E-unitary and almost factorizable orthodox semigroups

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Preliminaries: Inverse semigroups

A basic construction: the semidirect product

Let E be a semilattice and let G be a group acting on E on the left. Then the *semidirect product of* E *by* G is the set $E \times G$ endowed with the operation

$$(\alpha, g) \cdot (\beta, h) = (\alpha \cdot {}^{g}\beta, gh).$$

The semidirect product is denoted by $E * G$. It is an inverse semigroup.

An old question

How are inverse semigroups related to semidirect products?

Through two paths

E-unitary inverse semigroups

A regular semigroup S is E-unitary if for every $e, a \in S$, $(e, ea \in E(S)) \Rightarrow a \in E(S)$.

Factorizable inverse monoids

An inverse monoid M is *factorizable* if $M = E(M) \cdot U(M)$ where $U(M)$ denotes the group of units of M, that is, the two-sided divisors of 1.

Almost factorizable inverse semigroups

An inverse semigroup S is *almost factorizable* if for every $s \in S$ there exists $e \in E(S)$ and $\rho \in \Sigma(S)$ such that $s = e\rho$. Here $\Sigma(S)$ denotes the group of bijective right translations of S.

Glossary

1. D. B. McAlister, 1976: Every inverse semigroup has an E-unitary cover (that is, every inverse semigroup is an *idempotent separating* homomorphic image of an E-unitary semigroup).
2. L. O'Carroll, 1976: Every E-unitary inverse semigroup is embeddable into a semidirect product.
3. D. B. McAlister, 1976: Almost factorizable inverse semigroups are the homomorphic images of semidirect products.
4. D. B. McAlister, N. R. Reilly, 1977: Every inverse semigroup can be embedded into an almost factorizable inverse semigroup. Furthermore, every E-unitary cover arises this way.
5. M. V. Lawson, 1994: Almost factorisable inverse semigroups are closely related to factorizable inverse monoids.

Orthodox semigroups

Definition

A semigroup is *orthodox* if it is regular, and its idempotents form a subsemigroup (products of idempotents are idempotents).

Semidirect product

If B is a band, G is a group acting on the band on the left, then the *semidirect product of* B *by* G is the set $B \times G$ endowed with the operation

$$(e, g) \cdot (f, h) = (e \cdot {}^{g}f, gh).$$

The question

The question is the same: how are orthodox semigroups related to semidirect products.

Known results

1. M. B. Szendrei, K. Takizawa, 1979-80: Every orthodox semigroup has an E-unitary cover.
2. M. B. Szendrei, 1987: Every E-unitary orthodox semigroup having a regular band of idempotents is embeddable.
3. M. B. Szendrei, 1993: Every orthodox semigroup has an E-unitary cover which is embeddable.
4. B. Billhardt, 1998: Non-embeddable E-unitary orthodox semigroups exist.
5. M. Hartmann, 2007: Almost factorizable orthodox semigroups are just the idempotent separating homomorphic images of semidirect products. General homomorphic images differ from idempotent separating homomorphic images.
6. M. Hartmann, 2007: Every orthodox semigroup is embeddable into an almost factorizable orthodox semigroup.

Factorizable orthodox monoids

Definition

An orthodox monoid M is *factorizable* if $M = E(M) \cdot U(M)$. Recall that $U(M)$ is the group of units.

Theorem

An orthodox monoid M is factorizable if and only if M is an (idempotent separating) homomorphic image of a (monoid) semidirect product.

Proof (of one part): Let us suppose that M is factorizable. Then $U(M)$ acts on $E(M)$ by ${}^{u}e = ueu^{-1}$. The map $\varphi\colon E(M) * U(M) \to M,\ (e,u) \mapsto eu$ is a surjective idempotent separating homomorphism.

Almost factorizable othodox semigroups

The translational hull

Let S be a semigroup. A map $\rho\colon S \to S$ is a right translation if for every $s, t \in S$, $(st)\rho = s(t\rho)$. Similarly, a *right* map $\lambda\colon S \to S$ is a left translation if for every $s, t \in S$, $\lambda(st) = (\lambda s)t$. A pair (λ, ρ) is linked if for every $s, t \in S$, $(s\rho)t = s(\lambda t)$. The set of all pairs under the componentwise multiplication is a semigroup, denoted by $\Omega(S)$. If S is orthodox, $\Omega(S)$ is orthodox, too.
The group of units of $\Omega(S)$ is denoted by $\Sigma(S)$.

Definition

An orthodox semigroup S is *almost factorizable* if for every $s \in S$ there exist $e \in E(S)$ and $(\lambda, \rho) \in \Sigma(S)$ such that $s = e\rho$.

A. f. o. semigroups and semidirect products

Theorem

An orthodox semigroup is almost factorizable if and only if it is an idempotent separating homomorphic image of a semidirect product.

Proof(of the converse one part): Suppose that $\varphi \colon B * G \to S$ is an idempotent separating homomorphism. We define a left map $\lambda_g \colon S \to S$ by

$$\lambda_g((f,h)\varphi) = ({}^g f, gh)\varphi.$$

Using the fact that φ is idempotent separating, one can prove that λ_g is well-defined. Furthermore, it is easy to see that it is a left translation. Dually one can define a corresponding right tranlation ρ_g, and it is easy to see that $(\lambda_g, \rho_g) \in \Sigma(S)$.

Weakly coverable orthodox semigroups

Proposition

The class of almost factorizable orthodox semigroups is properly contained in the class of homomorphic images of semidirect products.

Definition

An orthodox semigroup is *weakly coverable* if it is a homomorphic image of a semidirect product.

Question

Can we characterize weakly coverable orthodox semigroups? Is it decidable if a finite orthodox semigroup is weakly coverable?

Splitting a homomorphism

Let $\eta\colon B' * G \to S$ be a homomorphism.
Then it induces a homomorphism $\chi\colon E' * G \to S/\gamma$ where E' is the structure semilattice of B' and γ is the smallest inverse semigroup congruence on S (thus, the greatest inverse semigroup homomorphic image of a weakly coverable orthodox semigroup is necessarily almost factorizable).
Furthermore, η can be restricted to B': we denote the restriction by φ. The following rule clearly connects χ and φ: for every $e \in B'$,

$$(e\varphi)\gamma = (e\mathcal{J}, 1)\chi. \tag{1}$$

Constructing a homomorphism

Let S be an orthodox semigroup having an almost factorizable greatest inverse semigroup homomorphic image. Furthermore, let B' be a band, let G be a group acting on B', and let $\chi\colon E' * G \to S/\gamma$ and $\varphi\colon B' \to B$ be surjective homomorphisms satisfying condition (1).
Then there exist at most one homomorphism $\eta\colon B' * G \to S$ such that φ and χ are induced by η, namely the following map - if it is a homomorphism:

$$(e, g) \mapsto s \text{ where } s\gamma = (e\mathcal{J}, g)\chi, s \mathrel{\mathcal{R}} e\varphi \text{ and } s \mathrel{\mathcal{L}} ({}^{g^{-1}}e)\varphi.$$

The generalized inverse case

Theorem

If S is a generalized inverse semigroup then the map defined previously is a homomorphism.

Lemma

If χ is given, a band B' and a homomorphism $\varphi B' \to B$ can always be constructed such that condition (1) is satisfied.

Theorem

A generalized inverse semigroup is weakly coverable if and only it its greatest inverse semigroup homomorphic image is almost factorizable.

The general case

Example

There exists an orthodox semigroup which is not weakly coverable although it has an almost factorizable greatest inverse semigroup homomorphic image.

An ugly theorem

Let S be a finite orthodox semigroup having a greatest inverse semigroup homomorphic image, and let G be a group. Then it is decidable if there exists a semidirect product $B' * G$ such that S is the homomorphic image of $B' * G$.

Where could we find such a group? Not in $\Sigma(S/\gamma)$...

There exists a weakly coverable orthodox semigroup S which cannot be covered by any semidirect product of the form $B' * G$ where G is a subgroup of $\Sigma(S/\gamma)$.