E$, via partial isomorphisms of $E$. The *Munn semigroup* $T_E$ of $E$ has semilattice of idempotents isomorphic to $E$ and is "maximal" in the sense that an inverse semigroup $S$ with semilattice of idempotents $E$ is fundamental if and only if it is isomorphic to a full subsemigroup of $T_E$. Further, if $S$ is an inverse semigroup with semilattice of idempotents $E$ then there exists a homomorphism $\phi : S \to T_E$ whose kernel is $\mu$, the maximum idempotent separating congruence on $S$ [M].

The founding work of Munn has been generalised in several directions. Dropping the condition of commutativity of idempotents leads to the study of *orthodox* semigroups, that is, regular semigroups whose idempotents form a subsemigroup. Semigroups of idempotents are called *bands*. The *Hall semigroup* $W_B$ of a band $B$ is an orthodox

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semigroup with band of idempotents isomorphic to $B$ and properties analogous to those described above for $T_E$ [Ha1]. Hall and Nambooripad took this still further to the case of regular semigroups in [Ha2] and [N] respectively.

Another direction has been taken by Fountain in [F1], where he considers *adequate* semigroups. The move from inverse to adequate semigroups is obtained by retaining the commutativity of the idempotents but weakening the condition of regularity. This is accomplished via consideration of Green's $*$-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ where elements $a, b$ of a semigroup $S$ are $\mathcal{L}^*$-related if and only if they are $\mathcal{L}$-related in an oversemigroup of $S$; the relation $\mathcal{R}^*$ is defined dually. In fact $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations [F1]. A semigroup $S$ is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of $S$ contains an idempotent and *adequate* if, in addition, the idempotents of $S$ form a commutative subsemigroup. In this case the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of $a \in S$ contains a unique idempotent, denoted by $a^*(a^+$, sometimes $a^\dagger)$. If $S$ is a regular semigroup then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$; clearly then a regular semigroup is abundant and an inverse semigroup is adequate, with $a^* = a^{-1}a$ and $a^+ = aa^{-1}$. In an adequate semigroup there need not be a greatest idempotent separating congruence. However, on an inverse semigroup $\mu$ is also the largest congruence contained in $\mathcal{H}$. Defining $\mathcal{H}^*$ to be $\mathcal{L}^* \cap \mathcal{R}^*$ we may without ambiguity denote by $\mu$ the largest congruence contained in $\mathcal{H}^*$. In [F1] Fountain shows that if $S$ is an adequate semigroup with semilattice of idempotents $E$, which in addition satisfies

$$ea = a(ea)^* \text{ and } ae = (ae)^+a \qquad (A)$$

for all $a \in S$ and for all idempotents $e \in E$, then there is a homomorphism $\phi : S \to T_E$ with kernel $\mu$. Such a semigroup is called *type A* in [F1] and more recently *ample* [G].

The work in this paper continues the approach of [F1]. First, we drop the 'ample' condition $(A)$, imposing a strictly weaker condition introduced in [F2]. In a second direction we weaken the adequacy condition and consider $E$-semiadequate semigroups, first defined by Lawson in [L]. A semigroup $S$ is *$E$-semiadequate*, where $E$ is a semilattice of idempotents and a subsemigroup of $S$, if every $\widetilde{\mathcal{L}}_E$-class and every $\widetilde{\mathcal{R}}_E$-class of $S$ contains a (necessarily unique) idempotent of $E$. Here $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ are generalisations of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, and are defined in Section 2. If $S$ is $E$-semiadequate then by a natural extension of our previous notation, we denote by $a^*$ $(a^+)$ the idempotent of $E$ in the $\widetilde{\mathcal{L}}_E$-class ($\widetilde{\mathcal{R}}_E$-class) of $a \in S$. If $E$ consists of all idempotents of $S$

and $S$ is adequate, then $\mathcal{L}^* = \widetilde{\mathcal{L}}_E$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$ so that no ambiguity arises.

Our interest in this class of semigroups arose from considering the Schützenberger product $M \diamond N$ of monoids $M$ and $N$. The monoid $M \diamond N$ is not adequate unless $M$ and $N$ are both cancellative. However, $M \diamond N$ is $E$-semiadequate for a certain subset $E$ of idempotents. If $M$ is left cancellative and $N$ is right cancellative then $E$ is the set of all idempotents of $M \diamond N$ and $M \diamond N$ has a number of other properties; it is an example of a *weakly hedged* monoid.

Lawson [L] establishes a strong connection between a class of $E$-semiadequate semigroups and small ordered categories. In Theorem 4.24 of [L] he shows that a certain category of $E$-semiadequate semigroups and admissible homomorphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors. The semigroups considered are called *Ehresmann* semigroups in [L]; in our terminology they are $E$-semiadequate semigroups satisfying conditions (CR) and (CL), defined in the next section. This paper concentrates on $E$-*fundamental* $E$-semiadequate semigroups. We describe an analogue of the Munn semigroup $T_E$ of a semilattice $E$. This semigroup, which we denote by $F_E$, plays the role for a class of $E$-semiadequate semigroups that $T_E$ plays for inverse semigroups having semilattice of idempotents $E$.

In Section 2 we define the class of semigroups under consideration, *weakly $E$-hedged* semigroups. They are $E$-semiadequate semigroups satisfying two conditions weaker than $(A)$. Trivially, every monoid is weakly $\{1\}$-hedged and it is not difficult to show that every inverse monoid is weakly $E$-hedged where $E$ is the semilattice of all its idempotents. As mentioned above, more interesting examples of weakly $E$-hedged semigroups are obtained from the Schützenberger product of a left cancellative monoid with a right cancellative monoid, discussed at length in Section 3. Further examples of semigroups satisfying the corresponding one-sided conditions are provided by graph expansions of monoid presentations of unipotent monoids. In Section 4 given a semilattice $E$ we construct an $\overline{E}$-semiadequate semigroup, $F_E$, containing a semilattice of idempotents $\overline{E}$ isomorphic to $E$. The semigroup $F_E$ is built using pairs of homomorphisms from $E^1$ to $E$. The need to consider *pairs* of homomorphisms arises from the fact that, unlike the case for inverse semigroups, the endomorphisms of $E^1$ obtained in a natural way from the elements of a weakly $E$-hedged semigroup $S$ do not come equipped with inverses on certain domains. That is, not unless $S$ satisfies condition $(A)$. The Munn semigroup $T_E$ is embedded in $F_E$

via an injection $\pi$. Defining $\mu_E$ to be the largest congruence contained in $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \cap \widetilde{\mathcal{R}}_E$, we show that $\mu_{\overline{E}}$ is trivial on $F_E$; accordingly, we say that $F_E$ is $\overline{E}$-*fundamental.* If $S$ is a weakly $E$-hedged semigroup then there is a homomorphism $\theta : S \to F_E$ with ker $\theta = \mu_E$.

In line with the new terminology of [G], we call a weakly $E$-hedged semigroup satisfying condition $(A)$ *weakly* $E$-*ample.* The imposition of $(A)$ is enough for us to be able to dispense in this case with $F_E$ and show there is a homomorphism $\phi : S \to T_E$ with ker $\phi = \mu_E$. This result also occurs in work of El-Qallali and Fountain [EF], where they consider $U$-*semiabundant* semigroups for a class of idempotents $U$ (not necessarily a semilattice) satisfying (CR), (CL) and the analogue of the 'ample' condition (A). We also show that $\phi\pi = \theta$, where $\pi : T_E \to F_E$ and $\theta : S \to F_E$ are the homomorphisms mentioned above.

After this consideration of weakly $E$-ample semigroups in Section 5, our final section is devoted to using the theory we have built to deduce some facts concerning weakly $E$-hedged and weakly $E$-ample semigroups. In particular, a weakly $E$-hedged (weakly $E$-ample) semigroup is $E$-fundamental if and only if it is $E$-isomorphic to a subsemigroup of $F_E$ $(T_E)$.

## 2. $E$-SEMIADEQUATE AND WEAKLY $E$-HEDGED SEMIGROUPS

In this section we define the above classes of semigroups and state a number of their elementary properties. Proofs are omitted where they are virtually identical to those in [F1]. In the following section we show how these ideas arise naturally from Schützenberger products of monoids satisfying cancellation properties. We use the terminology and notation of [Ho1]; in particular, the set of idempotents of a semigroup $S$ is denoted by $E(S)$.

We begin with the following alternative description of $\mathcal{L}^*$, which may be found in [F1].

**Lemma 2.1.** *Elements $a, b$ of a semigroup $S$ are $\mathcal{L}^*$-related if and only if for all $x, y \in S^1$*

$$ax = ay \text{ if and only if } bx = by.$$

From Lemma 2.1 it follows that $\mathcal{L}^*$ is an equivalence relation. It is then easy to see that $\mathcal{L}^*$ is a right congruence and dually, $\mathcal{R}^*$ is a left congruence.

Let $E$ be a semilattice and a subsemigroup of $S$. We say that $S$ is *right (left) $E$-adequate* if every $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of $S$ contains an idempotent of $E$. If $S$ is right (left) $E$-adequate the idempotent of $E$ in the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of $a \in S$ is unique and is denoted $a^*$ $(a^+)$. If

$S$ is right and left $E$-adequate then $S$ is *$E$-adequate. If $E = E(S)$ then here, as elsewhere, we may omit mention of $E$ in these definitions.*

Suppose now that $S$ is right $E$-adequate and $a \in S$. From Lemma 2.1, $aa^* = a$ and if $e \in E$ is such that $ae = a$ then $a^*e = a^*$, so that $a^* \leq e$ in the semilattice $E$. Thus

$$a_E = \{e \in E : ae = a\}$$

has minimum member $a^*$ and dually

$$_Ea = \{e \in E : ea = a\}$$

has minimum member $a^+$. These facts, together with a number of examples (see Section 3), lead us to consider $E$-semiadequate semigroups, defined by Lawson in [L].

Let $S$ be a semigroup such that $E(S)$ contains a semilattice $E$. We say that $S$ is *right $E$-semiadequate* if for each $a \in S$ the set $a_E$ contains a minimum member, which we denote by $a^*$. Note that for $e \in E, e = e^*$. The relation $\widetilde{\mathcal{L}}_E$ is defined on $S$ by the rule that for $a, b \in S$,

$$a \, \widetilde{\mathcal{L}}_E \, b \text{ if and only if } a^* = b^*.$$

For any $a \in S, (a^*)^* = a^*$ so that $a \, \widetilde{\mathcal{L}}_E \, a^*$; clearly $a^*$ is the unique idempotent of $E$ that is $\widetilde{\mathcal{L}}_E$-related to $a$. If $S$ is right $E$-adequate then $\mathcal{L}^* = \widetilde{\mathcal{L}}_E$ so that the notation $a^*$ is unambiguous. A *left $E$-semiadequate* semigroup is defined dually; for an element $a$ of such a semigroup $S$, $a^+$ denotes the minimum member of $_Ea$. The relation $\widetilde{\mathcal{R}}_E$ is defined on $S$ by the rule that for $a, b \in S$,

$$a \, \widetilde{\mathcal{R}}_E \, b \text{ if and only if } a^+ = b^+.$$

If $S$ is right and left $E$-semiadequate then $S$ is said to be *$E$-semiadequate*. This terminology and the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ were introduced in [L], with a slightly different approach. As commented in [L], these ideas are inherent in an earlier paper of Batbedat and Fountain [BF].

If $S$ is right $E$-semiadequate then for any $a \in S$ there is a mapping $\alpha_a : E^1 \to E$ given by $x\alpha_a = (xa)^*$.

**Lemma 2.2.** *Let $S$ be a right $E$-semiadequate semigroup. Then*

*(1) for all $a, b \in S, (ab)^* \leq b^*$;*

*(2) for all $a \in S$ the mapping $\alpha_a : E^1 \to E$ is order preserving.*

**Proof** (1) For $a, b \in S$ we have $(ab)b^* = ab$ so that $(ab)^* \leq b^*$ by definition of $(ab)^*$ as the minimum element in $(ab)_E$.

(2) Let $a \in S$ and $x, y \in E^1$ with $x \leq y$. Then using (1),

$$x\alpha_a = (xa)^* = (xya)^* \leq (ya)^* = y\alpha_a.$$

The condition that a semigroup be right $E$-semiadequate can be very weak. To make progress we require at least that the semigroup satisfies condition (CR). We say that a right $E$-semiadequate semigroup satisfies (CR) if $\widetilde{\mathcal{L}}_E$ is a right congruence. In view of earlier remarks this is always true for a right $E$-adequate semigroup. Condition (CR) together with its left-right dual (CL) are called the *congruence condition* [L].

**Lemma 2.3.** *Let $S$ be a right $E$-semiadequate semigroup satisfying (CR).*

*(1) For all $a, b \in S$, $(ab)^* = (a^*b)^*$.*
*(2) For all $a \in S$ and $e \in E$, $(ae)^* = a^*e$.*
*(3) For all $a, b \in S$, $\alpha_{ab} = \alpha_a\alpha_b$.*

**Proof** (1) and (2) follow from Proposition 3.7 of [L].
Using (CR) we have that for any $a, b \in S$ and $x \in E^1$

$$x\alpha_a\alpha_b = (xa)^*\alpha_b = ((xa)^*b)^* \ \widetilde{\mathcal{L}}_E \ (xa)^*b \ \widetilde{\mathcal{L}}_E \ xab \ \widetilde{\mathcal{L}}_E \ (xab)^* = x\alpha_{ab}$$

so that (3) holds.

If $S$ is a left $E$-semiadequate semigroup then for any $a \in S$ the map $\beta_a : E^1 \to E$ is defined by $x\beta_a = (ax)^+$. The dual of Lemma 2.2 gives that for each $a \in S$, $\beta_a$ is order preserving and if condition (CL) holds the dual of Lemma 2.3 gives that for all $a, b \in S, \beta_{ab} = \beta_b\beta_a$. We denote by $\mathcal{O}_1(E^1)$ the semigroup of order preserving maps $\alpha : E^1 \to E$. Combining the above results we may define a homomorphism $\theta$ from an $E$-semiadequate semigroup $S$ satisfying the congruence condition to $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ by $a\theta = (\alpha_a, \beta_a)$, for all $a \in S$. Here $\mathcal{O}_1^*(E^1)$ is the *dual* semigroup of $\mathcal{O}_1(E^1)$.

For an element $e$ of a semilattice $E$ we denote by $\rho_e$ the homomorphism $E^1 \to E$ induced by multiplication with $e$. If $\alpha, \beta$ are endomorphisms of $E^1$ such that $x\alpha \leq x\beta$ for all $x \in E^1$, then we write $\alpha \leq \beta$.

**Lemma 2.4.** *Let $S$ be an $E$-semiadequate semigroup satisfying the congruence condition. Then for all $a \in S$,*

*(1) $a^+\alpha_a = a^*$ and $a^*\beta_a = a^+$;*
*(2) $\rho_{a^+} \leq \alpha_a\beta_a$ and $\rho_{a^*} \leq \beta_a\alpha_a$.*

**Proof** (1) is immediate from the definitions of $\alpha_a$ and $\beta_a$. To prove (2), suppose that $x \in E^1$. Then

$$xa^+(a(xa)^*)^+ \ \widetilde{\mathcal{R}}_E \ xa^+a(xa)^* = xa(xa)^* = xa \ \widetilde{\mathcal{R}}_E \ xa^+$$

so that $(x\rho_{a^+})(x\alpha_a\beta_a) = x\rho_{a^+}$ and $\rho_{a^+} \leq \alpha_a\beta_a$. Dually, $\rho_{a^*} \leq \beta_a\alpha_a$.

Let $S$ be an $E$-semiadequate semigroup satisfying the congruence condition. We recall from the introduction that $\mu_E$ denotes the largest congruence contained in $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \cap \widetilde{\mathcal{R}}_E$. The congruence $\mu_E$ may be described in an analogous manner to that given for adequate semigroups in [F1]; the proof is essentially the same as that in [F1]. Lemma 2.5 and Proposition 2.6 were also noted in [E].

**Lemma 2.5.** *Let $S$ be an $E$-semiadequate semigroup satisfying the congruence condition. Then the congruence $\mu_E = ker\ \theta$, where*

$$\theta : S \to \mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$$

*is the homomorphism given by $a\theta = (\alpha_a, \beta_a)$. Thus*

$$\mu_E = \{(a,b) \in S \times S : \alpha_a = \alpha_b \text{ and } \beta_a = \beta_b\}$$

$$= \{(a,b) \in S \times S : a \; \widetilde{\mathcal{H}}_E \; b, \alpha_a|_{a^+E} = \alpha_b|_{a^+E} \text{ and } \beta_a|_{a^*E} = \beta_b|_{a^*E}\}.$$

Let $E$ be a semilattice and a subsemigroup of $T$. We say that the semigroup $T$ is an *$E$-semilattice of monoids* if $T$ is a semilattice $E$ of monoids $T_e, e \in E$, such that for all $e \in E, e$ is the identity of $T_e$. A standard argument gives that if $T$ is an $E$-semilattice of monoids then $T$ is a *strong* semilattice of monoids determined by homomorphisms $\phi_{e,f} : T_e \to T_f (e \geq f)$ where for $a \in T_e, a\phi_{e,f} = af$. It is an easy exercise to show that such a semigroup $T$ is $E$-semiadequate and satisfies the congruence condition. Further, $E$ is central in $T$. The following shows that the converse result is true.

**Proposition 2.6.** *Let $S$ be an $E$ semiadequate semigroup satisfying the congruence condition. Then the following conditions are equivalent:*

*(1) $S/\mu_E \cong E$;*
*(2) for all $a \in S, a^* = a^+$;*
*(3) $\widetilde{\mathcal{L}}_E = \widetilde{\mathcal{H}}_E = \widetilde{\mathcal{R}}_E$;*
*(4) each $\widetilde{\mathcal{H}}_E$-class contains a (unique) idempotent of $E$;*
*(5) $E$ is central in $S$;*
*(6) $S$ is an $E$-semilattice of monoids.*

**Proof** Similar to that of Proposition 2.9 of [F1].

The main results of this paper are contained in Section 4, where we give a 'Munn type' representation for a class of $E$-semiadequate semigroups, namely the class of weakly $E$-hedged semigroups. A right $E$-semiadequate semigroup is *right weakly $E$-hedged* if it satisfies conditions (CR) and (HR).

(HR) For all $x, y \in E^1$ and for all $a \in S, (xya)^* = (xa)^*(ya)^*$.

In view of Lemma 2.2, condition (HR) can be replaced by

(HR)$'$ for all $x, y \in E$ and for all $a \in S, (xya)^* = (xa)^*(ya)^*$.

Condition (HR) and its dual (HL) were introduced for right (left) adequate semigroups in [F2] where a right adequate semigroup satisfying

(HR) is said to be *right h-adequate*; in this paper, such a semigroup is called *right hedged*.

The next lemma follows immediately from the definitions.

**Lemma 2.7.** *Let $S$ be a right $E$-semiadequate semigroup. Then $S$ satisfies (HR) if and only if for each $a \in S, \alpha_a : E^1 \to E$ is a homomorphism.*

*Left weakly $E$-hedged* semigroups are defined dually and a semigroup $S$ is *weakly $E$-hedged* if it is both left and right weakly $E$-hedged. Denoting by $\mathrm{End}_1 E^1$ the semigroup of endomorphisms of $E^1$ with image contained in $E$, we may now restate Lemma 2.5 for weakly $E$-hedged semigroups as follows.

**Lemma 2.8.** *Let $S$ be a weakly $E$-hedged semigroup. Then the congruence $\mu_E = ker\ \theta$, where $\theta : S \to \mathrm{End}_1 E^1 \times \mathrm{End}_1^* E^1$ is the homomorphism given by $a\theta = (\alpha_a, \beta_a)$.*

We end this section by considering the 'ample' or 'type A' condition for $E$-semiadequate semigroups. Following the new terminology of [G] we say that a right $E$-semiadequate semigroup $S$ is *weakly right $E$-ample* if $S$ satisfies conditions (CR) and (AR).

(AR) For all $a \in S$ and $e \in E, ea = a(ea)^*$.

*Weakly left $E$-ample* and *weakly $E$-ample* semigroups are defined using the now standard convention. If $S$ is an inverse semigroup with semilattice of idempotents $E$, then as mentioned in the introduction, $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$; from this section we also have that $\mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$. It is then easy to see that $S$ is ample, hence certainly weakly ample.

Weakly right $E$-ample semigroups are weakly right $E$-hedged, as we now show.

**Lemma 2.9.** *Let $S$ be a weakly right $E$-ample semigroup. Then $S$ satisfies (HR) so that $S$ is weakly right $E$-hedged.*

**Proof** Let $x, y \in E$ and $a \in S$. Then

$$xya = xa(ya)^* = a(xa)^*(ya)^*$$

so that $(xya)^* = a^*(xa)^*(ya)^* = (xa)^*(ya)^*$, using Lemma 2.2.

In Section 3 we show that weakly right $E$-hedged semigroups need not be weakly right $E$-ample. We remark that an $E$-semilattice of monoids is weakly $E$-ample so that from Proposition 2.6, if $S$ is an $E$-semiadequate semigroup satisfying the congruence condition and $E$ is central in $S$, then $S$ is weakly $E$-ample.

## 3. Schützenberger products

Recall that the Schützenberger product $M \diamond N$ of semigroups $M$ and $N$ is the semigroup with underlying set

$$\{\begin{pmatrix} m & P \\ 0 & n \end{pmatrix} : m \in M, n \in N, P \subseteq M \times N\}$$

and multiplication given by

$$\begin{pmatrix} m & P \\ 0 & n \end{pmatrix}\begin{pmatrix} m' & P' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} mm' & mP' \cup Pn' \\ 0 & nn' \end{pmatrix}.$$

Here $mP = \{m(x,y) : (x,y) \in P\}$ and $Pn = \{(x,y)n : (x,y) \in P\}$. The action of $m \in M$ on the left of $M \times N$ is given by $m(x,y) = (mx, y)$; the action of $n \in N$ on the right of $M \times N$ is dual (see [MP]). Throughout this section $M$ and $N$ will denote monoids so that $M \diamond N$ is a monoid with identity $\left(\begin{smallmatrix} 1 & \emptyset \\ 0 & 1 \end{smallmatrix}\right)$. We put

$$E = \{\begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} : P \subseteq M \times N\}.$$

It is easy to see that, as a submonoid of $M \diamond N$, $E$ is a semilattice and $E$ is isomorphic to the semilattice of subsets of $M \times N$ under union. We will impose various cancellation conditions on $M$ and $N$ and show how this gives examples of the various kinds of (left, right) $E$-semiadequate semigroups introduced in the previous section.

Before stating our first result we list a number of facts concerning the actions of $M$ and $N$ on $M \times N$. With the exceptions of (6) and (7) they are immediate; (6) and (7) are easily verifiable. For $m \in M$ and $P \subseteq M \times N$ we denote by $m^{-1}P$ the set $\{(x,y) : m(x,y) \in P\}$.

For all $m, a \in M$, $P, Q \subseteq M \times N$ and $n \in N$

(1) $m(Pn) = (mP)n$,
(2) $m(P \cup Q) = mP \cup mQ$,
(3) $m^{-1}(P \cup Q) = m^{-1}P \cup m^{-1}Q$,
(4) $(ma)P = m(aP)$,
(5) $m^{-1}(a^{-1}P) = (am)^{-1}P$,
(6) $m(Pn^{-1}) = (mP)n^{-1}$,
(7) if $M$ is left cancellative then $m^{-1}(mP) = P$.

From (5) and (7) we have that if $M$ is left cancellative then $(am)^{-1}(aP) = m^{-1}(a^{-1}(aP)) = m^{-1}P$, for all $a, m \in M$ and $P \subseteq M \times N$.

**Lemma 3.1.** *The monoid $M \diamond N$ is $E$-semiadequate and satisfies conditions (HR) and (HL). The operations $^*$ and $^+$ are given by*

$$\begin{pmatrix} a & P \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} 1 & a^{-1}P \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & P \\ 0 & b \end{pmatrix}^+ = \begin{pmatrix} 1 & Pb^{-1} \\ 0 & 1 \end{pmatrix}.$$

**Proof** Given $A = \left(\begin{smallmatrix} a & P \\ 0 & b \end{smallmatrix}\right) \in M \times N$ and $F = \left(\begin{smallmatrix} 1 & Q \\ 0 & 1 \end{smallmatrix}\right) \in E$ we have

$$AF = \begin{pmatrix} a & aQ \cup P \\ 0 & b \end{pmatrix} = A$$

if and only if $Q \subseteq a^{-1}P$. It follows that $A^*$ exists and $A^* = \left(\begin{smallmatrix} 1 & a^{-1}P \\ 0 & 1 \end{smallmatrix}\right)$; dually, $A^+$ exists and $A^+ = \left(\begin{smallmatrix} 1 & Pb^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. An easy argument involving facts (2) and (3) and their duals gives that (HR) and (HL) hold.

The next lemma enables us to distinguish the monoids $M$ for which $M \diamond N$ is weakly right $E$-hedged.

**Lemma 3.2.** *The monoid $M \diamond N$ satisfies (CR), that is, $\widetilde{\mathcal{L}}_E$ is a right congruence, if and only if $M$ is left cancellative.*

**Proof** Suppose first that $M$ is left cancellative. Let $A = \left(\begin{smallmatrix} a & P \\ 0 & b \end{smallmatrix}\right), B = \left(\begin{smallmatrix} a' & P' \\ 0 & b' \end{smallmatrix}\right) \in M \diamond N$ where $A \;\widetilde{\mathcal{L}}_E\; B$ and let $C = \left(\begin{smallmatrix} m & Q \\ 0 & n \end{smallmatrix}\right) \in M \diamond N$. From Lemma 3.1, $a^{-1}P = (a')^{-1}P'$. We wish to show that $AC \;\widetilde{\mathcal{L}}_E\; BC$, which is equivalent to

$$(am)^{-1}(aQ \cup Pn) = (a'm)^{-1}(a'Q \cup P'n).$$

Now using fact (3),

$$(am)^{-1}(aQ \cup Pn) = (am)^{-1}(aQ) \cup (am)^{-1}(Pn).$$

By the comment following fact (7), $(am)^{-1}(aQ) = m^{-1}Q$ and by (5) and the dual of (6), $(am)^{-1}(Pn) = m^{-1}(a^{-1}(Pn)) = m^{-1}((a^{-1}P)n)$. But $a^{-1}P = (a')^{-1}P'$ so that

$$\begin{aligned}(am)^{-1}(aQ \cup Pn) &= m^{-1}Q \cup m^{-1}(((a')^{-1}P')n) \\ &= \cdots = (a'm)^{-1}(a'Q \cup P'n).\end{aligned}$$

Thus (CR) holds.

Conversely, suppose that $M$ is not left cancellative. Choose $a, x, y \in M$ with $ax = ay$ but $x \neq y$. Put $A = \left(\begin{smallmatrix} a & \emptyset \\ 0 & 1 \end{smallmatrix}\right), I = \left(\begin{smallmatrix} 1 & \emptyset \\ 0 & 1 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 1 & \{(x,1)\} \\ 0 & 1 \end{smallmatrix}\right)$.

As $a^{-1}\emptyset = 1^{-1}\emptyset$ we have $A \;\widetilde{\mathcal{L}}_E\; I$. But $AC = \left(\begin{smallmatrix} a & \{(ax,1)\} \\ 0 & 1 \end{smallmatrix}\right)$ and $(y, 1) \in a^{-1}\{(ax, 1)\} \setminus 1^{-1}\{(x, 1)\}$ so that $AC$ is not $\widetilde{\mathcal{L}}_E$-related to $IC = C$. Thus (CR) fails.

**Corollary 3.3.** *The monoid $M \diamond N$ is weakly right $E$-hedged if and only if $M$ is left cancellative.*

Recall that a semigroup $S$ is *unipotent* if it contains exactly one idempotent.

**Lemma 3.4.** *The idempotents of $M \diamond N$ form a semilattice if and only if $M$ and $N$ are unipotent. Moreover, in this case, $E(M \diamond N) = E$.*

**Proof** If $M$ and $N$ are unipotent, then $E(M \diamond N)$ is the semilattice $E$. For the converse, suppose that $e$ is a non-identity idempotent in $N$. It is easy to check that $\left(\begin{smallmatrix} 1 & \{(1,1),(1,e)\} \\ 0 & e \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \{(1,e)\} \\ 0 & e \end{smallmatrix}\right)$ are non-commuting idempotents. A similar argument works for $M$.

Lemma 3.4 shows that (2) implies (1) in the next result. Following the now standard pattern of terminology, a semigroup is *right $E$-hedged* if it is right $E$-adequate and satisfies condition (HR).

**Proposition 3.5.** *The following conditions are equivalent :*
*(1) $M \diamond N$ is right $E$-hedged;*
*(2) $M \diamond N$ is right hedged;*
*(3) $M$ and $N$ are left cancellative monoids.*

**Proof** (1) $\Rightarrow$ (3) If $M \diamond N$ is right $E$-hedged then, as noted in Section 2, $\widetilde{\mathcal{L}}_E = \mathcal{L}^*$ so that $\widetilde{\mathcal{L}}_E$ is a right congruence. Lemma 3.2 gives that $M$ is left cancellative. If $p, q, r \in N$ with $pq = pr$, then $A = \left(\begin{smallmatrix} 1 & \emptyset \\ 0 & p \end{smallmatrix}\right), X = \left(\begin{smallmatrix} 1 & \emptyset \\ 0 & q \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 1 & \emptyset \\ 0 & r \end{smallmatrix}\right)$ are elements of $M \diamond N$ with $AX = AY$. Since $A \,\mathcal{L}^*\, A^*$ we have $A^*X = A^*Y$. Now $A^*$ is the identity of $M \diamond N$ so that $X = Y$ and $q = r$. Thus $N$ is left cancellative.

(3) $\Rightarrow$ (2) As $M$ and $N$ are unipotent, $E = E(M \diamond N)$. Let $A = \left(\begin{smallmatrix} m & P \\ 0 & n \end{smallmatrix}\right) \in M \diamond N$; we must show that $A\mathcal{L}^*A^*$ where $A^* = \left(\begin{smallmatrix} 1 & m^{-1}P \\ 0 & 1 \end{smallmatrix}\right)$. Since $AA^* = A$ it is enough to show that for any $X, Y \in M \diamond N$, if $AX = AY$ then $A^*X = A^*Y$. Let $X = \left(\begin{smallmatrix} x & Q \\ 0 & y \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} x' & Q' \\ 0 & y' \end{smallmatrix}\right)$ be such that $AX = AY$. Then

$$\begin{pmatrix} mx & mQ \cup Py \\ 0 & ny \end{pmatrix} = \begin{pmatrix} mx' & mQ' \cup Py' \\ 0 & ny' \end{pmatrix}$$

so that $mx = mx', mQ \cup Py = mQ' \cup Py'$ and $ny = ny'$. As $M$ and $N$ are left cancellative we obtain $x = x'$ and $y = y'$. To show that $A^*X = A^*Y$ we must show that $Q \cup (m^{-1}P)y = Q' \cup (m^{-1}P)y'$. From $mQ \cup Py = mQ' \cup Py'$ we have, using fact (7) and the dual of fact (6), that

$$Q \cup (m^{-1}P)y = m^{-1}(mQ) \cup m^{-1}(Py) = m^{-1}(mQ \cup Py) =$$

$$m^{-1}(mQ' \cup Py') = \cdots = Q' \cup (m^{-1}P)y'$$

as required.

We now consider the conditions under which $M \diamond N$ is weakly right $E$-ample.

**Proposition 3.6.** *The monoid $M \diamond N$ is weakly right $E$-ample if and only if $M$ is a group.*

**Proof** Suppose first that $M$ is a group. By Corollary 3.3 the monoid $M \diamond N$ is weakly right $E$-hedged, it remains to show that (AR) holds.

Using the fact that $M$ is a group it is easy to check that for any $m \in M$ and $P \subseteq M \times N$, $m(m^{-1}P) = P$. Let $F = \left(\begin{smallmatrix} 1 & P \\ 0 & 1 \end{smallmatrix}\right) \in E$ and $A = \left(\begin{smallmatrix} m & Q \\ 0 & n \end{smallmatrix}\right) \in M \diamond N$. Then $FA = \left(\begin{smallmatrix} m & Q\cup Pn \\ 0 & n \end{smallmatrix}\right)$ so that

$$A(FA)^* = \begin{pmatrix} m & Q \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & m^{-1}(Q \cup Pn) \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} m & m(m^{-1}(Q \cup Pn)) \cup Q \\ 0 & n \end{pmatrix} = \begin{pmatrix} m & Q \cup Pn \\ 0 & n \end{pmatrix}$$

so that $A(FA)^* = FA$.

Conversely, if $M \diamond N$ is weakly right $E$-ample then $M$ is left cancellative by Lemma 2.9 and Corollary 3.3. Suppose that $M$ contains an element $a$ which lacks a right inverse. Put $G = \left(\begin{smallmatrix} 1 & \{(a,1)\} \\ 0 & 1 \end{smallmatrix}\right) \in E$ and $B = \left(\begin{smallmatrix} a^2 & \{(1,1)\} \\ 0 & 1 \end{smallmatrix}\right) \in M \diamond N$. Now $GB = \left(\begin{smallmatrix} a^2 & \{(1,1),(a,1)\} \\ 0 & 1 \end{smallmatrix}\right)$ so that $(GB)^* = \left(\begin{smallmatrix} 1 & (a^2)^{-1}\{(1,1),(a,1)\} \\ 0 & 1 \end{smallmatrix}\right)$. If $(x, y) \in (a^2)^{-1}\{(1, 1), (a, 1)\}$ then $a^2(x, y) = (1, 1)$ or $(a, 1)$ so that $a^2x = 1$ or $a^2x = a$. Since $M$ is left cancellative, if $a^2x = a$ then $ax = 1$, so that in either case $a$ has a right inverse. Hence $(a^2)^{-1}\{(1, 1), (a, 1)\} = \emptyset$. Thus $B(GB)^* = B \neq GB$, contradicting the fact that $M \diamond N$ satisfies (AR). Thus *every* element of $M$ has a right inverse. Consequently, the monoid $M$ is a group.

Propositions 3.5 and 3.6 yield

**Corollary 3.7.** *The monoid $M \diamond N$ is right $E$-ample if and only if $M$ is a group and $N$ is left cancellative.*

Of course, the left-right duals of Lemma 3.2, Corollary 3.3, Propositions 3.5 and 3.6 and Corollary 3.7 hold. In particular, we have the equivalence of the first two conditions of the following result. That the third condition follows from the first was noted by Margolis and Pin [MP, Proposition 1.1].

**Corollary 3.8.** *For monoids $M$ and $N$ the following conditions are equivalent.*

*(1) $M$ and $N$ are groups.*
*(2) $M \diamond N$ is weakly $E$-ample.*
*(3) $M \diamond N$ is an inverse monoid.*

## 4. The semigroup $F_E$

Recall that an inverse semigroup is fundamental if the largest congruence contained in $\mathcal{H}$ is trivial and an adequate semigroup is fundamental if the largest congruence contained in $\mathcal{H}^*$ is trivial. Accordingly, we define an $E$-semiadequate semigroup to be $E$-*fundamental* if $\mu_E$ is trivial.

The aim of this section is to construct from any given semilattice $E$ an $E$-fundamental weakly $E$-hedged semigroup $F_E$ that is maximal in the sense that if $S$ is any weakly $E$-hedged semigroup then there is a homomorphism $\theta : S \to F_E$ with kernel $\mu_E$. As we see below, $E(F_E) \neq E$ and $E(F_E)$ is not a semilattice (unless $E$ is trivial), so that $F_E$ is not weakly hedged.

Let $E$ be a semilattice and let $F_E$ be the subset of $\mathrm{End}_1 E^1 \times \mathrm{End}_1^* E^1$ given by

$$F_E = \{(\alpha, \beta) : \rho_{1\beta} \leq \alpha\beta, \rho_{1\alpha} \leq \beta\alpha\}.$$

In particular, if $(\alpha, \beta) \in F_E$ then

$$1\alpha = 1\rho_{1\alpha} \leq 1\beta\alpha \leq 1\alpha$$

so that $1\alpha = 1\beta\alpha$ and dually, $1\beta = 1\alpha\beta$. Thus $\alpha$ maps the maximum element of im $\beta$ to the maximum element of im $\alpha$, and $\beta$ maps the maximum element of im $\alpha$ to the maximum element of im $\beta$.

Observe first that $F_E \neq \emptyset$, since for any $e \in E$, $\overline{e} = (\rho_e, \rho_e) \in F_E$. Denoting by $c_e$ the constant map $E^1 \to E$ with image $\{e\}$ we also have that for any $e, f \in E$, $(c_f, c_e) \in F_E$. Note $(c_f, c_e) \in E(F_E)$ and $(c_f, c_e)(c_e, c_f) \neq (c_e, c_f)(c_f, c_e)$ unless $e = f$. This also illustrates that the image of $\alpha$ where $(\alpha, \beta) \in F_E$ *need not* be a principal ideal.

**Lemma 4.1.** *If $(\alpha, \beta) \in F_E$ then $\rho_{1\beta}\alpha = \alpha$ and $\rho_{1\alpha}\beta = \beta$.*

**Proof** For all $x \in E^1$,

$$x(\rho_{1\beta}\alpha) = (x \cdot 1\beta)\alpha = (x\alpha)(1\beta\alpha) = (x\alpha)(1\alpha) = x\alpha$$

so that $\rho_{1\beta}\alpha = \alpha$ and dually, $\rho_{1\alpha}\beta = \beta$.

**Lemma 4.2.** *The set $F_E$ is a subsemigroup of $\mathrm{End}_1 E^1 \times \mathrm{End}_1^* E^1$.*

**Proof** Let $(\alpha, \beta), (\gamma, \delta) \in F_E$. Then for any $x \in E^1$,

$$\begin{aligned} x(\alpha\gamma)(\delta\beta) = (x\alpha)(\gamma\delta)\beta \;&\geq\; ((x\alpha)(1\delta))\beta = (x\alpha\beta)(1\delta\beta) \\ &\geq\; (x \cdot 1\beta)(1\delta\beta) = x \cdot 1\delta\beta = x\rho_{1\delta\beta} \end{aligned}$$

so that $(\alpha\gamma)(\delta\beta) \;\geq\; \rho_{1\delta\beta}$ and dually, $(\delta\beta)(\alpha\gamma) \;\geq\; \rho_{1\alpha\gamma}$.

We put $\overline{E} = \{\overline{e} : e \in E\}$; it is easy to see that $e \mapsto \overline{e}$ is an isomorphism between $E$ and $\overline{E}$.

**Lemma 4.3.** *The semigroup $F_E$ is $\overline{E}$-semiadequate. If $(\alpha, \beta) \in F_E$ then*

$$(\alpha, \beta)^* = (\rho_{1\alpha}, \rho_{1\alpha}) \text{ and } (\alpha, \beta)^+ = (\rho_{1\beta}, \rho_{1\beta}).$$

**Proof** If $(\alpha, \beta) \in F_E$, then using Lemma 4.1,

$$(\alpha, \beta)(\rho_{1\alpha}, \rho_{1\alpha}) = (\alpha\rho_{1\alpha}, \rho_{1\alpha}\beta) = (\alpha, \beta).$$

Now for any $e \in E$, if $(\alpha, \beta)(\rho_e, \rho_e) = (\alpha, \beta)$ then certainly $\alpha\rho_e = \alpha$ so that $1\alpha = 1(\alpha\rho_e) = (1\alpha)\rho_e = (1\alpha)e$, giving $1\alpha \leq e$. Thus $\overline{1\alpha} \leq \overline{e}$ and $(\alpha, \beta)^*$ exists and equals $(\rho_{1\alpha}, \rho_{1\alpha})$. Dually, $(\alpha, \beta)^+$ exists and equals $(\rho_{1\beta}, \rho_{1\beta})$.

**Lemma 4.4.** *The semigroup $F_E$ is weakly $\overline{E}$-hedged.*

**Proof** Let $(\alpha, \beta), (\gamma, \delta)$ be $\widetilde{\mathcal{L}}_{\overline{E}}$-related elements of $F_E$. By Lemma 4.3, $1\alpha = 1\gamma$. For any $(\zeta, \xi) \in F_E$ we have $(\alpha, \beta)(\zeta, \xi) = (\alpha\zeta, \xi\beta)$ and $(\gamma, \delta)(\zeta, \xi) = (\gamma\zeta, \xi\delta)$. Now $1\alpha\zeta = 1\gamma\zeta$ so that $(\alpha, \beta)(\zeta, \xi) \; \widetilde{\mathcal{L}}_{\overline{E}} \; (\gamma, \delta)(\zeta, \xi)$ and (CR) holds.

Still with $(\alpha, \beta) \in F_E$, let $(\rho_e, \rho_e), (\rho_f, \rho_f) \in \overline{E}$. Then

$$((\rho_e, \rho_e)(\rho_f, \rho_f)(\alpha, \beta))^* = ((\rho_{ef}, \rho_{ef})(\alpha, \beta))^* = (\rho_{ef}\alpha, \beta\rho_{ef})^*.$$

Now $1(\rho_{ef}\alpha) = (ef)\alpha = e\alpha f\alpha$ so that

$$((\rho_e, \rho_e)(\rho_f, \rho_f)(\alpha, \beta))^* = (\rho_{e\alpha f\alpha}, \rho_{e\alpha f\alpha}) =$$

$$(\rho_{e\alpha}, \rho_{e\alpha})(\rho_{f\alpha}, \rho_{f\alpha}) = ((\rho_e, \rho_e)(\alpha, \beta))^*((\rho_f, \rho_f)(\alpha, \beta))^*.$$

Thus (HR) holds. Together with the dual arguments this gives that $F_E$ is weakly $\overline{E}$-hedged.

**Theorem 4.5.** *Let $E$ be a semilattice. The semigroup $F_E$ is an $\overline{E}$-fundamental weakly $\overline{E}$-hedged semigroup. If $S$ is any weakly $E$-hedged semigroup then there is a homomorphism $\theta : S \to F_E$ such that $e\theta = \overline{e}$ for all $e \in E$ and ker $\theta = \mu_E$.*

**Proof** Let $(\alpha, \beta), (\gamma, \delta)$ be $\mu_{\overline{E}}$-related elements of $F_E$. Certainly then $(\alpha, \beta)^* = (\gamma, \delta)^*$ so that by Lemma 4.3, $1\alpha = 1\gamma$. From Lemma 2.5, $\alpha_{(\alpha,\beta)} = \alpha_{(\gamma,\delta)}$ so that for any $e \in E$,

$$((\rho_e, \rho_e)(\alpha, \beta))^* = \overline{e}\alpha_{(\alpha,\beta)} = \overline{e}\alpha_{(\gamma,\delta)} = ((\rho_e, \rho_e)(\gamma, \delta))^*.$$

Again by Lemma 4.3, we have that for any $e \in E, 1(\rho_e\alpha) = 1(\rho_e\gamma)$ so that $e\alpha = e\gamma$. Together with $1\alpha = 1\gamma$ this gives that $\alpha = \gamma$. The dual argument yields that $\beta = \delta$ so that $\mu_{\overline{E}}$ is trivial and $F_E$ is $\overline{E}$-fundamental.

Let $S$ be weakly $E$-hedged and let $\theta$ be the homomorphism defined in Section 2. For any $e \in E$ we have $e\theta = (\alpha_e, \beta_e) = \overline{e}$. It only remains to show that im $\theta \subseteq F_E$. Let $a \in S$ so that $a\theta = (\alpha_a, \beta_a)$. We have

$$1\alpha_a = (1a)^* = a^*, \; 1\beta_a = (a1)^+ = a^+.$$

Using Lemma 2.4(2)

$$\rho_{1\beta_a} = \rho_{a^+} \ \leq \ \alpha_a\beta_a \text{ and } \rho_{1\alpha_a} = \rho_{a^*} \ \leq \ \beta_a\alpha_a.$$

Hence $(\alpha_a, \beta_a) \in F_E$ and im $\theta \ \subseteq \ F_E$ as required.

We end this section by showing that for any semilattice $E$, the Munn semigroup $T_E$ is embedded in $F_E$. Recall that the elements of $T_E$ are *partial* isomorphisms between principal ideals of $E$ and the operation in $T_E$ is composition of partial mappings. The idempotents of $T_E$ are the identity maps on principal ideals of $T_E$. For each $e \in E$ we put $\overline{\overline{e}} = I_{eE}$ so that $\overline{\overline{E}} = \{\overline{\overline{e}} : e \in E\} \cong E$ is the semilattice of idempotents of $T_E$.

The following lemma is straightforward to check.

**Lemma 4.6.** *Let $E$ be a semilattice. For $(\alpha, \beta) \in F_E$, put*

$$\overline{\alpha} = \alpha|_{(1\beta)E} \text{ and } \overline{\beta} = \beta|_{(1\alpha)E}.$$

*Then $\overline{\alpha} : (1\beta)E \to (1\alpha)E$ and $\overline{\beta} : (1\alpha)E \to (1\beta)E$ are homomorphisms such that*

$$I_{(1\beta)E} \leq \overline{\alpha}\overline{\beta}, I_{(1\alpha)E} \leq \overline{\beta}\overline{\alpha}.$$

*Conversely, if $e, f \in E$ and $\overline{\gamma} : eE \to fE, \overline{\delta} : fE \to eE$ are homomorphisms such that*

$$I_{eE} \leq \overline{\gamma}\overline{\delta}, I_{fE} \leq \overline{\delta}\overline{\gamma},$$

*then $(\rho_e\overline{\gamma}, \rho_f\overline{\delta}) \in F_E$.*

Let $E$ be a semilattice and let $\psi \in T_E$, so that $\psi : eE \to fE$ is an isomorphism of principal ideals $eE$ and $fE$ of $E$. By Lemma 4.6, $(\rho_e\psi, \rho_f\psi^{-1}) \in F_E$ and we define $\pi : T_E \to F_E$ by $\psi\pi = (\rho_e\psi, \rho_f\psi^{-1})$.

**Proposition 4.7.** *The function $\pi : T_E \to F_E$ is an embedding.*

**Proof** Let $\psi : eE \to fE$ and $\xi : gE \to hE$ be isomorphisms in $T_E$. The composition of partial mappings $\psi$ and $\xi$ yields the isomorphism $\psi\xi$ between principal ideals $(fg)\psi^{-1}E$ and $(fg)\xi E$. Thus

$$(\psi\xi)\pi = (\rho_{(fg)\psi^{-1}}\psi\xi, \rho_{(fg)\xi}(\psi\xi)^{-1})$$

and we must show that this is equal to

$$(\rho_e\psi, \rho_f\psi^{-1})(\rho_g\xi, \rho_h\xi^{-1}) = (\rho_e\psi\rho_g\xi, \rho_h\xi^{-1}\rho_f\psi^{-1}).$$

Let $x \in E^1$. Then

$$x\rho_{(fg)\psi^{-1}}\psi\xi = (x(fg)\psi^{-1})\psi\xi = (xe(fg)\psi^{-1})\psi\xi =$$
$$= ((xe)\psi fg)\xi = ((xe)\psi g)\xi = x\rho_e\psi\rho_g\xi.$$

Dually, $\rho_{(fg)\xi}(\psi\xi)^{-1} = (\rho_h\xi^{-1})(\rho_f\psi^{-1})$ so that $\pi$ is a homomorphism.

To see that $\pi$ is one-one, let $\psi, \xi$ be as above and suppose that $\psi\pi = \xi\pi$. Then $(\rho_e\psi, \rho_f\psi^{-1}) = (\rho_g\xi, \rho_h\xi^{-1})$, giving

$$e = f\psi^{-1} = 1\rho_f\psi^{-1} = 1\rho_h\xi^{-1} = h\xi^{-1} = g.$$

Now for all $x \in E$,

$$(ex)\psi = x\rho_e\psi = x\rho_g\xi = (xg)\xi = (ex)\xi$$

so that $\psi = \xi$ as required.

As a consequence of Proposition 5.3, if $S$ is an inverse semigroup with semilattice of idempotents $E$, then $\phi\pi = \theta$, where $\phi : S \to T_E$ is the standard homomorphism from $S$ to $T_E$ and $\theta$ is the homomorphism from $S$ to $F_E$ given in Theorem 4.5.

## 5. Weakly $E$-ample semigroups

If $S$ is a weakly $E$-hedged semigroup, then as shown in the previous section, there is a homomorphism $\theta : S \to F_E$ with $\ker \theta = \mu_E$. We also know that for some classes of weakly $E$-hedged semigroups, namely those that are inverse [M], ample [F1] or weakly ample [E], we can dispense with consideration of *pairs* of endomorphisms of $E^1$ and make use of isomorphisms between principal ideals of $E$, in other words we look at $T_E$. This is essentially because the endomorphisms $\alpha_a, \beta_a$ of $E^1$ arising from an element $a$ of a weakly ample semigroup $S$ are mutually inverse when restricted to the domains $a^+E, a^*E$ respectively. The corresponding result is true for *weakly* $E$-ample semigroups, as we now show. At this point we recall some notation introduced in the previous section: if $S$ is weakly $E$-hedged and $a \in S$, so that $(\alpha_a, \beta_a) \in F_E$, put $\overline{\alpha_a} = \alpha_a|_{(1\beta_a)E}$ and $\overline{\beta_a} = \beta_a|_{(1\alpha_a)E}$. Now $1\alpha_a = a^*$ and $1\beta_a = a^+$, so that in view of Lemmas 2.2, 2.4 and their duals,

$$\overline{\alpha_a} = \alpha_a|_{a^+E} : a^+E \to a^*E$$

and

$$\overline{\beta_a} = \beta_a|_{a^*E} : a^*E \to a^+E.$$

**Lemma 5.1.** *Let $S$ be a weakly $E$-hedged semigroup. Then the following conditions are equivalent:*

*(1) $S$ is weakly $E$-ample;*

*(2) for all $a \in S, \overline{\alpha_a}$ and $\overline{\beta_a}$ are one-one;*

*(3) for all $a \in S, \overline{\alpha_a}$ and $\overline{\beta_a}$ are inverse isomorphisms.*

**Proof** Suppose first that $S$ is weakly $E$-ample. Let $x, y \in a^+E$ and suppose that $x\overline{\alpha_a} = y\overline{\alpha_a}$. Thus $(xa)^* = (ya)^*$ and using the fact that $S$ satisfies condition (AR),

$$xa = a(xa)^* = a(ya)^* = ya.$$

Now

$$xa^+ \, \widetilde{\mathcal{R}}_E \, xa = ya \, \widetilde{\mathcal{R}}_E \, ya^+$$

so that $xa^+ = ya^+$ and as $x, y \leq a^+$ we deduce $x = y$. Hence $\overline{\alpha_a}$ is one-one; the dual argument works for $\overline{\beta_a}$.

The proof of (2) implies (3) and (3) implies (1) is the same as that given in Proposition 4.4 of [F1].

Our next result follows closely Proposition 4.5 of [F1]. As remarked in the introduction, this also appears in [E] and [EF].

**Proposition 5.2.** [EF] *Let $S$ be a weakly $E$-ample semigroup. Define $\phi : S \to T_E$ by $a\phi = \overline{\alpha_a}$. Then $\phi$ is a homomorphism onto a full subsemigroup of $T_E$ with ker $\phi = \mu_E$ and $e\phi = I_{eE} = \overline{\overline{e}}$ for each $e \in E$.*

**Proof** If $a \in S$ then by Lemma 5.1

$$\overline{\alpha_a} : a^+E \to a^*E \text{ and } \overline{\beta_a} : a^*E \to a^+E$$

are inverse isomorphisms. Exactly as in [F1], if $b \in S$ the domain of $\overline{\alpha_a}\,\overline{\alpha_b}$ is $(ab)^+E$, that is, the domain of $\overline{\alpha_{ab}}$. Lemma 2.3 now gives that $\phi$ is a homomorphism. Clearly $e\phi = \rho_e|_{eE} = \overline{\overline{e}}$, and so $E\phi = \overline{\overline{E}}$ and im $\phi$ is full.

Suppose now that $a, b \in S$ and $a\phi = b\phi$ so that $\overline{\alpha_a} = \overline{\alpha_b}$. Thus $\overline{\alpha_a}$ and $\overline{\alpha_b}$ have the same domains $a^+E = b^+E$ and the same images $a^*E = b^*E$. Hence $a^+ = b^+$ and $a^* = b^*$, giving that $a \, \widetilde{\mathcal{H}}_E \, b$. We also have that $(\overline{\alpha_a})^{-1} = (\overline{\alpha_b})^{-1}$ and so $\overline{\beta_a} = \overline{\beta_b}$. From Lemma 2.5, $a\mu_E b$ so that ker $\phi \subseteq \mu_E$. The opposite inclusion follows immediately from the same lemma.

We now connect the two representations of a weakly $E$-ample semigroup.

**Proposition 5.3.** *Let $S$ be a weakly $E$-ample semigroup. For the homomorphisms $\theta : S \to F_E$, $\pi : T_E \to F_E$ and $\phi : S \to T_E$ defined above, we have $\phi\pi = \theta$.*

**Proof** Let $a \in S$ so that $a\phi = \overline{\alpha_a} : a^+E \to a^*E$. The prescription for $\pi$ given in Section 4 says that $(a\phi)\pi = (\rho_{a^+}\overline{\alpha_a}, \rho_{a^*}(\overline{\alpha_a})^{-1})$ so that by Lemma 5.1, $(a\phi)\pi = (\rho_{a^+}\overline{\alpha_a}, \rho_{a^*}\overline{\beta_a})$. For any $x \in E^1$,

$$x\rho_{a^+}\overline{\alpha_a} = (xa^+)\overline{\alpha_a} = (xa^+a)^* = (xa)^* = x\alpha_a$$

so that $\rho_{a^+}\overline{\alpha_a} = \alpha_a$ and dually, $\rho_{a^*}\overline{\beta_a} = \beta_a$. Hence $a\phi\pi = (\alpha_a, \beta_a) = a\theta$ and $\phi\pi = \theta$ as required.

## 6. Some applications

The aim of this section is to apply the material developed in Sections 4 and 5 to deduce some facts concerning weakly $E$-hedged and weakly $E$-ample semigroups.

**Lemma 6.1.** *Let $S$ be an $E$-semiadequate semigroup and let $T$ be a subsemigroup of $S$ containing $E$. Then*
*(1) $T$ is $E$-semiadequate;*
*(2) if $S$ satisfies the congruence condition then so does $T$;*
*(3) if $S$ is weakly $E$-hedged then so is $T$;*
*(4) if $S$ is weakly $E$-ample then so is $T$;*
*(5) if $S$ satisfies the congruence condition and is $E$-fundamental, then so is $T$.*

**Proof** Note that the restriction of the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ in $S$ to $T$ are respectively the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ in $T$. The first four statements are then clear. It follows from Lemma 2.5 that if $S$ satisfies the congruence condition then the restriction of the congruence $\mu_E$ on $S$ to $T$ is the congruence $\mu_E$ on $T$. Thus if $S$ is also $E$-fundamental, then so is $T$.

Let $S$ be a weakly $E$-hedged semigroup. Since by definition $\mu_E$ is contained in $\widetilde{\mathcal{H}}_E$, $\mu_E$ is idempotent separating, so that the set of idempotents $E\mu_E = \{e\mu_E : e \in E\}$ is a subsemilattice of $S/\mu_E$ isomorphic to $E$.

**Corollary 6.2.** *Let $S$ be a weakly $E$-hedged semigroup. Then $S/\mu_E$ is an $E\mu_E$-fundamental weakly $E\mu_E$-hedged semigroup. Further, if $S$ is weakly $E$-ample, then $S/\mu_E$ is weakly $E\mu_E$-ample.*

**Proof** Theorem 4.5 says there is a homomorphism $\theta : S \to F_E$ such that $e\theta = \overline{e}$ for all $e \in E$ and $\ker \theta = \mu_E$. Thus there is a one-one homomorphism $\overline{\theta} : S/\mu_E \to F_E$ such that $(e\mu_E)\overline{\theta} = \overline{e}$. Also by Theorem 4.5, $F_E$ is $\overline{E}$-fundamental so that by Lemma 6.1, $(S/\mu_E)\overline{\theta}$ is $\overline{E}$-fundamental weakly $\overline{E}$-hedged. Hence $S/\mu_E$ is an $E\mu_E$-fundamental, weakly $E\mu_E$-hedged semigroup.

Suppose now that $S$ is weakly $E$-ample. Using Proposition 5.2 in place of Theorem 4.5, there is a one-one homomorphism $\overline{\phi} : S/\mu_E \to T_E$ such that $(e\mu_E)\overline{\phi} = \overline{\overline{e}}$ for each $e \in E$. The inverse semigroup $T_E$ has semilattice of idempotents $\overline{\overline{E}}$. Being inverse, $T_E$ is weakly $(\overline{\overline{E}})$-ample. The result now follows from Lemma 6.1.

In order to state our next two corollaries we introduce some useful terminology. We say that a homomorphism (isomorphism) $\nu$ from a weakly $E$-hedged semigroup $S$ to a subsemigroup of $F_E$ or $T_E$ is an *$E$-homomorphism* (*$E$-isomorphism*) if $e\nu = \overline{e}$ or $e\nu = \overline{\overline{e}}$ for each $e \in E$.

**Corollary 6.3.** *A weakly $E$-hedged semigroup $S$ is $E$-fundamental if and only if it is $E$-isomorphic to a subsemigroup of $F_E$.*

**Proof** If $S$ is $E$-fundamental, then $\mu_E$ is trivial on $S$, so that the $E$-homomorphism $\theta$ given in Theorem 4.5 is an embedding.

Conversely, suppose that $\nu : S \to F_E$ is a one-one $E$-homomorphism. As above, im $\nu$ is $\overline{E}$-fundamental weakly $\overline{E}$-hedged, so that $S$ is $E$-fundamental.

The proof of the following corollary is analogous to that of Corollary 6.3.

**Corollary 6.4.** *A weakly $E$-ample semigroup $S$ is $E$-fundamental if and only if it is $E$-isomorphic to a subsemigroup of $T_E$. Consequently, if $S$ is an $E$-fundamental weakly $E$-ample semigroup, then $E = E(S)$.*

Recall that a semilattice $E$ is *anti-uniform* if $eE \cong fE$ implies $e = f$. The definition of an *$E$-semilattice of monoids* is given in Section 2. Corollary 6.5 is analogous to Corollary 4.9 of [F1], which is concerned with ample semigroups.

**Corollary 6.5.** *A semilattice $E$ has the property that every weakly $E$-ample semigroup is an $E$-semilattice of monoids if and only if $E$ is anti-uniform.*

**Proof** If $E$ is anti-uniform and $S$ is weakly $E$-ample, then by Lemma 5.1, $\overline{\alpha_a} : a^+E \to a^*E$ is an isomorphism. Hence $a^+ = a^*$ so that by Proposition 2.6, $S$ is an $E$-semilattice of monoids.

Conversely, suppose that $E$ is *not* anti-uniform. As in Theorem **V**.5.2 of [Ho1], $T_E$ is *not* a semilattice of groups; neither then can $T_E$ be an $\overline{\overline{E}}$-semilattice of monoids. But as previously remarked, $T_E$ is weakly $\overline{\overline{E}}$-ample.

Imposing the condition that every weakly $E$-hedged semigroup is an $E$-semilattice of monoids emerges as a much stronger restriction.

**Corollary 6.6.** *A semilattice $E$ has the property that every weakly $E$-hedged semigroup is an $E$-semilattice of monoids if and only if $E$ is trivial.*

**Proof** If $E = \{e\}$ is trivial and $S$ is weakly $E$-hedged, then $S$ is a monoid with identity $e$.

Conversely, suppose that every weakly $E$-hedged semigroup is an $E$-semilattice of monoids. According to Proposition 2.6, if $S$ is a weakly $E$-hedged semigroup, then $a^* = a^+$ for all $a \in S$. In particular, $(\alpha, \beta)^* = (\alpha, \beta)^+$ for all $(\alpha, \beta) \in F_E$.

Let $e, f \in E$. As remarked at the beginning of Section 4, $(c_f, c_e) \in F_E$ where $c_e(c_f)$ is the constant map with image $\{e\}$ $(\{f\})$. By Lemma 4.3,

$$(c_f, c_e)^* = (\rho_{1c_f}, \rho_{1c_f}) = (\rho_f, \rho_f)$$

and

$$(c_f, c_e)^+ = (\rho_{1c_e}, \rho_{1c_e}) = (\rho_e, \rho_e)$$

so that $(\rho_f, \rho_f) = (\rho_e, \rho_e)$ and $e = f$. Thus $E$ is trivial.

We end this paper by considering those semilattices $E$ having the property that $\widetilde{\mathcal{H}}_E$ is a congruence on every weakly $E$-ample (-hedged) semigroup. Again there is a sharp split between the two cases.

Recall that a semilattice $E$ is *rigid* if for each $e \in E$ there is only one automorphism of $eE$. Equivalently, there is at most one isomorphism between $eE$ and $fE$ for each pair $e, f \in E$. Corollary 6.7 is analogous to Corollary 4.10 of [F1].

**Corollary 6.7.** *A semilattice $E$ has the property that $\widetilde{\mathcal{H}}_E$ is a congruence on every weakly $E$-ample semigroup if and only if $E$ is rigid.*

**Proof** If $\widetilde{\mathcal{H}}_E$ is a congruence on every weakly $E$-ample semigroup then $\widetilde{\mathcal{H}}_{\overline{E}} = \mathcal{H}^* = \mathcal{H}$ is a congruence on $T_E$. Thus $T_E$ is $\mathcal{H}$-trivial and it is well known that in this case $E$ is rigid.

Conversely, suppose that $E$ is rigid and $S$ is a weakly $E$-ample semigroup. If $a, b \in S$ and $a \,\widetilde{\mathcal{H}}_E\, b$ then $a^* = b^*$ and $a^+ = b^+$, so that $\overline{\alpha_a}, \overline{\alpha_b}$ are both isomorphisms between $a^+E$ and $a^*E$. Since $E$ is rigid we have that $\overline{\alpha_a} = \overline{\alpha_b}$. Consequently, $\overline{\beta_a} = \overline{\beta_b}$ also and $a \mu_E b$ by Lemma 2.5. Thus $\mu_E = \widetilde{\mathcal{H}}_E$ and $\widetilde{\mathcal{H}}_E$ is a congruence on $S$.

**Corollary 6.8.** *A semilattice $E$ has the property that $\widetilde{\mathcal{H}}_E$ is a congruence on every weakly $E$-hedged semigroup if and only if $E$ is trivial.*

**Proof** If $E$ is trivial and $S$ is weakly $E$-hedged, then $\widetilde{\mathcal{H}}_E$ is the universal congruence on $S$.

Conversely, suppose that $\widetilde{\mathcal{H}}_E$ is a congruence on every weakly $E$-hedged semigroup. Certainly then $\widetilde{\mathcal{H}}_{\overline{E}}$ is a congruence on $F_E$ so that $\mu_{\overline{E}} = \widetilde{\mathcal{H}}_{\overline{E}}$ on $F_E$ and $\widetilde{\mathcal{H}}_{\overline{E}}$ is the trivial congruence.

Let $e \in E$. Consider the endomorphisms $\rho_e$ and $c_e$ of $E^1$. Then $1\rho_e = 1c_e = e$ and for any $x \in E^1$,

$$xc_e\rho_e = x\rho_e c_e = e \geq xe = x\rho_e.$$

Thus $(\rho_e, c_e)$ and $(c_e, \rho_e)$ are elements of $F_E$. By Lemma 4.3, $(\rho_e, c_e)\ \widetilde{\mathcal{H}}_{\overline{E}}$ $(c_e, \rho_e)$ so that $(\rho_e, c_e) = (c_e, \rho_e)$. For any $y \in eE, y = y\rho_e = yc_e = e$

so that the ideal $eE$ is trivial. As this is true for *any* $e \in E$ it follows that $E$ is trivial.

## References


[BF] A. Batbedat and J. Fountain, 'Connections between left adequate semigroups and $\gamma$-semigroups', *Semigroup Forum* 22 (1981), 59-65.

[E] A. El-Qallali, 'Structure Theory for Abundant and Related Semigroups', D. Phil. thesis, York (1980).

[EF] A. El-Qallali and J. Fountain, Fundamental representations of semiadbundant semigroups, in preparation.

[F1] J. Fountain, 'Adequate semigroups', *Proc. Edinburgh Math. Soc.* 22 (1979), 113-125.

[F2] J. Fountain, 'Free right h-adequate semigroups', *Semigroups, theory and applications*, Lecture Notes in Mathematics 1320, Springer (1988), 97-120.

[G] V. Gould, 'Graph expansions of right cancellative monoids', *Int. J. Algebra Comp.*, to appear.

[Ha1] T.E. Hall, 'Orthodox semigroups', *Pacific J. Math.* 39 (1971), 677-686.

[Ha2] T.E. Hall, 'On regular semigroups', *J. Algebra* 24 (1973), 1-24.

[Ho1] J.M. Howie, *An Introduction to Semigroup Theory*, Academic Press (1976).

[Ho2] J.M. Howie, *Automata and Languages*, Oxford Science Publications (1991).

[L] M.V. Lawson, 'Semigroups and ordered categories I. The reduced case', *J. Algebra* 141 (1991), 422-462.

[M] W.D. Munn, 'Fundamental inverse semigroups', *Quart. J. Math, Oxford* 21 (1970), 157-170.

[MP] S.W. Margolis and J.E. Pin, 'Expansions, free inverse semigroups, and Schützenberger product', *J. Algebra* 110 (1987), 298-305.

[N] K.S.S. Nambooripad, 'Structure of regular semigroups I', *Mem. American Math. Soc.* 22 (1979), No. 224.



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