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FAITHFUL FUNCTORS FROM CANCELLATIVE CATEGORIES TO CANCELLATIVE MONOIDS WITH AN APPLICATION TO ABUNDANT SEMIGROUPS

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We prove that any small cancellative category admits a faithful functor to a cancellative monoid. We use our result to show that any primitive ample semigroup is a full subsemigroup of a Rees matrix semigroup $\mathcal{M}^0(M; I, I; P)$ where M is a cancellative monoid and P is the identity matrix. On the other hand a consequence of a recent result of Steinberg is that it is undecidable whether a finite ample semigroup embeds as a full subsemigroup of an inverse semigroup.

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1. Introduction

The question of which categories embed into groupoids, that is, small categories in which every morphism has an inverse, is old and awkward. Clearly a necessary condition is that the category be cancellative; various authors, for example Hasse and Michler [7] and Karger [10] have given sufficient conditions for embeddability. The failure of any set of necessary and sufficient conditions to emerge is explained by recent results in papers of Hall, Kublanovsky, Margolis, Sapir and Trotter [6, Theorem 1.3] and Steinberg [14, Theorem 7.1]. Steinberg shows explicitly that it is undecidable whether a finite cancellative category embeds into a finite groupoid or into a groupoid; this result is implicit in [6].

It is straightforward to show that a small category admits a faithful functor to a group if and only if it embeds into a groupoid [12] (see also [7]). Given that it is therefore undecidable whether a (necessarily cancellative) finite category admits a faithful functor to a group, can we decide whether it admits a faithful functor to a cancellative monoid? Our first aim, achieved in Sec. 2, is to answer this positively. We show that a small category admits a faithful functor to a cancellative monoid if and only if it is cancellative.

Our motivation for considering this problem arose from an embeddability question for abundant semigroups. Fountain introduced abundant semigroups in [2] as an analog of PP rings, that is, rings in which every principal one-sided ideal is projective. In an abundant semigroup, every principal right (left) ideal is isomorphic as a right (left) ideal to one generated by an idempotent. Abundant semigroups generalize regular semigroups; roughly speaking, in the theory of abundant semigroups, cancellative monoids play the role that groups fulfill in the regular case. For example, an abundant semigroup is unipotent (that is, possesses exactly one idempotent,) if and only if it is a cancellative monoid; a regular semigroup is unipotent if and only if it is a group.

Certainly full subsemigroups of regular semigroups are abundant. The question therefore arises of deciding whether an abundant semigroup of a certain type embeds in a nice way in a regular semigroup of the same type. The latter sections of the paper consider this question for *ample* semigroups; these are the non-regular analog of inverse semigroups. In Sec. 3 we discuss what we might mean by a "nice" embedding; in Sec. 4 we show that as a consequence of [14] it is undecidable whether a finite ample semigroup is a full subsemigroup of an inverse semigroup. In our final section we use a structure theorem of Fountain [3] together with the positive embeddability result from the first part of the paper to show that any primitive ample semigroup is fully embeddable into a Rees matrix semigroup of the form $\mathcal{M}^0 = \mathcal{M}^0(M; I, I; P)$, where M is a cancellative monoid and P is the identity matrix over M^0. Of course, $\mathcal{M}^0$ is not inverse unless M is a group, but we have reduced our problem to that of embedding M into a group.

2. Faithful Functors from Cancellative Categories to Cancellative Monoids

Throughout this paper we denote a small category by C and regard C as a generalization of a monoid; we consider only *covariant* functors. We denote the objects and morphisms of C by Ob C and Mor C respectively. We identify each object with the identity morphism at that object so that Ob $C \subseteq$ Mor C. For $\alpha, \beta \in$ Ob C we let $M_{\alpha\beta}$ be the set of morphisms from α to β. We put $T_\alpha = M_{\alpha\alpha}$ and denote the identity of T_α by e_α. We denote the source and target of a morphism $x \in$ Mor C by $\mathbf{s}(x)$ and $\mathbf{t}(x)$ respectively, so that if $x \in M_{\alpha\beta}$ then $\mathbf{s}(x) = \alpha$ and $\mathbf{t}(x) = \beta$. If $x \in M_{\alpha\beta}$ has a (necessarily unique) inverse, then we denote this by x^{-1}; notice that x^{-1} must lie in $M_{\beta\alpha}$. A category C is *locally unipotent* if each local submonoid is unipotent, that is, contains exactly one idempotent. Certainly any cancellative category is locally unipotent.

We are concerned with the question of finding a faithful functor from a category C into a monoid of specific type. Our positive results hinge on the following construction.

Let C be a category. Let

$$\bar{C} = \{\bar{m} \mid m \in \text{Mor } C \backslash \text{Ob } C\}$$

be a new set of symbols in bijective correspondence with the set Mor $C\backslash$Ob C of non-identity morphisms in C. We consider the free monoid $\bar{C}^*$ on $\bar{C}$, denoting its identity by ϵ. We define $\bar{e} = \epsilon$ for each identity $e \in$ Ob C. Let ρ be the binary relation on $\bar{C}^*$ given by

$$\rho = \{(\bar{a}\ \bar{b}, \overline{ab}) \mid a, b \in \text{Mor } C\backslash\text{Ob } C, \mathbf{t}(a) = \mathbf{s}(b)\}.$$

Let ρ^* be the minimum congruence on $\bar{C}^*$ containing ρ, and let M be the monoid $\bar{C}^*/\rho^*$. We denote by $[\alpha]$ the ρ^*-class of a word $\alpha \in \bar{C}^*$. We shall show that the function σ : Mor $C \to M$ given by $x\sigma = [\bar{x}]$ for all $x \in$ Mor C is a faithful functor from C to M, and that if C is cancellative, then so also is M.

We say that a (possibly empty) word $\overline{a_1} \cdots \overline{a_n} \in \bar{C}^*$ (where each $a_i \in$ Mor $C\backslash$Ob C) is in *normal form* if for $1 \le i < n$ we have $\mathbf{t}(a_i) \ne \mathbf{s}(a_{i+1})$, that is, if no consecutive letters correspond to morphisms which can be multiplied together, in the appropriate order, in C.

Given a (possibly empty) word $\alpha \in \bar{C}^*$, we define a sequence $\left(\alpha^{(k)}\right)_{k\in\mathbb{N}}$ recursively as follows. First we define $\alpha^{(0)} = \alpha$. Now suppose $k \ge 1$ and $\alpha^{(k-1)} = \overline{a_1} \cdots \overline{a_n}$ where each $a_i \in$ Mor $C \setminus$ Ob C. We obtain $\alpha^{(k)}$ from $\alpha^{(k-1)}$ by replacing each maximal sequence of consecutive letters which correspond to morphisms which can be multiplied in C, with the single letter or empty word corresponding to their products. More formally, let

$$P = \{p \in \mathbb{N} \mid 1 \le p < n, \mathbf{t}(a_p) \ne \mathbf{s}(a_{p+1})\}$$

and suppose that

$$P = \{p_1, p_2, \ldots, p_q\}$$

where $q \ge 0$ and

$$1 \le p_1 < p_2 < \cdots < p_q < n.$$

We define

$$\alpha^{(k)} = (\overline{a_1 \cdots a_{p_1}})(\overline{a_{p_1+1} \cdots a_{p_2}}) \cdots (\overline{a_{p_q+1} \cdots a_n}),$$

noting that for each i with $0 \le i \le q$, $\overline{a_{p_i+1} \cdots a_{p_{i+1}}}$ (interpreting p_0 as 0 and p_{q+1} as n) is either the empty word or a letter in $\bar{C}$.

It follows easily from the definition of ρ that each $\alpha^{(k)}$ is ρ^*-related to $\alpha^{(k-1)}$, and hence that all words in the sequence represent the same element of M. Note that $\alpha^{(k)} = \alpha^{(k-1)}$ exactly if $\alpha^{(k-1)}$ is in normal form, and that if $\alpha^{(k)} \neq \alpha^{(k-1)}$ then $\alpha^{(k)}$ is strictly shorter than $\alpha^{(k-1)}$. It follows that there must exist some $n \geq 1$ such that $\alpha^{(n)}$ is in normal form and $\alpha^{(j)} = \alpha^{(n)}$ for all $j \geq n$. We denote this $\alpha^{(n)}$ by α^∞.

Lemma 2.1. *Let $a \in \mathrm{Mor}\ C$ and $\gamma \in \bar{C}^*$. Then*

$$(\bar{a}\gamma)^\infty = (\bar{a}(\gamma^\infty))^\infty.$$

Proof. First, if $a \in \mathrm{Ob}\ C$ then $\bar{a} = \epsilon$ and since $(\gamma^\infty)^\infty = \gamma^\infty$ the given equation is satisfied, so assume $a \notin \mathrm{Ob}\ C$. Observe also that if γ is in normal form then $\gamma = \gamma^\infty$, so that the required equation is again satisfied. We prove the result in the remaining cases by induction over the reduction process described above.

Let

$$\gamma = \overline{c_1} \cdots \overline{c_n} \in \bar{C}^*$$

be a word not in normal form, where $n > 0$ (since the empty word is in normal form) and each $c_i \in \mathrm{Mor}\ C \backslash \mathrm{Ob}\ C$. Assume for induction that $(\bar{a}\gamma^{(1)})^\infty = (\bar{a}(\gamma^{(1)})^\infty)^\infty$.

For convenience, we write $c_0 = a$. Let $j \in \mathbb{N}$ be maximal such that $0 \leq j \leq n$ and $\mathbf{t}(c_i) = \mathbf{s}(c_{i+1})$ for all $0 \leq i < j$. Notice that $j = 0$ exactly if $\mathbf{t}(a) \neq \mathbf{s}(c_1)$.

Let $\beta \in \bar{C}^*$ be the (possibly empty) word $\overline{c_{j+1}} \cdots \overline{c_n}$, and consider the word $\beta^{(1)}$, obtained by applying the first step of the reduction process to β. Suppose $\beta^{(1)} = \overline{d_1} \cdots \overline{d_m}$ where $m \geq 0$ and each $d_i \in \mathrm{Mor}\ C \backslash \mathrm{Ob}\ C$. For convenience, set $d_0 = c_j$ (noting that $d_0 = c_0 = a$ if $j = 0$). Now let $k \in \mathbb{N}$ be maximal such that $0 \leq k \leq m$ and $\mathbf{t}(d_i) = \mathbf{s}(d_{i+1})$ for all $0 \leq i < k$. Let $\delta \in \bar{C}^*$ be the (possibly empty) word $\overline{d_{k+1}} \cdots \overline{d_m}$.

Now we have

$$\begin{aligned}
(\bar{a}(\gamma^\infty))^\infty &= (\bar{a}(\gamma^{(1)})^\infty)^\infty \\
&= (\bar{a}\gamma^{(1)})^\infty && \text{(by the inductive hypothesis)} \\
&= [\bar{a} \quad \overline{c_1 \cdots c_j} \quad \beta^{(1)}]^\infty && \text{(since } \gamma^{(1)} = \overline{c_1 \cdots c_j}\ \beta^{(1)}) \\
&= [(\bar{a} \quad \overline{c_1 \cdots c_j} \quad \beta^{(1)})^{(1)}]^\infty \\
&= [\overline{ac_1 \cdots c_j d_1 \cdots d_k} \quad \delta^{(1)}]^\infty \\
&= [(\overline{ac_1 \cdots c_j} \quad \overline{d_1} \cdots \overline{d_k} \quad \delta)^{(1)}]^\infty \\
&= [(\overline{ac_1 \cdots c_j} \quad \beta^{(1)})^{(1)}]^\infty \\
&= [\overline{ac_1 \cdots c_j} \quad \beta^{(1)}]^\infty \\
&= [(\bar{a}\gamma)^{(1)}]^\infty \\
&= (\bar{a}\gamma)^\infty
\end{aligned}$$

as required. ☐

Lemma 2.2. *Every word $\alpha \in \bar{C}^*$ is ρ^*-related to a unique word (namely α^∞) in normal form.*

Proof. We define a binary relation τ on $\bar{C}^*$ by $\alpha \;\tau\; \beta$ if and only if $\alpha^\infty = \beta^\infty$. It is immediate from the method of definition that τ is an equivalence relation. We shall show that τ is in fact a congruence containing ρ, from which it will follow that τ contains the congruence ρ^*.

First, consider a relation $(\bar{x}\,\bar{y}, \overline{xy}) \in \rho$. For such to exist, we must have $\mathbf{t}(x) = \mathbf{s}(y)$. Now it is easy to see that

$$(\bar{x}\,\bar{y})^\infty = \overline{xy} = (\overline{xy})^\infty$$

so that $(\bar{x}\,\bar{y}, \overline{xy}) \in \tau$. We have shown that ρ is contained in τ.

Next, we show that τ is a congruence on $\bar{C}^*$. By symmetry of assumption, it will suffice to show that τ is left compatible. Let $\bar{a} \in \bar{C}$ and $\alpha, \beta \in \bar{C}^*$ be such that $\alpha \;\tau\; \beta$, that is, such that $\alpha^\infty = \beta^\infty$. By Lemma 2.1 we have

$$(\bar{a}\alpha)^\infty = (\bar{a}\alpha^\infty)^\infty = (\bar{a}\beta^\infty)^\infty = (\bar{a}\beta)^\infty.$$

Since $\bar{C}^*$ is generated by $\bar{C}$, this suffices to show that τ is left compatible. Thus, τ is a congruence. Since ρ^* is by definition the minimum congruence containing ρ, it follows that $\rho^* \subseteq \tau$. (Indeed, by our earlier observation that each word α is ρ^*-related to all words of the form $\alpha^{(i)}$, it follows that $\tau = \rho^*$.)

Now suppose $\alpha \;\rho^*\; \beta$ with β in normal form. Then $\alpha \;\tau\; \beta$, so $\alpha^\infty = \beta^\infty$. But since β is in normal form, we have $\beta = \beta^\infty = \alpha^\infty$. Thus, α^∞ is the unique element in normal form which is ρ^*-related to α. $\square$

We are now ready to prove the main theorem of this section.

Theorem 2.3. *Let C be a category and let M be the monoid constructed as above. Then*

(i) *C admits a faithful functor σ to M;*
(ii) *if $\tau : C \to N$ is a functor to a monoid N, then there exists a morphism $\theta : M \to N$ such that $\sigma\theta = \tau$;*
(iii) *C admits a faithful functor to a unipotent monoid if and only if C is locally unipotent;*
(iv) *C admits a faithful functor to a cancellative monoid if and only if C is cancellative.*

Proof. (i) Let $\sigma : \mathrm{Mor}\ C \to M$ be given by $x\sigma = [\bar{x}]$. We claim that σ is a faithful functor.

To see that σ is a functor, let $x, y \in \mathrm{Mor}\ C$ be such that $\mathbf{t}(x) = \mathbf{s}(y)$, and observe that

$$(xy)\sigma = [\overline{xy}] = [\bar{x}\,\bar{y}] = [\bar{x}][\bar{y}] = (x\sigma)(y\sigma).$$

To see that σ is faithful, suppose $x, y \in \mathrm{Mor}\ C$ are morphisms with $\mathbf{s}(x) = \mathbf{s}(y), \mathbf{t}(x) = \mathbf{t}(y)$ and $x\sigma = y\sigma$. From $[\bar{x}] = [\bar{y}]$ we deduce that $\bar{x}^\infty = \bar{y}^\infty$; if this word is empty then $x = y$ is an identity, otherwise, $\bar{x} = \bar{x}^\infty = \bar{y}^\infty = \bar{y}$ and again, $x = y$.

(ii) Define $\theta : M \to N$ by

$$[\overline{a_1} \cdots \overline{a_n}]\theta = a_1\tau \cdots a_n\tau$$

where $a_i \in \bar{C}, 1 \leq i \leq n$. To see that θ is well defined, we note that if $(\bar{x}\,\bar{y}, \overline{xy}) \in \rho$, then $\mathbf{t}(x) = \mathbf{s}(y)$ so that $x\tau y\tau = (xy)\tau$ in N. Since ρ^* is generated by ρ, it follows that θ is well defined; clearly θ is a morphism. For $x \in \text{Mor } C$ we have that $x\sigma\theta = [\bar{x}]\theta = x\tau$ as required.

(iii) Clearly a necessary condition for C to admit a faithful functor to a unipotent monoid is that C be locally unipotent.

Suppose conversely that C is locally unipotent, and assume for a contradiction that M is not unipotent. Then there is a non-empty word α in normal form such that $[\alpha] = [\alpha]^2$ in M. Suppose $\alpha = \overline{a_1} \cdots \overline{a_n}$ where $n \geq 1$ and $a_i \in \bar{C}, 1 \leq i \leq n$. Then as $\alpha\,\rho^*\,\alpha^2$ we have that

$$\alpha = ((\overline{a_1} \cdots \overline{a_n})(\overline{a_1} \cdots \overline{a_n}))^\infty$$

in the free monoid $\bar{C}^*$. If n is even, then since $\mathbf{t}(a_i) \neq \mathbf{s}(a_{i+1})$ for $1 \leq i < n$ we must have that

$$a_n = a_1^{-1}, \quad a_{n-1} = a_2^{-1}, \ldots, a_{\frac{n}{2}+1} = a_{\frac{n}{2}}^{-1}.$$

But then $\mathbf{s}(a_{\frac{n}{2}+1}) = \mathbf{t}(a_{\frac{n}{2}})$, a contradiction. On the other hand, if n is odd, then we must have that

$$a_n = a_1^{-1}, \quad a_{n-1} = a_2^{-1}, \ldots, a_{\frac{n+3}{2}} = a_{\frac{n-1}{2}}^{-1}$$

and

$$\alpha = \overline{a_1} \cdots \overline{a_{\frac{n-1}{2}}}\ \overline{a_{\frac{n+1}{2}} a_{\frac{n+1}{2}}}\ \overline{a_{\frac{n+3}{2}}} \cdots \overline{a_n}$$

in the free monoid $\bar{C}^*$. It follows that $a_{\frac{n+1}{2}}$ is idempotent in the category C; since C is locally unipotent, we must therefore have that $a_{\frac{n+1}{2}}$ is a local identity, which is again a contradiction. Consequently, M is unipotent.

(iv) Clearly for C to admit a faithful functor to a cancellative monoid it is necessary for C to be cancellative. To show the converse, we suppose C is cancellative and show that the monoid M is cancellative.

To this end, suppose $\bar{x} \in \bar{C}$ and $\alpha = \overline{a_1} \cdots \overline{a_n} \in \bar{C}^*$ and $\beta = \overline{b_1} \cdots \overline{b_m} \in \bar{C}^*$ are in normal form, where each $a_i \in \text{Mor } C \backslash \text{Ob } C$ and each $b_i \in \text{Mor } C \backslash \text{Ob } C$. Suppose further that $[\bar{x}][\alpha] = [\bar{x}][\beta]$. We shall show that $\alpha = \beta$, from which it will follow that $[\alpha] = [\beta]$. Since M is generated by elements represented by letters in $\bar{C}$ and every element of M has a representative in normal form, this will suffice to show that M is left cancellative. By symmetry of assumption, it will follow also that M is right cancellative.

Consider first the case in which $\mathbf{t}(x) \neq \mathbf{s}(a_1)$ and $\mathbf{t}(x) \neq \mathbf{s}(b_1)$. Then the words $\bar{x}\alpha$ and $\bar{x}\beta$ are easily verified to be in normal form and both represent the same element. But normal forms are unique, so we have $\bar{x}\alpha = \bar{x}\beta$ in $\bar{C}^*$, and since free monoids are cancellative, $\alpha = \beta$.

Next, consider the case where $\mathbf{t}(x) = \mathbf{s}(a_1) = \mathbf{s}(b_1)$. Now

$$\bar{x}\alpha = \bar{x}\ \overline{a_1}\cdots\overline{a_n} \quad \text{and} \quad \bar{x}\beta = \bar{x}\ \overline{b_1}\cdots\overline{b_m}$$

are reduced by the algorithm to

$$(\overline{xa_1})\overline{a_2}\cdots\overline{a_n} \quad \text{and} \quad (\overline{xb_1})\overline{b_2}\cdots\overline{b_m}$$

respectively, where each of $\overline{xa_1}$ and $\overline{xb_1}$ is either the empty word or a letter in $\bar{C}$.

Now $(\overline{xa_1})\overline{a_2}\cdots\overline{a_n}$ and $(\overline{xb_1})\overline{b_2}\cdots\overline{b_m}$ are both words in normal form, and represent the same element of M, so we must have

$$(\overline{xa_1})\overline{a_2}\cdots\overline{a_n} = (\overline{xb_1})\overline{b_2}\cdots\overline{b_m} \tag{1}$$

in the free monoid $\bar{C}^*$.

We claim now that $\overline{xa_1}$ and $\overline{xb_1}$ are the same length, that is, they are either both the empty word, or both letters in $\bar{C}$. Suppose for a contradiction that they have different lengths. Assume without loss of generality that $\overline{xa_1} = \epsilon$ and $\overline{xb_1} \neq \epsilon$. Then xa_1 is the identity at $\mathbf{s}(x)$, so we must have $\mathbf{t}(a_1) = \mathbf{s}(x)$. Also, we must have $xb_1 = a_2$. But then

$$\mathbf{s}(a_2) = \mathbf{s}(xb_1) = \mathbf{s}(x) = \mathbf{t}(a_1)$$

which contradicts the assumption that α is in normal form. Thus, we conclude that $\overline{xa_1}$ and $\overline{xb_1}$ are of the same length.

It now follows from (1) that $m = n$ and $a_i = b_i$ for $2 \leq i \leq m$. Furthermore, if $\overline{xa_1}$ and $\overline{xb_1}$ are non-empty then they are equal so we have $xa_1 = xb_1$ in the category C. If, on the other hand, they are both empty, then xa_1 and xb_1 are both the (unique) identity at $\mathbf{s}(x)$. But C is cancellative, so in both cases it follows that $a_1 = b_1$, and hence that $\alpha = \beta$ as required.

Finally, suppose for a contradiction that $\mathbf{t}(x) = \mathbf{s}(a_1)$ but $\mathbf{t}(x) \neq \mathbf{s}(b_1)$. Here, $\bar{x}\beta = \bar{x}\overline{b_1}\cdots\overline{b_m}$ is already in normal form, but $\bar{x}\alpha = \bar{x}\overline{a_1}\cdots\overline{a_n}$ is reduced by the algorithm to $(\overline{xa_1})\overline{a_2}\cdots\overline{a_n}$, where $\overline{xa_1}$ is either empty or a letter in $\bar{C}^*$. Once again employing the fact that normal forms are unique, we deduce that

$$(\overline{xa_1})\overline{a_2}\cdots\overline{a_n} = \bar{x}\overline{b_1}\cdots\overline{b_m} \tag{2}$$

in the free monoid $\bar{C}^*$.

Now if $\overline{xa_1}$ is a letter in $\bar{C}$ then we have $\overline{xa_1} = \bar{x}$, so that $x = xa_1$. Since C is cancellative, it follows that a_1 is an identity, which contradicts the assumption that $\overline{a_1}$ is a letter in $\bar{C}$.

If, on the other hand $\overline{xa_1}$ is empty then xa_1 is the local identity at $\mathbf{s}(x)$. It follows that $\mathbf{t}(a_1) = \mathbf{s}(x)$. But now by (2) it follows that $\bar{x} = \overline{a_2}$, so that $x = a_2$. Thus, we have $\mathbf{t}(a_1) = \mathbf{s}(x) = \mathbf{s}(a_2)$, which contradicts the assumption that α is in normal form.

We have shown that we cannot have $\mathbf{t}(x) = \mathbf{s}(a_1)$ and $\mathbf{t}(x) \neq \mathbf{s}(b_1)$, and by symmetry of assumption, it follows also that we cannot have $\mathbf{t}(x) = \mathbf{s}(b_1)$ and $\mathbf{t}(x) \neq \mathbf{s}(a_1)$. This completes the proof of Theorem 2.3. □

We end this section with some elementary observations. First, if C is a finite locally unipotent category, then each local submonoid T_α is a group. For any $\alpha, \beta \in$ Ob C, if T_α is a group, and $M_{\alpha\beta}$ and $M_{\beta\alpha}$ are both non-empty, then any $a \in M_{\alpha\beta}$ has an inverse in $M_{\beta\alpha}$. In fact, in view of the following routine technique, we can concentrate on categories C in which for distinct $\alpha, \beta \in$ Ob C, no morphism in $M_{\alpha\beta}$ has an inverse.

We define a relation $\equiv$ on Ob C by the rule

$$\alpha \equiv \beta \Leftrightarrow \exists a \in M_{\alpha\beta}, \quad a^{-1} \in M_{\beta\alpha}.$$

Clearly $\equiv$ is an equivalence relation on Ob C. Choose a transversal Γ of the $\equiv$-classes, and let D be the full subcategory of C with Ob $D = \Gamma$. For $\alpha \in$ Ob C let $\alpha_\Gamma \in \Gamma$ be such that $\alpha \equiv \alpha_\Gamma$, and choose and fix $a_{\alpha_\Gamma\alpha} \in M_{\alpha_\Gamma\alpha}$ having an inverse. Define $\theta : C \to D$ by

$$\alpha\theta = \alpha_\Gamma$$

for $\alpha \in$ Ob C, and

$$c_{\alpha\beta} = a_{\alpha_\Gamma\alpha} c_{\alpha\beta} a^{-1}_{\beta_\Gamma\beta},$$

for $\alpha, \beta \in$ Ob C and $c_{\alpha\beta} \in M_{\alpha\beta}$. The following result is now clear.

Lemma 2.4. *With definition as above, θ is a faithful functor from C to D. Consequently, for any monoid M, C admits a faithful functor to M if and only if D does likewise.*

3. Full Embeddings versus $(2, 1, 1)$-Embeddings

The most convenient approach to abundant semigroups is via the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, which weaken Green's relations $\mathcal{L}$ and $\mathcal{R}$ respectively.

The relation $\mathcal{L}^*$ is the equivalence relation associated with the preorder $\leq_{\mathcal{L}^*}$. We recall that $\leq_{\mathcal{L}^*}$ is defined on a semigroup S by the rule that $a \leq_{\mathcal{L}^*} b$ if and only if for all $x, y \in S^1$,

$$bx = by \text{ implies that } ax = ay.$$

It is well known (see, for example, [13]) that $a \leq_{\mathcal{L}^*} b$ ($a \,\mathcal{L}^*\, b$) if and only if $a \leq_{\mathcal{L}} b$ ($a \,\mathcal{L}\, b$) in some oversemigroup of S, where $\leq_{\mathcal{L}}$ is the preorder associated with Green's relation $\mathcal{L}$. Corresponding statements apply to the duals $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ of $\leq_{\mathcal{L}^*}$ and $\mathcal{L}^*$. Notice that $\leq_{\mathcal{L}^*}$, and consequently $\mathcal{L}^*$, are right compatible relations; dually, $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ are left compatible.

The intersection and the join of $\mathcal{L}^*$ and $\mathcal{R}^*$ are denoted by $\mathcal{H}^*$ and $\mathcal{D}^*$ respectively; unlike the case for Green's (unstarred) relations, $\mathcal{L}^*$ and $\mathcal{R}^*$ do not, in general, commute. Clearly for any semigroup we have that $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$; for a regular semigroup, these inclusions are equalities.

A semigroup S is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent and *adequate* if, in addition, the set $E(S)$ of idempotents is a semilattice. It is easy to see that for an element a of an adequate semigroup S the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of a contains a *unique* idempotent, denoted by a^* (a^+). Clearly abundant and adequate semigroups generalize regular and inverse semigroups respectively. Many of the structure theorems of regular semigroup theory have their analog for special classes of abundant and adequate semigroups known as *IC-abundant* and *ample* respectively. We concentrate here on ample semigroups.

An adequate semigroup S is ample (formerly, type A) if

$$ae = (ae)^+a \quad \text{and} \quad ea = a(ea)^*$$

for each $a \in S$ and $e \in E(S)$. The ample condition enables us to move the position of idempotents in products. Many adequate semigroups satisfy the ample condition, including inverse semigroups and primitive adequate semigroups. The latter are adequate semigroups with zero in which every non-zero idempotent is primitive. Regarding ample semigroups as algebras of type $(2, 1, 1)$, where the unary operations are $a \mapsto a^*$, $a \mapsto a^+$, ample semigroups form a *quasivariety* and as such are closed under subalgebra and direct product. Clearly then since inverse semigroups are ample any $(2, 1, 1)$-subalgebra of an inverse semigroup is an ample semigroup. On the other hand, given an ample semigroup S there is a $(2, 1)$-embedding of S into $\mathcal{I}_S$, that is, a semigroup embedding that respects $^+$ [2] (see also [4]). Of course, the dual result holds for *. A natural question is to ask whether any ample semigroup is a $(2, 1, 1)$-subalgebra of an inverse semigroup. Our task in this section is to show that this is equivalent to S being a full subsemigroup of an inverse semigroup.

Let S and T be semigroups and $\theta : S \to T$ be a (semigroup) morphism. We say that θ *preserves* $\mathcal{L}^*$ (in the terminology of [11], θ is *good*), if for any $a, b \in S$,

$$a\,\mathcal{L}^*\,b \text{ implies that } a\theta\,\mathcal{L}^*\,b\theta;$$

dually for $\mathcal{R}^*$. The proof of the following lemma is straightforward.

Lemma 3.1. *Let S and T be ample semigroups and let $\theta : S \to T$ be a morphism. Then θ preserves $\mathcal{L}^*$ if and only if for all $a \in S, (a\theta)^* = a^*\theta$; dually for $\mathcal{R}^*$.*

Let S be an ample semigroup. It follows from [2, Proposition 1.2] that $\phi : S \to \mathcal{I}_S$ given by $s\phi = \rho_s$ where dom $\rho_s = Ss^+$ and $x\rho_s = xs$ for all $x \in Ss^+$ is an embedding that preserves $\mathcal{R}^*$.

Lemma 3.2. *Let S and ϕ be as above. Then ϕ is a $(2, 1, 1)$-embedding if and only if S is regular.*

Proof. Our only concern is whether ϕ preserves $\mathcal{L}^*$.

If S is regular then it is well known that any embedding of S into a regular semigroup preserves $\mathcal{L} = \mathcal{L}^*$ [9].

Conversely, if ϕ preserves $\mathcal{L}^*$ and $s \in S$, then by Lemma 3.1

$$s^*\phi = (s\phi)^* = (s\phi)^{-1}s\phi = I_{\mathrm{im}\ s\phi} = I_{Ss}.$$

Consequently,

$$Ss^* = \mathrm{dom}\ s^*\phi = Ss$$

so that $s\,\mathcal{L}\,s^*$ and S is regular. □

In view of the above (and similar results for related embeddings) it is clear that embeddings of non-regular ample semigroups into inverse semigroups via translations are not going to preserve both $\mathcal{L}^*$ and $\mathcal{R}^*$, that is, they are not going to be $(2,1,1)$-embeddings. On the positive side we have the following.

Lemma 3.3. *Let S be a full subsemigroup of an ample semigroup T. Then S is a $(2,1,1)$-subalgebra of T (and hence an ample semigroup).*

Proof. Since S is full in T it is clear that S is closed under * and $^+$, so that S is a $(2,1,1)$-subalgebra. By the comments above, S is ample. □

Lemma 3.3 yields one direction of the following result. Here, as elsewhere, if T is a regular oversemigroup of S, then (unless stated otherwise) the relations $\mathcal{L}^*, \mathcal{R}^*, \leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}^*}$ are relations on S, and $\mathcal{L}, \mathcal{R}, \leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are relations on T.

Theorem 3.4. *Let S be an ample semigroup. Then S is a $(2,1,1)$-subalgebra of an inverse semigroup if and only if S is a full subsemigroup of an inverse semigroup.*

Proof. Let S be a $(2,1,1)$-subalgebra of an inverse semigroup U. We remark that for $a, b \in S$,

$$a^* = b^* \Leftrightarrow a\,\mathcal{L}^*\,b \Leftrightarrow a\,\mathcal{L}\,b \Leftrightarrow a^{-1}a = b^{-1}b,$$

and dually for $\mathcal{R}^*$.

Put

$$V = \{a_1^{\epsilon_1}a_2^{\epsilon_2}\cdots a_n^{\epsilon_n} : n \in \mathbb{N}\backslash\{0\}, a_i \in S, \epsilon_i \in \{1,-1\} \text{ for } 1 \leq i \leq n\},$$

that is, V is generated as an inverse subsemigroup of U by S. Hence for any $p, q \in V$,

$$p\,\mathcal{L}\,q \text{ in V} \Leftrightarrow p\,\mathcal{L}\,q \text{ in } U,$$

and dually for $\mathcal{R}$. We show by induction on the length of words q in V that S has non-empty intersection with the $\mathcal{L}$-class of q in V.

Clearly for any $a \in S$ we have that $S \cap L_a \neq \emptyset$. Now

$$a^{-1}a\,\mathcal{L}\,a\,\mathcal{L}\,a^*$$

in V so that $a^{-1}a = a^*$ and dually, $aa^{-1} = a^+$. Hence in V,

$$a^{-1}\,\mathcal{L}\,aa^{-1} = a^+,$$

so that $S \cap L_{a^{-1}} \neq \emptyset$.

Let $n > 2$ and make the inductive assumption that for any element $q = a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}}$ in V (with $a_i \in S, \epsilon_i \in \{1,-1\}, 1 \leq i \leq n-1$) we have that $S \cap L_q \neq \emptyset$.

Consider $w = a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^{\epsilon_n}$ where $a_i \in S, \epsilon_i \in \{1,-1\}, 1 \leq i \leq n$. Put $q = a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}}$ and $a_n = a$; by the inductive assumption there exists $t \in S$ with $q\,\mathcal{L}\,t$.

If $\epsilon_n = 1$, then

$$w = qa\,\mathcal{L}\,ta \in S.$$

Suppose now that $\epsilon_n = -1$. We have that

$$qa^{-1}a = qa^*\,\mathcal{L}\,ta^*.$$

Noticing that $ta^* \leq_{\mathcal{L}^*} a$ we call upon [5, Proposition 2.3] to obtain an element $b \in S$ such that $ta^*\,\mathcal{L}^*\,ba$. Hence $qa^*\,\mathcal{L}\,ba$ in V so that

$$w = qa^{-1} = qa^{-1}aa^{-1} = qa^*a^{-1}\,\mathcal{L}\,baa^{-1} = ba^+ \in S.$$

Thus in either case we can show that $S \cap L_w \neq \emptyset$. Induction yields that $S \cap L_q \neq \emptyset$ for every $q \in V$.

Let $e \in E(V)$. Then $e\,\mathcal{L}\,s$ for some $s \in S$ and consequently, $e\,\mathcal{L}\,s^{-1}s = s^*$. But V is inverse so that $e = s^* \in S$. Hence S is a full subsemigroup of V. $\square$

We end this section with some positive results of a partial nature.

The relation σ is defined on an ample semigroup S by the rule that $a\,\sigma\,b$ if and only if $ea = eb$ for some $e \in E(S)$. The relation σ is the least cancellative monoid congruence on S, and S is *proper*, if

$$\sigma \cap \mathcal{L}^* = \sigma \cap \mathcal{R}^* = \iota.$$

Lawson shows in [11] that a proper ample semigroup is fully embeddable into an inverse semigroup if and only if S/σ is embeddable into a group.

Corollary 3.5. *Any finite proper ample semigroup is fully embeddable into an inverse semigroup.*

On the other hand, for an ample semigroup with zero, σ is universal, so S fails to be proper unless both $\mathcal{L}^*$ and $\mathcal{R}^*$ are trivial.

A subsemigroup S of a regular semigroup T is *fully stratified* in T if

$$\leq_{\mathcal{L}^*} = \leq_{\mathcal{L}} \cap (S \times S) \quad \text{and} \quad \leq_{\mathcal{R}^*} = \leq_{\mathcal{R}} \cap (S \times S).$$

We recall from [5] that $a \in S$ is *square cancellable* if $a\,\mathcal{H}^*\,a^2$.

Lemma 3.6. *Let S be an ample semigroup and a $(2,1,1)$-subalgebra of an inverse semigroup T. Then*

(i) *S is fully stratified in T;*
(ii) *for any $a \in S$, a is square cancellable if and only if a lies in a subgroup of T; in this case the $\mathcal{H}^*$-class H_a^* of a is a cancellative monoid embedded in the subgroup H_a of T;*
(iii) *if a is a square cancellable element of S, and $b, c \in S$ are such that $b, c \leq_{\mathcal{R}^*} a$ and $ab\,\mathcal{R}^*\,ac$, then $b\,\mathcal{R}^*\,c$;*
(iv) *if S intersects every $\mathcal{H}$-class of T, then $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$.*

Proof. We know that

$$\mathcal{L}^* = \mathcal{L} \cap (S \times S) \quad \text{and} \quad \mathcal{R}^* = \mathcal{R} \cap (S \times S)$$

as S is a $(2,1,1)$-subalgebra. If $a, b \in S$ and $a \leq_{\mathcal{L}^*} b$, then $a^* \leq_{\mathcal{L}^*} b^*$, so that $a^*b^* = b^*a^* = a^*$. Consequently, in $T, a^* \leq_{\mathcal{L}} b^*$ so that

$$a\,\mathcal{L}\,a^* \leq_{\mathcal{L}} b^*\,\mathcal{L}\,b$$

and (i) holds.

From [5, Proposition 2.5], if a is square cancellable, then $a\,\mathcal{H}^*\,e$ for some $e \in E(S)$. Thus $H_a^* = H_e^*$ is a cancellative monoid embedded in H_a. The remaining assertions follow easily. □

Using [5, Theorem 4.3], we obtain a partial converse to Lemma 3.6. We recall that a monoid M is *right* (*left*) *reversible* if for any $a, b \in M, Ma \cap Mb \neq \emptyset$ ($aM \cap bM \neq \emptyset$). It is well known that a right reversible cancellative semigroup has a group of left quotients [1].

Proposition 3.7. *Let S be an ample semigroup such that $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$, condition* (iii) *of* Lemma 3.6 *holds, and the $\mathcal{H}^*$-class of any square cancellable element of S is right reversible* (*so has a group of left quotients*). *Then S is fully embeddable into* (*indeed, is a left order in*) *an inverse semigroup.*

If S is ample, $\mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$, and the $\mathcal{H}^*$-class of any square cancellable element of S is right and left reversible, then again S is fully embeddable into an inverse semigroup.

4. Undecidability

The aim of this section is to show that it is undecidable whether a finite ample semigroup is a full subsemigroup of an inverse semigroup. Effectively this result can be deduced from Lemma 2.4 and those in our subsequent section. The latter makes heavy use of a structure theorem of Fountain [3] for primitive adequate semigroups. We give here a short and direct argument.

The proof of the next lemma is omitted since it is entirely routine. Part (i) may be found in [8].

Lemma 4.1. *Let C be a category such that each local submonoid is unipotent. Then C^0 is primitive and has semilattice of idempotents $\{e_\alpha : \alpha \in \text{Ob } C\} \cup \{0\}$. Further,*

(i) *C^0 is inverse if and only if C is a groupoid; and*
(ii) *C^0 is ample if and only if C is cancellative.*

If C^0 is ample then for any $a \in M_{\alpha\beta}$ we have that

$$e_\alpha \, \mathcal{R}^* \, a \, \mathcal{L}^* \, e_\beta.$$

We denote the Brandt semigroup with index set I over a group G by $\mathcal{B}^0(G; I)$. More generally, for any monoid M, we let $\mathcal{B}^0(M; I)$ denote the semigroup with underlying set

$$(I \times M \times I) \cup \{0\}$$

and with binary operation given by

$$(i, m, j)(k, n, l) = \begin{cases} (i, mn, l) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and

$$0x = x = x0 \quad \text{for all } x \in \mathcal{B}^0(M; I).$$

Clearly $\mathcal{B}^0(M; I)$ may be viewed as the categorical at zero semigroup associated with a category $\mathcal{B}(M; I)$. In the case that M is a group, the latter is called a *Brandt groupoid.*

Proposition 4.2. *Let C be a category such that each T_α is unipotent. Then C^0 embeds as a full subsemigroup of a (finite) inverse semigroup if and only if C embeds into a (finite) groupoid.*

Proof. If C embeds into a groupoid D, then (by restricting the object set of D), C is a subcategory of a groupoid E with $\text{Ob } C = \text{Ob } E$. Now C^0 embeds into E^0 and the latter is inverse by Lemma 4.1. Clearly C^0 is full in E^0.

Conversely, suppose that C^0 is a full subsemigroup of an inverse semigroup T. By Lemma 4.1 we know that the non-zero idempotents of C^0 are primitive. Since C^0 is full in T it is easy to see that the zero of C^0 is the zero of T and clearly T is primitive. Hence T is a 0-direct union of Brandt semigroups and C embeds into the union of the corresponding Brandt groupoids. $\square$

Corollary 4.3. *It is undecidable whether a finite ample semigroup embeds as a full subsemigroup of a finite inverse semigroup, or of an inverse semigroup.*

Proof. Let C^0 be the categorical at zero semigroup associated with a finite cancellative category C. By Lemma 4.1, C^0 is ample. By Proposition 4.2 C^0 embeds as a full subsemigroup into a (finite) inverse semigroup if and only if C embeds into a (finite) groupoid. By [14, Theorem 7.1] the latter is undecidable. $\square$

5. Primitive Adequate Semigroups

In this section we use a structure theorem of Fountain [3] to find necessary and sufficient conditions for a primitive adequate semigroup to be a full subsemigroup of an inverse semigroup. We recall from Sec. 3 that a primitive adequate semigroup is of necessity ample.

Note first that if S is a primitive adequate semigroup *without* zero, then S is a cancellative monoid and our embeddability question becomes the classic problem of embedding a cancellative monoid in a group. Hence we confine our attention to primitive adequate semigroups with zero. For convenience we briefly give the details of Fountain's construction that we require, simplifying his notation for our purposes.

Let C be a cancellative category such that if α, β are distinct elements of $\Gamma = \mathrm{Ob}\, C$ then no element of $M_{\alpha\beta}$ has an inverse. Let I be a non-empty set and suppose that Γ indexes a partition $\{I_\alpha : \alpha \in \Gamma\}$ of I. We denote by $\mathcal{M}^0 = \mathcal{M}^0(C; I, \Gamma)$ the semigroup with underlying set

$$\left(\bigcup_{\alpha,\beta\in\Gamma} I_\alpha \times M_{\alpha\beta} \times I_\beta \right) \cup \{0\}$$

and binary operation given by

$$(i, m, j)(k, n, l) = \begin{cases} (i, mn, l) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and

$$0x = x = x0 \quad \text{for all } x \in \mathcal{M}^0(C; I, \Gamma).$$

The semigroup $\mathcal{M}^0$ is special case of a *PA blocked Rees matrix semigroup.*

Theorem 5.1 [3, Proposition 5.5]. *Let S be a semigroup with zero. Then S is primitive adequate if and only if S is isomorphic to a semigroup of the form $\mathcal{M}^0(C; I, \Gamma)$.*

Lemma 5.2. *Let $\mathcal{M}^0 = \mathcal{M}^0(C; I, \Gamma)$ be a primitive adequate semigroup. Suppose that $\theta : C \to M$ is a faithful functor to a unipotent monoid M. Then $\mathcal{M}^0$ is fully embeddable in $\mathcal{B}^0(M; I)$.*

Proof. Define $\phi : \mathcal{M}^0 \to \mathcal{B}^0$ by

$$0\phi = 0, \quad (i, a, j)\phi = (i, a\theta, j).$$

It is routine to check that ϕ is an embedding. Since θ is a functor it follows that $\mathrm{Im}\, \phi$ is full in $\mathcal{B}^0$. □

In the above result, if $|I_\alpha| = 1$ for each $\alpha \in \Gamma$, then effectively $\mathcal{M}^0$ is the categorical at zero semigroup associated with C. The condition that for any distinct $\alpha, \beta \in \Gamma$, no element of $M_{\alpha\beta}$ has an inverse is akin to the simplification provided by Lemma 2.4. In the proof of Lemma 5.2, we do not utilize this condition. Now Theorem 2.3 yields the following.

Corollary 5.3. *Let C be a cancellative category. Then C is embeddable in a category $\mathcal{B}(M;I)$, where M is a cancellative monoid.*

Proposition 5.4. *Let $\mathcal{M}^0 = \mathcal{M}^0(C;I,\Gamma)$ be a primitive adequate semigroup. Then $\mathcal{M}^0$ is a full subsemigroup of an inverse semigroup if and only if there is a faithful functor $\theta : C \to G$ for some group G.*

Proof. One direction of the result follows from Lemma 5.2.

Suppose that $\mathcal{M}^0$ is a full subsemigroup of an inverse semigroup T. Then T is primitive and hence a 0-direct union of Brandt semigroups. Consequently, T is a full subsemigroup of a Brandt semigroup U.

Using the fact that $\mathcal{M}^0$ is full in U, we can use I to index the non-zero $\mathcal{L}$-classes and $\mathcal{R}$-classes of U. We know that U is isomorphic to a Brandt semigroup $\mathcal{B}^0(G;J)$ for some group G and set J with $|J| = |I|$. We may therefore assume that $I = J$ and $\theta : U \to \mathcal{B}^0$ is an isomorphism such that $(i,a,j)\theta$ is of the form (i,b,j) for each non-zero $(i,a,j) \in \mathcal{M}^0$.

For each $\alpha \in \Gamma$, choose and fix $i_\alpha \in I_\alpha$. Define $\phi : C \to G$ by the rule that for $a \in M_{\alpha\beta}$,

$$a\phi = b \quad \text{where } (i_\alpha, a, i_\beta)\theta = (i_\alpha, b, i_\beta).$$

Straightforward checks show that ϕ is a faithful functor. □

It is easy to see from the characterization of $\mathcal{L}^*$ and $\mathcal{R}^*$ on $\mathcal{M}^0(C;I,\Gamma)$ given in [3] that $\mathcal{R}^* \circ \mathcal{L}^* = \mathcal{L}^* \circ \mathcal{R}^*$ if and only if for any $\alpha,\beta \in \Gamma$, $M_{\alpha\beta} \neq \emptyset$ if and only if $M_{\beta\alpha} \neq \emptyset$. Consequently, if $\mathcal{M}^0$ is finite and $\mathcal{R}^* \circ \mathcal{L}^* = \mathcal{L}^* \circ \mathcal{R}^*$, then C is a disjoint union of groups and $\mathcal{M}^0$ is inverse.

A primitive adequate semigroup S with zero is *primitively indecomposable* if and only if S is a 0-disjoint union of principal right ideals $e_iS, i \in J$ such that $e_i \mathcal{D} e_j$ for each $i,j \in J$. From [3, Theorem 4.9], primitive adequate primitively indecomposable semigroups are precisely the semigroups of the form $\mathcal{B}^0(M;I)$ where M is a cancellative monoid. Putting together Theorem 2.3 and Lemma 5.2 we deduce the following.

Corollary 5.5. *A primitive adequate semigroup is fully embeddable into a primitive adequate primitively indecomposable semigroup.*

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