FUNDAMENTAL EHRESMANN SEMIGROUPS

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ABSTRACT. The celebrated construction by Munn of a fundamental inverse semigroup T_E from a semilattice E provides an important tool in the study of inverse semigroups. We present here a semigroup C_E that plays the T_E role for Ehresmann semigroups. Inverse semigroup are Ehresmann, as are those that are adequate, weakly ample or weakly hedged. We describe explicitly the semigroups C_E for some specific semilattices E and extract information relating to the corresponding classes of Ehresmann semigroups.

1. INTRODUCTION

To what extent is the structure of a semigroup S determined by a subset E of its set of idempotents? Of all the many possible approaches to this question we take the path first laid out by Munn. Munn considered *fundamental* inverse semigroups, that is, inverse semigroups having no non-trivial idempotent separating congruences. Munn [M] showed how an important fundamental inverse semigroup T_E could be constructed from any semilattice E, via partial isomorphisms of E. The *Munn semigroup* T_E of E has semilattice of idempotents isomorphic to E and is 'maximal' in the sense that an inverse semigroup S with semilattice of idempotents E is fundamental if and only if it is isomorphic to a full subsemigroup of T_E. Further, if S is an inverse semigroup with semilattice of idempotents E then there exists a homomorphism $\phi : S \to T_E$ whose kernel is μ, the maximum idempotent separating congruence on S [M].

In this paper we develop Munn's approach for a class of semigroups named *Ehresmann* by Lawson [L]. These semigroups arose from his study of the connection between semigroups and the classes of ordered small categories introduced by Ehresmann [EH]. Specifically, Theorem 4.24 of [L] states that the category of Ehresmann semigroups and admissible homomorphisms (that is, homomorphisms preserving a given

Date: June 5, 2000.
1991 *Mathematics Subject Classification.* 20 M 10.
Key words and phrases. semilattice, Munn semigroup, adequate, ample.
This work was supported by project PRAXIS 2/2.1/MAT/73.94.

unary operation,) is isomorphic to the category of Ehresmann categories and strongly ordered functors. This is analogous to the relation between inverse semigroups and inductive groupoids. From our standpoint, Ehresmann semigroups are arrived at via equivalence relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$, defined on a semigroup S containing a semilattice E as a subsemigroup. These relations contain Green's relations $\mathcal{L}$ and $\mathcal{R}$ and share some of their properties. Following the terminology of Lawson, we say that the pair (S, E) is an *Ehresmann semigroup* if every $\widetilde{\mathcal{L}}_E$-class and every $\widetilde{\mathcal{R}}_E$-class contains an idempotent and if, in addition, $\widetilde{\mathcal{L}}_E$ is a right congruence and $\widetilde{\mathcal{R}}_E$ is a left congruence. We remark that if S is inverse then $\widetilde{\mathcal{L}}_{E(S)} = \mathcal{L}$ and $\widetilde{\mathcal{R}}_{E(S)} = \mathcal{R}$, and clearly, $(S, E(S))$ is Ehresmann. Similarly, any adequate semigroup is Ehresmann. Further examples abound; we give details of these as we proceed.

For an Ehresmann semigroup (S, E) we denote by μ_E be the largest congruence contained in $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \cap \widetilde{\mathcal{R}}_E$. We say that (S, E) is *fundamental* if μ_E is trivial. The aim of this paper is to construct from a given semilattice E a semigroup C_E containing a semilattice of idempotents $\overline{E}$ isomorphic to E such that $(C_E, \overline{E})$ is fundamental Ehresmann. Further, for any Ehresmann semigroup (S, E), there is a homomorphism $\theta_E : S \to C_E$ such that θ_E restricts to an isomorphism from E to $\overline{E}$, and such that the kernel of θ_E is the relation μ_E on S. It follows that an Ehresmann semigroup (S, E) is fundamental if and only if θ_E is injective.

The structure of the paper is as follows.

In Section 2 we give details of the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ and collect together some preliminary results from [FGG]. We show that there is a homomorphism θ_E with kernel μ_E from an Ehresmann semigroup (S, E) to $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$, where $\mathcal{O}_1(E^1)$ consists of all order preserving functions from E^1 to E and here, as elsewhere, a * denotes the dual of a semigroup. Since μ_E is contained in $\widetilde{\mathcal{H}}_E$, it follows that θ_E is one-one on E. We write $\overline{E}$ for $E\theta_E$.

Section 3 contains the main results of the paper, namely the construction of a fundamental Ehresmann semigroup $(C_E, \overline{E})$ from a given semilattice E. The semigroup C_E is a subsemigroup of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$; as remarked above, the semilattice $\overline{E}$ is isomorphic to E. We show that the image of θ_E for any Ehresmann semigroup (S, E) is contained in C_E and consequently, (S, E) is fundamental if and only if θ_E is an embedding.

Section 4 concentrates on the special case of weakly E-hedged semigroups, which formed the topic of [FGG]. Weakly E-hedged semigroups are Ehresmann semigroups having the property that the image of θ_E

consists of pairs of *endomorphisms* of E. A fundamental weakly E-hedged semigroup F_E was contructed in [FGG]. We show that F_E is a subsemigroup of C_E and make some comparisons between F_E and C_E.

A discussion of the path that led us from inverse semigroups, through type A or ample semigroups [F1] and the weakly E-hedged and weakly E-ample semigroups of [FGG], to Ehresmann semigroups, is delayed until Section 5. By this point sufficient details are in place for a deliberation of the obstacles to progress to be meaningful.

In the final section we take some small semilattices E and describe explicitly the semigroups F_E and C_E. This enables us to find small examples of Ehresmann semigroups that do not fall into any of the classes of semigroups previously considered.

2. E-SEMIADEQUATE SEMIGROUPS AND THE MAPS α_a, β_a

Throughout this paper E denotes a semilattice, and E^1 denotes E with identity adjoined *if necessary.* If E is a commutative subsemigroup of idempotents of a semigroup S, we say simply 'E is a subsemilattice of S'. Note that we do not insist that E consist of all idempotents of S. However, in the case that $E = E(S)$ *we may omit mention of E from our definitions and statements.*

Ehresmann semigroups form a subclass of the class of E-semiadequate semigroups. The latter are approached via the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$, which we now define.

Let E be a subsemilattice of a semigroup S. For any $a \in S$ we put

$$a_E = \{e \in E : ae = a\} \text{ and } {}_Ea = \{e \in E : ea = a\}.$$

Notice that if ${}_Ea$ is not empty, it is a subsemilattice and filter in E; dually for a_E. The relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ are defined by

$$a\,\widetilde{\mathcal{L}}_E\,b \Leftrightarrow a_E = b_E \text{ and } a\,\widetilde{\mathcal{R}}_E\,b \Leftrightarrow {}_Ea =_E b$$

for any $a, b \in S$. Clearly, $\widetilde{\mathcal{L}}_E, \widetilde{\mathcal{R}}_E$ and hence their intersection $\widetilde{\mathcal{H}}_E$ are equivalences.

Recall from [FGG] that S is *E-semiadequate* if every $\widetilde{\mathcal{L}}_E$-class and every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of S. The following result is straightforward but useful enough to be highlighted as a lemma.

Lemma 2.1. *Let E be a subsemilattice of S. For any $a \in S$ and $e \in E$, $a\,\widetilde{\mathcal{L}}_E\,e$ if and only if e is the minimum element of a_E. Consequently, a is $\widetilde{\mathcal{L}}_E$-related to* at most one *idempotent of E.*

Together with its dual the lemma gives us

Corollary 2.2. *Let E be a subsemilattice of S. Then S is E-semiadequate if and only if for each $a \in S$, the sets a_E and ${}_Ea$ contain a minimum element.*

For an E-semiadequate semigroup S we denote by a^* (a^+) the *unique* idempotent in the $\widetilde{\mathcal{L}}_E$-class ($\widetilde{\mathcal{R}}_E$-class) of a. Thus a^* (a^+) is the minimum element of a_E (${}_Ea$, respectively). Notice that for any $e \in E$, $e^* = e$ so that for any $a \in S, (a^*)^* = a^*$ and for any $b, c \in S$,

$$b \,\widetilde{\mathcal{L}}_E\, c \text{ if and only if } b^* = c^*.$$

The dual remarks hold for $\widetilde{\mathcal{R}}_E$.

If S is an E-semiadequate semigroup then for any $a \in S$ there are functions

$$\alpha_a : E^1 \to E, \;\; \beta_a : E^1 \to E$$

given by

$$x\alpha_a = (xa)^*, \;\; x\beta_a = (ax)^+.$$

As commented in the introduction, if S is inverse, then $\widetilde{\mathcal{L}} = \mathcal{L}$ and $\widetilde{\mathcal{R}} = \mathcal{R}$. In this case, for any $x \in E^1$,

$$x\alpha_a = (xa)^* = (xa)^{-1}(xa) = a^{-1}xa,$$

so that, except for the domain, our function α_a is the same as that introduced by Munn [M] in his representation of inverse semigroups. Moreover, with domains restricted to Eaa^{-1} and $Ea^{-1}a$ respectively, the maps α_a and β_a are mutually inverse isomorphisms.

Although much is lost in moving away from the inverse case, α_a and β_a retain enough useful properties.

Lemma 2.3. [FGG] *Let S be an E-semiadequate semigroup. Then*
(1) for all $a, b \in S, (ab)^* \leq b^*$ and $(ab)^+ \leq a^+$;
(2) for all $a \in S$ the mappings $\alpha_a, \beta_a : E^1 \to E$ are order preserving.

The condition that a semigroup be E-semiadequate can be very weak. To make progress we require that the semigroup satisfies the *congruence condition* [L], which says that $\widetilde{\mathcal{L}}_E$ is a right congruence and $\widetilde{\mathcal{R}}_E$ is a left congruence. An E-semiadequate semigroup S satisfying the congruence condition is an *Ehresmann* semigroup. For convenience and with considerable abuse of notation we follow the lead of [L] and refer to 'the Ehresmann semigroup (S, E)' and say '(S, E) is Ehresmann'.

Lemma 2.4. [FGG] *Let (S, E) be an Ehresmann semigroup.*
(1) For all $a, b \in S$, $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$.
(2) For all $a \in S$ and $e \in E$, $(ae)^* = a^*e$ and $(ea)^+ = ea^+$.
(3) For all $a, b \in S$, $\alpha_{ab} = \alpha_a\alpha_b$ and $\beta_{ab} = \beta_b\beta_a$.

For any semilattice E we denote by $\mathcal{O}_1(E^1)$ the semigroup of order preserving functions from E^1 to E and by $\mathrm{End}_1 E^1$ the subsemigroup of endomorphisms of E^1 with image contained in E. The dual semigroups are denoted by $\mathcal{O}_1^*(E^1)$ and $\mathrm{End}_1^* E^1$. For any $e \in E$ the endomorphism in $\mathrm{End}_1 E^1$ induced by multiplication with e is written ρ_e. Notice that

$$\overline{E} = \{\overline{e} = (\rho_e, \rho_e) : e \in E\}$$

is a semilattice contained in $\mathrm{End}_1 E^1 \times \mathrm{End}_1^* E^1$ and *$e \mapsto \overline{e}$ is an isomorphism from E to $\overline{E}$.*

Recall that μ_E is the largest congruence contained in $\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \cap \widetilde{\mathcal{R}}_E$. The congruence μ_E may be described in an analogous manner to that given for adequate semigroups in [F1]; the proof is essentially the same as that in [F1].

Lemma 2.5. *Let (S, E) be an Ehresmann semigroup. Then*

$$\theta_E : S \to \mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$$

given by

$$a\theta_E = (\alpha_a, \beta_a)$$

is a homomorphism with kernel μ_E. Thus

$$\mu_E = \{(a, b) \in S \times S : \alpha_a = \alpha_b \text{ and } \beta_a = \beta_b\}.$$

Further, for any $e \in E$, $e\theta_E = \overline{e}$ so that $\theta_E|_E : E \to \overline{E}$ is an isomorphism.

Proof In view of Lemmas 2.3 and 2.4, θ_E exists as given and is a homomorphism. With the exception of the last statement, the remainder of the lemma is taken from Lemma 2.5 [FGG]. For any $x \in E^1$ and $e \in E$ we have

$$x\alpha_e = (xe)^* = xe = x\rho_e$$

and dually, $x\beta_e = x\rho_e$, so that

$$e\theta = (\alpha_e, \beta_e) = (\rho_e, \rho_e) = \overline{e}$$

as required.

3. Fundamental Ehresmann Semigroups

In this section we construct from a semilattice E a fundamental Ehresmann semigroup $(C_E, \overline{E})$ which is 'maximal' in the sense that any Ehresmann semigroup (S, E) is fundamental if and only if the homomorphism θ_E given in the previous section embeds S in C_E.

We remark that the semigroup $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ is *not* $\overline{E}$-semiadequate unless E is a lattice, so the obvious choice for C_E fails miserably.

The semigroups $\mathcal{O}_1(E^1)$ and $\mathcal{O}_1^*(E^1)$ are partially ordered by $\leq$ where

$$\alpha \leq \beta \text{ if and only if } x\alpha \leq x\beta \text{ for all } x \in E^1.$$

It is easy to see that $\leq$ is compatible with multiplication. The subset C_E of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$ is then defined by

$$C_E = \{(\alpha, \beta) \in \mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1) : \forall x \in E^1,$$

$$\rho_{x\alpha} \leq \beta\rho_x\alpha \text{ and } \rho_{x\beta} \leq \alpha\rho_x\beta\}.$$

Before we show that C_E is the semigroup we seek, we make some minor remarks. First, C_E is symmetric in the sense that a pair (α, β) is in C_E if and only if the pair (β, α) is in C_E. Second, for any $e \in E$ and $x, y \in E^1$,

$$y\rho_e\rho_x\rho_e = y\rho_{xe} = y\rho_{x\rho_e}$$

so that $\overline{e} = (\rho_e, \rho_e) \in C_E$ and $\overline{E} \subseteq C_E$.

Lemma 3.1. *The set C_E is a subsemigroup of $\mathcal{O}_1(E^1) \times \mathcal{O}_1^*(E^1)$.*

Proof We have seen that $C_E \neq \emptyset$. Let $(\alpha, \beta), (\gamma, \delta) \in C_E$. Then

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \delta\beta).$$

We have

$$\rho_{x\alpha} \leq \beta\rho_x\alpha \text{ and } \rho_{y\gamma} \leq \delta\rho_y\gamma$$

for all $x, y \in E^1$, so that with $y = x\alpha$

$$\rho_{x\alpha\gamma} \leq \delta\rho_{x\alpha}\gamma \leq \delta\beta\rho_x\alpha\gamma.$$

Together with the dual argument this gives that $(\alpha\gamma, \delta\beta) \in C_E$.

Proposition 3.2. *The ordered pair $(C_E, \overline{E})$ is a fundamental Ehresmann semigroup. Further, for any $(\alpha, \beta) \in C_E$,*

$$(\alpha, \beta)^* = (\rho_{1\alpha}, \rho_{1\alpha}) \text{ and } (\alpha, \beta)^+ = (\rho_{1\beta}, \rho_{1\beta}).$$

Proof Let $(\alpha, \beta) \in C_E$. As α is order preserving, clearly $\alpha\rho_{1\alpha} = \alpha$. We show that $\rho_{1\alpha}\beta = \beta$ so that

$$(\alpha, \beta)(\rho_{1\alpha}, \rho_{1\alpha}) = (\alpha, \beta).$$

Let $x \in E^1$. Then

$$x\beta = 1\rho_{x\beta} \leq 1\alpha\rho_x\beta = (1\alpha x)\beta = x\rho_{1\alpha}\beta \leq x\beta$$

so that $\rho_{1\alpha}\beta = \beta$ as required.

Suppose now that $(\alpha, \beta)(\rho_e, \rho_e) = (\alpha, \beta)$ for some $e \in E$. Then $\alpha\rho_e = \alpha$ so that $1\alpha e = 1\alpha$ and $1\alpha \leq e$. As $e \mapsto \overline{e}$ is an isomorphism, $\overline{1\alpha} \leq \overline{e}$, that is, $(\rho_{1\alpha}, \rho_{1\alpha}) \leq (\rho_e, \rho_e)$. From Lemma 2.1, $(\alpha, \beta)^*$ exists and $(\alpha, \beta)^* = (\rho_{1\alpha}, \rho_{1\alpha})$.

The dual argument gives that $(\alpha,\beta)^+$ exists and is $(\rho_{1\beta},\rho_{1\beta})$. It is then a routine matter to check that the congruence condition holds, so that $(C_E,\overline{E})$ is an Ehresmann semigroup.

It remains to show that $(C_E,\overline{E})$ is fundamental. Suppose that (α,β) is $\mu_{\overline{E}}$-related to (γ,δ). As $\mu_{\overline{E}} \subseteq \widetilde{\mathcal{H}}_{\overline{E}} \subseteq \widetilde{\mathcal{L}}_{\overline{E}}$ we have that $(\alpha,\beta)^* = (\gamma,\delta)^*$; by the above, $1\alpha = 1\gamma$.

Let $e \in E$. As $\mu_{\overline{E}}$ is a congruence,

$$(\rho_e,\rho_e)(\alpha,\beta)\, \mu_{\overline{E}}\, (\rho_e,\rho_e)(\gamma,\delta)$$

so that $(\rho_e\alpha,\beta\rho_e)^* = (\rho_e\gamma,\delta\rho_e)^*$ and so

$$e\alpha = 1\rho_e\alpha = 1\rho_e\gamma = e\gamma.$$

Thus $\alpha = \gamma$. Dually, $\beta = \delta$ so that $(\alpha,\beta) = (\gamma,\delta)$ and $\mu_{\overline{E}}$ is trivial as required.

The following lemma is taken from Lemma 6.1 of [FGG]; note that in that paper the term ‘Ehresmann semigroup’ is not used.

Lemma 3.3. *Let S be an E-semiadequate semigroup and let T be a subsemigroup of S containing E. Then*

(1) T is E-semiadequate;

(2) if (S,E) is Ehresmann, then so is (T,E);

(3) if (S,E) is Ehresmann and fundamental, then so is (T,E).

Let (S,E) be an Ehresmann semigroup. Since by definition the congruence μ_E is contained in $\widetilde{\mathcal{H}}_E$, we have that μ_E is idempotent separating, so that the set of idempotents $E\mu_E = \{e\mu_E : e \in E\}$ is a subsemilattice of S/μ_E isomorphic to E.

A homomorphism (isomorphism) ν from an E-semiadequate semigroup S to C_E is an *E-homomorphism* (*E-isomorphism*) if $e\nu = \overline{e}$ for each $e \in E$.

We are now in a position to state our main result.

Theorem 3.4. *Let E be a semilattice. Then $(C_E,\overline{E})$ is a fundamental Ehresmann semigroup.*

For any Ehresmann semigroup (S,E), there is an E-homomorphism $\theta_E : S \to C_E$ given by

$$a\theta_E = (\alpha_a,\beta_a)$$

with kernel μ_E. Consequently,

(1) $(S/\mu_E, E\mu_E)$ is a fundamental Ehresmann semigroup;

(2) (S,E) is fundamental if and only if it is E-isomorphic to a subsemigroup of C_E.

Proof The first statement is Proposition 3.2. Concerning the second statement, in view of Lemma 2.5 it remains only to show that $S\theta_E \subseteq C_E$. Let $a \in S$. Then $a\theta_E = (\alpha_a, \beta_a)$ and for any $x, y \in E^1$

$$y\rho_{x\alpha_a} = y(xa)^* = ((xa)^*y)^* = (xay)^*$$

by Lemma 2.4. Again using Lemma 2.4,

$$y\rho_{x\alpha_a} = (x(ay)^+ay)^* = ((x(ay)^+a)^*y)^* = (x(ay)^+a)^*y$$

so that

$$y\rho_{x\alpha_a} \leq (x(ay)^+a)^* = ((ay)^+xa)^* = y\beta_a\rho_x\alpha_a$$

and $\rho_{x\alpha_a} \leq \beta_a\rho_x\alpha_a$. Dually, $\rho_{x\beta_a} \leq \alpha_a\rho_x\beta_a$ so that $(\alpha_a, \beta_a) \in C_E$ and $\theta_E : S \to C_E$.

By the fundamental theorem of homomorphisms for semigroups, there is a one-one homomorphism $\overline{\theta_E} : S/\mu_E \to C_E$ such that $(e\mu_E)\overline{\theta_E} = e\theta_E = \overline{e}$. By Lemma 3.3, $((S/\mu_E)\overline{\theta_E}, \overline{E})$ is fundamental Ehresmann, hence so is the semigroup $(S\mu_E, E\mu_E)$.

To prove (2), notice first that if (S, E) is fundamental then the E-homomorphism θ_E is one-one. Conversely, if $\nu : S \to C_E$ is a one-one E-homomorphism, then as by Lemma 3.3, the semigroup $(S\nu, \overline{E})$ is fundamental Ehresmann, so also then is (S, E).

4. THE HEDGED CASE

An Ehresmann semigroup (S, E) which satisfies the 'hedged' conditions

(HR) for all $x, y \in E$ and for all $a \in S$,

$$(xya)^* = (xa)^*(ya)^*$$

and its dual (HL) is called *weakly E-hedged*. Fundamental weakly E-hedged semigroups were the topic of [FGG]. We remark that for any order preserving function $\alpha : E^1 \to E$, α is an endomorphism if and only if $(xy)\alpha = x\alpha y\alpha$ for all $x, y \in E$.

Lemma 4.1. *Let (S, E) be an Ehresmann semigroup. Then S is weakly E-hedged if and only if α_a and β_a are endomorphisms for all $a \in S$.*

In [FGG] we showed how to construct a weakly $\overline{E}$-hedged semigroup F_E from any given semilattice E. For convenience we recall here that

$$F_E = \{(\alpha, \beta) \in \mathrm{End}_1 E^1 \times \mathrm{End}_1^* E^1 : \rho_{1\beta} \leq \alpha\beta, \rho_{1\alpha} \leq \beta\alpha\}.$$

Lemma 4.2. *Let $\alpha \in End_1 E^1$ and $\beta \in \mathcal{O}_1(E^1)$. Then*

$$\rho_{1\alpha} \leq \beta\alpha$$

if and only if

$$\rho_{x\alpha} \leq \beta\rho_x\alpha \text{ for all } x \in E^1.$$

Proof One direction is clear, since ρ_1 is the identity mapping in E^1.

Suppose now that $\rho_{1\alpha} \leq \beta\alpha$ and take $x, y \in E^1$. We have

$$y\beta\rho_x\alpha = (y\beta\, x)\alpha = (y\beta\alpha)(x\alpha)$$

as α is a homomorphism. By assumption,

$$y\beta\rho_x\alpha \geq (y\rho_{1\alpha})(x\alpha) = y(1\alpha)(x\alpha) = y(x\alpha) = y\rho_{x\alpha}$$

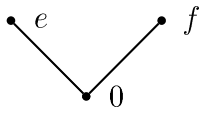
as α is order preserving. Thus $\beta\rho_x\alpha \geq \rho_{x\alpha}$ as required.

Corollary 4.3. *The semigroup F_E is a subsemigroup of C_E. Further,*

$$F_E = C_E \cap (\operatorname{End}_1 E^1 \times \operatorname{End}_1^* E^1).$$

Corollary 4.4. *If $F_E = C_E$ then every Ehresmann semigroup (S, E) is weakly E-hedged.*

If E is a chain, it is easy to see that $\operatorname{End}_1 E^1 = \mathcal{O}_1(E^1)$ and consequently, $F_E = C_E$. Curiously, we can have $F_E = C_E$ *without* $\operatorname{End}_1 E^1 = \mathcal{O}_1(E^1)$. If E is the three element semilattice with Hasse diagram



then $\alpha : E^1 \to E$ given by

$$1\alpha = e\alpha = f\alpha = e,\ 0\alpha = 0$$

is order preserving but not an endomorphism. However, we show in Section 6 that, nevertheless, $F_E = C_E$ for this semilattice.

Corollary 4.5. *If $|E| \leq 3$, then $F_E = C_E$ and every Ehresmann semigroup (S, E) is weakly E-hedged.*

Finally in this section we consider the internal structure of C_E with regard to the conditions (HR) and (HL).

Proposition 4.6. *For any $(\alpha, \beta) \in C_E$ we have*

$$(\overline{e}\overline{f}(\alpha, \beta))^* = (\overline{e}(\alpha, \beta))^*(\overline{f}(\alpha, \beta))^*$$

for all $\overline{e}, \overline{f} \in \overline{E}$ if and only if α is a homomorphism. Dually,

$$((\alpha, \beta)\overline{e}\overline{f})^+ = ((\alpha, \beta)\overline{e})^+((\alpha, \beta)\overline{f})^+$$

for all $\overline{e}, \overline{f} \in \overline{E}$ if and only if β is a homomorphism.

Proof For any $\overline{x} \in \overline{E}$,

$$(\overline{x}(\alpha,\beta))^* = ((\rho_x,\rho_x)(\alpha,\beta))^* = (\rho_x\alpha,\beta\rho_x)^*$$

so that from Proposition 3.2,

$$(\overline{x}(\alpha,\beta))^* = (\rho_{1\rho_x\alpha},\rho_{1\rho_x\alpha}) = (\rho_{x\alpha},\rho_{x\alpha}).$$

Bearing in mind the remark preceding Lemma 4.1, the result follows easily.

Corollary 4.7. *For any semilattice E, $F_E = C_E$ if and only if C_E is weakly E-hedged.*

Corollary 4.8. *For any semilattice E, $F_E = C_E$ if and only if every Ehresmann semigroup (S,E) is weakly E-hedged.*

Corollary 4.9. *If every Ehresmann semigroup (S,E) satisfies one of (HR) or (HL), then every Ehresmann semigroup (S,E) is weakly E-hedged.*

Proof Suppose that C_E satisfies (HR). By Proposition 4.6, for any $(\alpha,\beta) \in C_E$ we have that α is an endomorphism. Now if $(\alpha,\beta) \in C_E$, then, by an earlier remark, $(\beta,\alpha) \in C_E$ so that β is also an endomorphism. Consequently, $C_E = F_E$ and the result follows from Corollary 4.4.

5. The route from T_E to C_E

As remarked in the introduction, the pair $(S,E(S))$ is an Ehresmann semigroup for any inverse semigroup S. Munn's celebrated result [M] builds a fundamental inverse semigroup T_E from *partial isomorphisms* of E; if S is inverse then S/μ is isomorphic to a subsemigroup of $T_{E(S)}$. The founding work of Munn has been generalised in several directions. One way is to drop the condition that the idempotents commute but retain regularity of S. This route has been successfully trodden by Hall and Nambooripad [Ha1],[Ha2],[N].

Another direction, and the one we follow, is to retain commutativity of the idempotents, but loosen the regularity condition. This was first achieved by Fountain in [F1], where he considers *adequate* semigroups; more particularly, a special class of adequate semigroups called in [F1] *type A* (latterly, *ample*). Adequate semigroups may be arrived at via the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, where elements a,b of a semigroup S are $\mathcal{L}^*$-related ($\mathcal{R}^*$-related) in S if they are $\mathcal{L}$-related ($\mathcal{R}$-related) in an oversemigroup of S. For any semigroup S we have

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_E \text{ and } \mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$$

for any subsemilattice E of S. If S is regular, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$, but in general these relations are distinct. The easiest way to see this is to consider a *unipotent* monoid S, that is, a monoid whose only idempotent is the identity. Clearly, $\widetilde{\mathcal{L}}$ is universal, but unless S is left cancellative, $\mathcal{L}^*$ is not universal, and unless S is a group, $\mathcal{L}$ is not universal.

Let E be a semilattice. A semigroup S is *E-adequate* if E is a subsemilattice of S and every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains a (unique) idempotent of E. It is easy to see that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are right and left congruences respectively. Further, if S is E-adequate then $\mathcal{L}^* = \widetilde{\mathcal{L}}_E$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}_E$. Thus if S is E-adequate it is E-semiadequate, indeed (S, E) is Ehresmann. The ample condition is essentially a weak commutativity condition on idempotents. An E-adequate semigroup S is called *ample* (formerly, type A), if the ample condition

$$(AR) \text{ for all } e \in E, a \in S,\ ea = a(ea)^*$$

and its dual (AL) hold. We remark that if S is ample for a set of idempotents E, then $E = E(S)$, so that no ambiguity arises from the terminology. An Ehresmann semigroup (S, E) satisfying the ample condition is called *weakly E-ample*; in this case E need not be equal to $E(S)$. Any inverse semigroup is ample, as is any cancellative monoid, indeed any semilattice of cancellative monoids [F1]. Semilattices of unipotent monoids need not be ample but are weakly ample [GG].

Adequate semigroups are a natural generalisation of inverse semigroups and an extensive theory has been built up around them. However, the obvious question 'is there a T_E theorem for adequate semigroups?' runs into problems before it is even asked. The difficulty is, that S can be adequate without S/μ being adequate or even $E(S)\mu$-adequate [F1, Example 2.4]; naturally, in a representation theorem one wants S and the image of S to have the same defining properties. The insight of Fountain in [F1] was in spotting that if S is adequate and satisfies the ample conditions, then S/μ is adequate, satisfies the ample conditions and *moreover* is isomorphic to a subsemigroup of T_E. Thus the ample condition negotiates *two* problems. First, it ensures that S/μ is adequate, and second, S/μ is represented by partial isomorphisms of E.

An E-adequate semigroup satisfying the hedged conditions (see Section 4) is said to be E-hedged. From Proposition 3.5 of [FGG], the Schützenberger product of two cancellative monoids is hedged. We remark also that the free *left* ample monoid [F2, G] on a set with at least two generators is not ample but is hedged. In this terminology and with trivial adjustments, part (3) of Lemma 2.1 of [F1] says that an ample semigroup is hedged. The hedged condition ensures that α_a

and β_a are endomophisms, but attempts to move away from representations of ample semigroups by partial isomorphisms of E to representations of E-hedged semigroups by (partial) endomorphisms of E^1 proved fruitless. This was because, as pointed out above, S/μ need not be $E(S)\mu$-adequate if S is not ample. In fact Example 2.4 of [F1] is a hedged semigroup S such that S/μ is not $E(S)\mu$-adequate.

Switching perspective, it is certainly true that for any E-hedged semigroup S, the quotient S/μ_E is weakly $E\mu_E$-hedged; indeed if S is *weakly* E-hedged then S/μ_E is weakly $E\mu_E$-hedged. This gave rise to our study in [FGG] of fundamental weakly E-hedged semigroups and the discovery of F_E. The loss in exchanging $\mathcal{L}^*$ and $\mathcal{R}^*$ for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ is counterbalanced by the gain in that the congruence and hedged conditions are preserved by quotienting with μ.

As in the adequate case, it is easy to see that a weakly E-ample semigroup is weakly E-hedged. Fountain and El-Qallali [EF] have shown that if S is weakly E-ample then S/μ_E is isomorphic to a full subsemigroup of T_E. In [FGG] we showed that there is an embedding from T_E to F_E which respects the natural image of E and the representation theorems mentioned above.

The aim of this paper is to push Munn's representation theory as far as it will go in this direction. Given that weakly E-hedged semigroups are manageable in this regard, the next natural step is to look at E-semiadequate semigroups. These must, however, be Ehresmann for the theory to work (without resorting to extra quotienting procedures) since if the congruence condition does not hold, the map θ_E will not be a homomorphism.

6. The semigroups F_E and C_E

In this final section we describe explicitly the semigroups F_E and C_E for some small semilattices E. Our calculations will show that if E has less than four elements, then $F_E = C_E$ but, in general, F_E and C_E are distinct. Curiously, one can find a four element semilattice E such that $\mathcal{O}_1(E^1)$ and F_E are both be regular but C_E is not. We use these results to extract some information concerning Ehresmann semigroups (S, E).

It is worth recalling that a pair of functions (α, β) is in F_E (respectively C_E) if and only if (β, α) is in F_E (respectively C_E). Attention needs to be paid to whether E has an identity or not: if E has a 1 then $E = E^1$ so that, for example, the identity function in E lies in $\mathrm{End}_1 E^1$. If E does not have an identity so that $E \neq E^1$, then the latter statement is not true.

For any $e \in E$, $\rho_e : E^1 \to E$ denotes multiplication by e and $c_e : E^1 \to E$ is the constant map on e. Notice that ρ_e and c_e are endomorphisms. The first lemma of this section is straightforward.

Lemma 6.1. *For any $e, f \in E$,*

$$\rho_e \leq c_f \text{ if and only if } e \leq f.$$

In view of Corollary 4.3, as ρ_e and c_e are endomorphisms, the pair (α, β) is in F_E if and only if it is in C_E, for any $\alpha, \beta \in \{\rho_e, c_e : e \in E\}$.

Lemma 6.2. *For any $e, f \in E$,*
(1) $(\rho_e, \rho_f) \in F_E$ if and only if $e = f$;
(2) $(\rho_e, c_f) \in F_E$ if and only if $e \leq f$;
(3) $(c_e, c_f) \in F_E$ for any $e, f \in E$.

Proof (1) If $e = f$ then $(\rho_e, \rho_f) = \overline{e} \in F_E$ by Corollary 4.3.
Conversely, if $(\rho_e, \rho_f) \in F_E$ then

$$\rho_{1\rho_e} \leq \rho_f \rho_e \text{ and } \rho_{1\rho_f} \leq \rho_e \rho_f$$

giving that

$$\rho_e \leq \rho_{fe} \text{ and } \rho_f \leq \rho_{ef}.$$

As $e \mapsto \rho_e$ is an embedding of E into $\mathrm{End}_1 E^1$, it follows that $e = fe = ef = f$.
(2) By Lemma 6.1,

$$\rho_{1c_f} = \rho_f \leq c_f = \rho_e c_f$$

for any $e, f \in E$. Thus $(\rho_e, c_f) \in F_E$ if and only if

$$\rho_e = \rho_{1\rho_e} \leq c_f \rho_e = c_{fe}.$$

Again by Lemma 6.1, this is equivalent to $e \leq fe$ and hence to $e \leq f$.
(3) Notice simply that

$$\rho_{1c_e} = \rho_e \leq c_e = c_f c_e$$

by Lemma 6.1.

In our investigation of F_E and C_E for specific semilattices E, we remark first that if $E = \{e\}$ is trivial, then $\mathcal{O}_1(E^1) = \mathrm{End}_1 E^1$ is also trivial, hence so is $F_E = C_E$.

Corollary 6.3. *If E is non-trivial then F_E and C_E are neither weakly $\overline{E}$-ample nor $\overline{E}$-adequate.*

Proof Given that E is non-trivial, there exist $e, f \in E$ with $f < e$. By Lemma 4.2, $(c_f, c_e) \in F_E$ and $\overline{f} = (\rho_f, \rho_f) \in \overline{E} \subseteq F_E$. Now

$$\overline{f}(c_f, c_e) = (\rho_f, \rho_f)(c_f, c_e) = (c_f, c_f)$$

and so by Proposition 3.2,

$$(c_f, c_e)(\overline{f}(c_f, c_e))^* = (c_f, c_e)(\rho_{1c_f}, \rho_{1c_f}) = (c_f, c_e)(\rho_f, \rho_f) = (c_f, c_e)$$

so that

$$(c_f, c_e)(\overline{f}(c_f, c_e))^* \neq \overline{f}(c_f, c_e)$$

and (AR) does not hold. Thus F_E is not weakly $\overline{E}$-ample. According to Lemma 6.1 of [FGG], neither then is C_E.

Still with $f < e$, we have

$$(c_e, c_f), (c_e, c_e), \overline{e} = (\rho_e, \rho_e) \in F_E$$

and

$$(c_e, c_f)(\rho_e, \rho_e) = (c_e, c_f) = (c_e, c_f)(c_e, c_e).$$

From Proposition 3.2, $(c_e, c_f)^* = (\rho_e, \rho_e)$ but

$$(\rho_e, \rho_e)(\rho_e, \rho_e) = (\rho_e, \rho_e) \neq (c_e, c_e) = (\rho_e, \rho_e)(c_e, c_e).$$

Thus (c_e, c_f) is not $\mathcal{L}^*$-related to $(c_e, c_f)^*$ so that F_E is not $\overline{E}$-adequate. By the definition of $\mathcal{L}^*$, it follows that C_E is not $\overline{E}$-adequate.

Our next lemma is again concerned with constant maps.

Lemma 6.4. *Let E be finite with least element 0. Then*
(1) $(c_0, \alpha) \in C_E$ if and only if α is constant;
(2) if $E = E^1$, then $(c_1, \alpha) \in C_E$ for all $\alpha \in \mathcal{O}_1(E^1)$.

Proof (1) Using Lemma 4.2, we have $(c_0, \alpha) \in C_E$ if and only if

$$\rho_{1c_0} \leq \alpha c_0 \text{ and } \rho_{x\alpha} \leq c_0 \rho_x \alpha$$

for all $x \in E^1$. Now $\rho_{1c_0} = \rho_0 = c_0 = \alpha c_0$ always. Thus $(c_0, \alpha) \in C_E$ if and only if $\rho_{x\alpha} \leq c_0 \rho_x \alpha = c_{0\alpha}$ for all $x \in E^1$. Using Lemma 6.1, this is equivalent to $x\alpha = 0\alpha$ for all $x \in E^1$, that is, α is constant.

(2) If $E = E^1$, then certainly

$$\rho_{1c_1} = \rho_1 \leq c_1 = \alpha c_1$$

for any $\alpha \in \mathcal{O}_1(E^1)$. Further, for any $x \in E$,

$$c_1 \rho_x \alpha = c_x \alpha = c_{x\alpha} \geq \rho_{x\alpha}$$

by Lemma 6.1. Thus $(c_1, \alpha) \in C_E$.

An order preserving function $\alpha : E \to E$ is *order increasing* if $x \leq x\alpha$ for all $x \in E$. Clearly α is order increasing if and only if $I \leq \alpha$, where I is the identity map on E.

Lemma 6.5. *Let E be finite and let $\alpha, \beta \in \mathcal{O}_1(E^1)$. Then*
(1)

$$\rho_{0\alpha} \leq \beta\rho_0\alpha;$$

(2) if $E = E^1$ and $1\alpha = 1$ then

$$\rho_{1\alpha} \leq \beta\alpha \text{ if and only if } \beta\alpha \text{ is order increasing};$$

(3) if $E = E^1$, $(\alpha, \beta) \in C_E$ and $x\alpha = 1$ if and only if $x = 1$, then $1\beta = 1$so that $\alpha\beta$ and $\beta\alpha$ are both order increasing;
(4) if $E = E^1$ and $\alpha \in \operatorname{End}_1 E^1$, then $(\alpha, I) \in F_E$ if and only if α is order increasing.

Proof (1) We have

$$\beta\rho_0\alpha = \beta c_0\alpha = c_{0\alpha} \geq \rho_{0\alpha}$$

by Lemma 6.1.
(2) Remark that if $1\alpha = 1$, then $\rho_{1\alpha} = \rho_1 = I$.
(3) Given that $(\alpha, \beta) \in C_E$ we have by (2) that $I \leq \beta\alpha$. Thus

$$1 = 1I = (1\beta)\alpha \leq 1$$

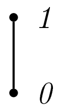
so that $1 = (1\beta)\alpha$ and then $1\beta = 1$.
(4) If $(\alpha, I) \in F_E$, then by (3) α is order increasing. Conversely, if α is order increasing, then $1\alpha = 1$ so that

$$\rho_{1\alpha} \leq I\alpha \text{ and } \rho_{1I} \leq \alpha I$$

so that $(\alpha, I) \in F_E$ as required.

We now consider the semigroups F_E where E is a chain with 2 or 3 elements. As E is a finite chain, $E = E^1$ and $\mathcal{O}_1(E^1) = \operatorname{End}_1 E^1$, so that from Corollary 4.3, $F_E = C_E$ in these cases.

Example 6.6. *Let E be the chain*

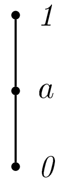


Clearly $\operatorname{End}_1 E^1 = \{I, c_1, c_0\}$ and $\overline{E} = \{(I, I), (c_0, c_0)\}$. The pairs $(I, c_1), (c_1, I), (c_1, c_1), (c_1, c_0)$ and (c_0, c_1) are in F_E by Lemma 6.4. The remaining two pairs (I, c_0) and (c_0, I) are not in F_E, by the same lemma. Thus

$$F_E = \{(I, I), (I, c_1), (c_1, I), (c_1, c_1), (c_1, c_0), (c_0, c_1), (c_0, c_0)\}$$

and so F_E is a 7 element band. It is not a semilattice as (c_0, c_0) and (c_1, c_1) do not commute.

Example 6.7. *Let E be the chain*



We first list all endomorphisms of E. It is easy to see that they are

$$I, c_0, c_1, c_a, \rho_a,$$

and

$$\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ a & 0 & 0 \end{pmatrix}.$$

We use the technical results developed above to find which pairs (α, β) where $\alpha = \beta$ or α *precedes* β in the list above are in F_E.

By (4) of Lemma 6.5, we have that

$$(I, I), (I, c_1), (I, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}), (I, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix})$$

and

$$(I, \begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix})$$

are in F_E.

Considering c_0, by (1) of Lemma 6.4,

$$(c_0, c_0), (c_0, c_1) \text{ and } (c_0, c_a)$$

are in F_E. By (2) of that lemma

$$(c_1, c_1) \text{ and } (c_1, \alpha)$$

are in F_E for the seven functions α succeeding c_1.

Considering now c_a, $(c_a, \alpha) \in F_E$ if and only if $\rho_{1c_a} \leq \alpha c_a$ and $\rho_{1\alpha} \leq c_a\alpha$. Thus $(c_a, \alpha) \in F_E$ is equivalent to

$$\rho_a \leq c_a \text{ and } \rho_{1\alpha} \leq c_{a\alpha}$$

and hence by Lemma 6.1, to $1\alpha = a\alpha$. Thus

$$(c_a, c_a), (c_a, \rho_a), (c_a, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}), (c_a, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix})$$

are in F_E.

The pairs (ρ_a, α) require slightly more thought. If $(\rho_a, \alpha) \in F_E$ then from the defining condition of F_E one deduces that $a \leq a\alpha = 1\alpha$ and so the possibilities for α are

$$\rho_a, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We know that $\overline{a} = (\rho_a, \rho_a) \in F_E$. A hands on check gives that

$$(\rho_a, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}), (\rho_a, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}) \in F_E.$$

The remaining cases are speedily dealt with by direct calculation. We find

$$(\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}), (\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}), (\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix})$$

$$(\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}) \text{ and } (\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ a & 0 & 0 \end{pmatrix})$$

are in F_E;

$$(\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}) \text{ and } (\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix})$$

are in F_E; finally,

$$(\begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix}, \begin{pmatrix} 1 & a & 0 \\ 1 & a & a \end{pmatrix})$$

is in F_E.

Counting the above elements (and remembering to count twice a pair (α, β) where $\alpha \neq \beta$, we have shown that F_E has 54 elements. From [AG1] we know that $\operatorname{End}_1 E^1 = \mathcal{O}_1(E^1)$ is regular and hence so is $\operatorname{End}_1 E^1 \times \operatorname{End}_1^* E^1$. The condition that a pair (α, β) be in F_E is that in some sense α and β be weak inverses of each other. However some peculiar behaviour arises at this point.

The semigroup F_E is not regular.

We know that $(I, \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix})$ is in F_E. If F_E were regular, there would be a pair (I, α) in F_E with

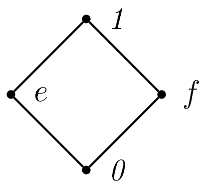
$$\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix} \alpha \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}.$$

This would necessitate $a\alpha = 0$ so that also $0\alpha = 0$. But no such pair (I, α) lies in F_E.

Notice also that $\begin{pmatrix} 1 & a & 0 \\ 1 & 1 & a \end{pmatrix}$ is not an inverse of I. On the other hand, if $\alpha = \begin{pmatrix} 1 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then α is idempotent, so that α is an inverse of α. Nevertheless, $(\alpha, \alpha) \notin F_E$.

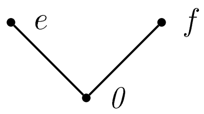
The semilattices in our final two examples are not chains. In enumerating the possible order preserving maps $\alpha : E^1 \to E$ it is useful to remember that if E is finite with least element 0, 1α is the greatest and 0α the least element of the image of α.

Lemma 6.8. *Let E be the four element semilattice*



If $\alpha \in \mathcal{O}_1(E^1)$, then α is an endomorphism if and only if $(ef)\alpha = e\alpha f\alpha$.

Example 6.9. *Let E be the semilattice*



As remarked before Lemma 6.8, if $\alpha \in \mathcal{O}_1(E^1)$, then the image of α must have a greatest and a least element. As the image of α is contained in E, α is constant or the image of α is $\{e, 0\}$ or $\{f, 0\}$. It follows that the elements of $\mathcal{O}_1(E^1)$ are

$$c_0, c_e, c_f, \rho_e, \rho_f, c_0^e, c_0^f, \alpha, \beta, \gamma \text{ and } \delta$$

where

$$\begin{aligned}
c_0^e &= \begin{pmatrix} 1 & e & f & 0 \\ e & 0 & 0 & 0 \end{pmatrix}, & c_0^f &= \begin{pmatrix} 1 & e & f & 0 \\ f & 0 & 0 & 0 \end{pmatrix} \\
\alpha &= \begin{pmatrix} 1 & e & f & 0 \\ e & e & e & 0 \end{pmatrix}, & \beta &= \begin{pmatrix} 1 & e & f & 0 \\ e & 0 & e & 0 \end{pmatrix} \\
\gamma &= \begin{pmatrix} 1 & e & f & 0 \\ f & f & 0 & 0 \end{pmatrix}, & \delta &= \begin{pmatrix} 1 & e & f & 0 \\ f & f & f & 0 \end{pmatrix}.
\end{aligned}$$

According to Lemma 6.8, the only elements of $\mathcal{O}_1(E^1)$ that are *not* endomorphisms are α and δ. We show that $(\alpha, \epsilon) \notin C_E$ for any $\epsilon \in \mathcal{O}_1(E^1)$.

Suppose that $(\alpha, \epsilon) \in C_E$ for some $\epsilon \in \mathcal{O}_1(E^1)$. By the defining condition for C_E,

$$\rho_e = \rho_{f\alpha} \leq \epsilon\rho_f\alpha \text{ and } \rho_e = \rho_{e\alpha} \leq \epsilon\rho_e\alpha.$$

Thus

$$e = e\rho_e \leq (e\epsilon\, f)\alpha$$

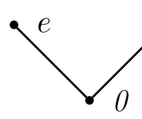
so that $e\epsilon = f$, and

$$e = e\rho_e \leq (e\epsilon\, e)\alpha$$

so that $e\epsilon = e$, a contradiction.

Similarly, there is no pair (δ, ϵ) in C_E.

Corollary 6.10. *For the semilattice*



$C_E = F_E$ and so by Corollary 4.4, every Ehresmann semigroup (S, E) is weakly E-hedged.

Returning now to Example 6.9, calculations of a now familiar nature give that

$$(c_0, c_0), (c_e, c_e), (c_f, c_f), (\rho_e, \rho_e), (\rho_f, \rho_f)$$

are all elements of $C_E = F_E$ as are

$$(c_0, c_e), (c_0, c_f), (c_e, c_f), (c_e, \rho_e), (c_e, \gamma), (c_f, \rho_f), (c_f, \beta), (\beta, \gamma)$$

and all the pairs (ϵ, η) where (η, ϵ) is in the previous list. Thus $C_E = F_E$ has 21 elements.

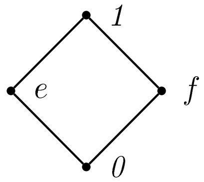
Concerning regularity, a curious fact emerges. The semigroup $\mathcal{O}_1(E^1)$ is not regular, since, for example, the element c_0^e is not regular. Nevertheless, C_E is regular, as we now show. On the other hand in our next example, $\mathcal{O}_1(E^1)$ is regular but C_E is not.

The non-idempotent elements of F_E are

$$(c_e,\gamma),(\gamma,c_e),(c_f,\beta),(\beta,c_f)(\beta,\gamma) \text{ and } (\gamma,\beta).$$

It is easy to check that β and γ are mutually inverse so that (β,γ) and (γ,β), (c_e,γ) and (c_f,β), and (γ,c_e) and (β,c_f) are mutually inverse pairs. Thus $C_E = F_E$ is regular.

Example 6.11. *Let E be the semilattice*



In this case, $C_E \neq F_E$ and C_E is not regular. The elements of C_E may be determined as in previous examples. The procedure is now more lengthy, since it emerges that C_E has 183 elements, of which 108 lie in F_E.

To see that C_E is not regular, check first that the pair $(\alpha,\beta)\in C_E$ where

$$\alpha = \begin{pmatrix} 1 & e & f & 0 \\ 1 & 1 & e & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & e & f & 0 \\ 1 & 1 & 1 & f \end{pmatrix}.$$

Suppose that $(\gamma,\delta)\in C_E$ and

$$(\alpha,\beta)(\gamma,\delta)(\alpha,\beta) = (\alpha,\beta).$$

Then

$$\alpha\gamma\alpha = \alpha \text{ and } \beta\delta\beta = \beta.$$

These give

$$0 = 0\alpha = 0\alpha\gamma\alpha = 0\gamma\alpha$$

so that $0\gamma = 0$. Similarly, $e\gamma = f$. Now

$$1 = 1\alpha = 1\alpha\gamma\alpha = 1\gamma\alpha$$

so that $1\gamma = 1$ or e. But $f = e\gamma \leq 1\gamma$ so that $1\gamma = 1$. From $\beta\delta\beta = \beta$ we obtain $f\delta = 0$. By assumption, $(\gamma, \delta) \in C_E$ so that $\rho_{1\gamma} \leq \delta\gamma$ and

$$f = f\rho_{1\gamma} \leq f\delta\gamma = 0\gamma = 0,$$

a contradiction. Thus C_E is not regular.

Notice that the pair (α, β) above does not lie in F_E. In fact, F_E is regular.

To show that there are semilattices E with F_E not regular, we look to the work of Adams and M. Gould for examples of semilattices with identity having non-regular endomorphism monoids. In [AG1] they show that for any finite chain E, $\operatorname{End}_1 E^1$ is regular. In Lemma 1 of [AG1] they characterise those infinite chains having non-regular endomorpism monoid. Their technique can be adapted to find a chain with identity having a non-regular endomorphism monoid.

Another paper of Adams and M. Gould [AG2] describes those *finite* semilattices having non-regular endomorphism monoids. The free semilattice monoid on three generators is one such.

For *any* semilattice monoid E such that $\operatorname{End}_1 E^1$ is not regular, choose a non-regular map $\alpha \in \operatorname{End}_1 E^1$. By Lemma 6.4, $(\alpha, c_1) \in F_E$ and (α, c_1) cannot be regular. Thus F_E is not regular.

According to Corollary 4.5, if we wish to find an Ehresmann semigroup that is not weakly E-hedged, the semilattice E must contain at least four elements. Considering Example 6.11, the pair (α, α) where

$$\alpha = \begin{pmatrix} 1 & e & f & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

is in C_E so that by Lemma 3.3, the subsemigroup

$$S = \langle \{(\alpha, \alpha)\} \cup \overline{E} \rangle$$

of C_E is Ehresmann. However, as α is not an endomorphism, Proposition 4.6 gives that S is not weakly $\overline{E}$-hedged. It is easy to calculate that S has 11 elements.

Similarly, to find small non-regular fundamental Ehresmann semigroups or weakly E-hedged semigroups, it is enough to consider the subsemigroup of C_E or F_E generated by $\overline{E} \cup \{(\alpha, \beta)\}$ for any non-regular pair (α, β), and call on Lemma 6.1 of [FGG].

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