MEMBERSHIP OF $\mathcal{A} \vee \mathcal{G}$ FOR CLASSES OF FINITE WEAKLY ABUNDANT SEMIGROUPS

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Abstract. We consider the question of membership of $\mathbf{A} \vee \mathbf{G}$, where $\mathbf{A}$ and $\mathbf{G}$ are the pseudovarieties of finite aperiodic semigroups, and finite groups, respectively. We find a straightforward criterion for a semigroup S lying in a class of finite semigroups that are weakly abundant, to be in $\mathbf{A} \vee \mathbf{G}$.

The class of weakly abundant semigroups contains the class of regular semigroups, but is much more extensive; we remark that any finite monoid with semilattice of idempotents is weakly abundant. To study such semigroups we develop a number of techniques that may be of interest in their own right.

1. Introduction

We consider the celebrated question of membership for a finite semigroup in the pseudovariety $\mathbf{A} \vee \mathbf{G}$. For orthodox semigroups there is an elegant criterion provided by McAlister in [17], namely, a finite orthodox semigroup lies in $\mathbf{A} \vee \mathbf{G}$ if and only if $\mathcal{H}$ is a congruence on S. This paved the way for McAlister to characterise in [18] regular semigroups in $\mathbf{A} \vee \mathbf{G}$. Steinberg [23] has shown that the work of Tilson and Rhodes [20] allows McAlister's results to be extended to the question of membership of $\mathbf{A} \vee \mathbf{H}$ for an arbitrary pseudovariety $\mathbf{H}$ of groups $\mathbf{H}$. In a different direction, the first two authors have shown [8] that a finite bountiful semigroup lies in $\mathbf{A} \vee \mathbf{G}$ if and only if $\mathcal{H} \subseteq \mu$, where μ is the analogue for a bountiful semigroup of the greatest idempotent separating congruence on a regular semigroup. Bountiful semigroups are approached via the generalisations $\mathcal{L}^*$ and $\mathcal{R}^*$ of Green's relations $\mathcal{L}$ and $\mathcal{R}$, details of which are given in Section 2. A semigroup S is *bountiful* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent, $E(S)$ is a band and the *idempotent connected* condition (IC) holds. For

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such semigroups, the greatest congruence contained in $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ is denoted by μ.

If S is regular and bountiful, then S is orthodox and $\mathcal{H}^* = \mathcal{H}$, so that the above result from [8] simply says that $\mathcal{H}$ is a congruence, as in McAlister's original result.

The aim of the current article is to extend the result of Fountain and Gomes. We will consider classes of semigroups arrived at via considerations of the relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ on a semigroup S (defined in the next section), which are distant analogues of $\mathcal{L}$ and $\mathcal{R}$ and which contain the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, respectively. A semigroup is *weakly abundant* if every $\widetilde{\mathcal{L}}$-class and every $\widetilde{\mathcal{R}}$-class contains an idempotent. We consider two classes of weakly abundant semigroups. For one class we can show that if a semigroup S lies in the class and is also in $\mathbf{A} \vee \mathbf{G}$, then $\mathcal{H} \subseteq \mu$, where here μ is the largest congruence contained in $\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}}$. For semigroups in the other class we can show the converse. A finite bountiful semigroup lies in both classes, and for such a semigroup $\mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$.

Our approach is inspired by that of [8], but there are two issues. First, we are working with semigroups with weaker properties, and second, the argument of [8] calls upon many techniques already established for the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. We have the task of developing their analogues here for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$, many of which are of independent interest.

The structure of the paper is as follows. In Section 2 we define the relations $\mathcal{L}^*, \widetilde{\mathcal{L}}, \mathcal{R}^*$ and $\widetilde{\mathcal{R}}$, and quote a number of results concerning *fundamental* semigroups where, in this context, a semigroup is fundamental if and only if the largest congruence μ contained in $\widetilde{\mathcal{H}}$ is trivial. We also discuss conditions we call (C) and (WIC) and demonstrate their independence in Section 3.

Section 4 gives the proof of one of our results: that if S is finite weakly abundant such that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ are respectively right and left congruences and the regular elements of S form a subsemigroup, then if $S \in \mathbf{A} \vee \mathbf{G}$ we must have $\mathcal{H} \subseteq \mu$. It is notable that to prove this result we must make use of that of McAlister in the regular case.

In Section 5 we first define *weakly bountiful* semigroups; these are weakly abundant semigroups with band of idempotents, and satisfying condition (WIC). Such semigroups are a generalisation of bountiful semigroups, which themselves include the class of orthodox semigroups. We make a careful analysis of the congruence δ on a weakly bountiful semigroup, where δ is the analogue of the least inverse congruence on an orthodox semigroup. If S is weakly bountiful, then S/δ lies in the appropriate class of semigroups analogous to inverse semigroups, the class of weakly adequate semigroups with condition (A). Some of the results of Section 5 have also appeared in [19], where different terminology is employed. In Section 6 we analyse the least monoid unipotent congruence, σ, on a weakly bountiful semigroup.

Section 7 considers the existence of finite *proper covers* for weakly adequate semigroups with (A). Our covers must be carefully stitched together from covers that are known to exist for semigroups having only analogous one-sided properties.

Finally in Section 8 we present our main result, Theorem 8.13, which states that a finite weakly bountiful semigroup for which $\mathcal{H} \subseteq \mu$ must lie in $\mathbf{A} \vee \mathbf{G}$. For such a semigroup S, we construct a cover T from V and S/μ, where V is a cover of the weakly adequate semigroup with (A), S/δ, proven to exist in Section 7. The semigroup T has the property that $\sigma \cap \mu$ is trivial, and it is this that allows us to prove Theorem 8.13. An immediate consequence of this result and Theorem 4.2 is that $\mathcal{H} \subseteq \mu$ is a necessary and sufficient condition for a finite weakly bountiful semigroup with (C) to be a member of $\mathbf{A} \vee \mathbf{G}$.

2. Preliminaries

For the convenience of the reader we gather together in this section some basic definitions and elementary observations concerning abundant and weakly abundant semigroups. Further details may be found in [4], [6] and [16]. For basic semigroup notation and terminology we follow [14].

Let S be a semigroup with subset of idempotents U. The relation $\widetilde{\mathcal{L}}_U$ is defined by the rule that for any $a, b \in S$, $a \,\widetilde{\mathcal{L}}_U\, b$ if and only if for all $e \in U$,

$$ae = a \text{ if and only if } be = b.$$

Recall that the relation $\mathcal{L}^*$ is defined on S by the rule that $a \,\mathcal{L}^*\, b$ if and only if for any $x, y \in S^1$,

$$ax = ay \text{ if and only if } bx = by.$$

It is easy to see that

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U,$$

with equality if S is regular and $U = E(S)$. Another useful observation is that if $a \in S$ and $e \in U$, then $a \,\widetilde{\mathcal{L}}_U\, e$ if and only if $ae = a$ and for any $f \in U$, $af = f$ implies that $ef = e$. The relations $\widetilde{\mathcal{R}}_U$ and $\mathcal{R}^*$ are defined dually; we continue the analogy with the notation for Green's relations by denoting $\widetilde{\mathcal{L}}_U \cap \widetilde{\mathcal{R}}_U$ by $\widetilde{\mathcal{H}}_U$ and $\mathcal{L}^* \cap \mathcal{R}^*$ by $\mathcal{H}^*$. Clearly $\widetilde{\mathcal{L}}_U, \mathcal{L}^*, \widetilde{\mathcal{R}}_U$ and $\mathcal{R}^*$ are equivalences, hence so also are $\widetilde{\mathcal{H}}_U$ and $\mathcal{H}^*$.

It is easy to see that (as is the case for $\mathcal{L}$ and $\mathcal{R}$) $\mathcal{L}^*$ and $\mathcal{R}^*$ are, respectively, right and left congruences. On the other hand, $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ need not be; if they are, then we say that *Condition (C) holds (with respect to U).*

The semigroup S is *weakly U-abundant* if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class contains an idempotent of U. If a is an element of such an S, then we commonly denote idempotents of U in the $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class of a by a^* and a^+ respectively. Beware, however, that there may not be a unique choice for a^* or a^+. The following lemma is immediate.

Lemma 2.1. [4, Lemma 2.1] *Let S be a weakly U-abundant semigroup. Then for any $a, b \in S$,*

$$(ab)^* \leq_{\mathcal{L}} b^* \text{ and } (ab)^+ \leq_{\mathcal{R}} a^+.$$

When $U = E(S)$ we drop the subscript $_U$ from our notation and terminology; for example we write $\widetilde{\mathcal{L}}_{E(S)}$ as $\widetilde{\mathcal{L}}$ and refer to a weakly $E(S)$-abundant semigroup as *weakly abundant*.

A semigroup S is *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent. It is well known, and easy to see, that in such a semigroup $\mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$. Consequently, an abundant semigroup is weakly abundant with (C).

Morphic images of regular and inverse semigroups are regular and inverse respectively. The same is not true even for abundant semigroups with semilattice of idempotents [7]. With this in mind we make the following definition. Let S be a semigroup, U a subset of $E(S)$ and $\varphi : S \to T$ be a morphism. Then φ is *U-admissible* if for any $a, b \in S$,

$$a \,\widetilde{\mathcal{L}}_U\, b \text{ implies that } a\varphi \,\widetilde{\mathcal{L}}_{U\varphi}\, b\varphi$$

and

$$a \,\widetilde{\mathcal{R}}_U\, b \text{ implies that } a\varphi \,\widetilde{\mathcal{R}}_{U\varphi}\, b\varphi.$$

If, in addition, the reverse implications hold we say that φ is *strongly U-admissible*.

We denote by μ_U the largest congruence contained in $\widetilde{\mathcal{H}}_U$, and say that S is *U-fundamental* if μ_U is trivial. If S is abundant, then $\mathcal{H}^* = \widetilde{\mathcal{H}}$ and so μ is the largest congruence contained in $\mathcal{H}^*$; if moreover S is regular, then $\widetilde{\mathcal{H}} = \mathcal{H}^* = \mathcal{H}$ and μ is the largest congruence contained in $\mathcal{H}$ and so it is the maximum idempotent separating congruence on S. Thus our notation is consistent with the standard notation for regular semigroups. When more than one semigroup is involved, we may write μ^S for the relation μ on S.

We now quote three results from [4], essential for the sequel.

Lemma 2.2. [4, Lemma 2.3] *Let S be a semigroup with $U \subseteq E(S)$, and let $\varphi : S \to T$ be an onto morphism. Then φ is strongly U-admissible if and only if the kernel of φ is contained in $\widetilde{\mathcal{H}}_U$. In this case, S is weakly U-abundant if and only if T is weakly $U\varphi$-abundant, and S satisfies (C) with respect to U if and only if T satisfies (C) with respect to $U\varphi$.*

Proposition 2.3. [4, Lemma 2,4] *Let S be a semigroup and let $U \subseteq E(S)$. The natural morphism $\mu_U^\natural$ associated with μ_U is strongly U-admissible and restricts to an injection on U. Denoting the image of U under $\mu_U^\natural$ by $\overline{U}$, we have that S/μ_U is $\overline{U}$-fundamental.*

If S is weakly U-abundant, then S/μ_U is weakly $\overline{U}$-abundant; if S satisfies (C), then so does S/μ_U.

We say that a subsemigroup T of S is *U-full* if $U \subseteq T$.

Lemma 2.4. [4, Lemma 2.7] *Let T be a U-full subsemigroup of S. Then for any $a, b \in T$,*

$$a\, \widetilde{\mathcal{L}}_U\, b \text{ in } T \text{ if and only if } a\, \widetilde{\mathcal{L}}_U\, b \text{ in } S$$

and

$$a\, \widetilde{\mathcal{R}}_U\, b \text{ in } T \text{ if and only if } a\, \widetilde{\mathcal{R}}_U\, b \text{ in } S.$$

Consequently, if S is weakly U-abundant, then so is T; if S satisfies (C) with respect to U, then so does T.

If S is U-fundamental weakly U-abundant with (C), then so is T.

It remains in this section to discuss the weak idempotent connected condition. A fuller version of some of the ideas we present here is contained in [21] and in [4]. Essentially, all of the idempotent connected and ample (formerly, type A) conditions extant give some control over the position of idempotents in products, usually facilitating results for abundant or weakly abundant semigroups reminiscent of those in the regular case.

For a band B and element e of B we denote by $\langle e \rangle$ the principal order ideal generated by e; so that

$$\langle e \rangle = \{x \in B : x \leq e\} = \{x \in B : ex = xe = x\}.$$

Let S be a weakly B-abundant semigroup where B is a band. We say that S satisfies the *weak idempotent connected* condition (WIC) (with respect to B) if for any $a \in S$ and some a^*, a^+, if $x \in \langle a^+ \rangle$ then there exists $y \in B$ with $xa = ay$; and dually, if $z \in \langle a^* \rangle$ then there exists $t \in B$ with $ta = az$.

Some observations concerning this definition are in order. First, it is easy to see that a regular semigroup satisfies (WIC) with respect to $E(S)$. Second, we can replace 'some' in (WIC) by 'any'. For suppose that S has (WIC), $a \in S$, a^+ is the chosen idempotent of B in the $\widetilde{\mathcal{R}}_B$-class of a, and $a^\dagger$ is another element of B in the same $\widetilde{\mathcal{R}}_B$-class. If $x \in \langle a^\dagger \rangle$, we certainly have that $xa^+ = a^+xa^+ \in \langle a^+ \rangle$ and so by (WIC),

$$xa = (xa^+)a = ay$$

for some $y \in B$. Similarly, we can take z to lie in $\langle a^\circ \rangle$ for any $a^\circ \in B$ lying in the $\widetilde{\mathcal{L}}_B$-class of a. Finally, if $a \in S$, and $x, y \in B$ with $xa = ay$, then for any a^* we have that $xa = a(a^*ya^*)$. Thus in the definition of (WIC) we may choose the y to lie in any given $\langle a^* \rangle$, and dually, the t to lie in any given $\langle a^+ \rangle$.

In the case where B is a semilattice, (WIC) simplifies considerably. Let U be a subset of idempotents of a semigroup S; it is easy to see that for any $e, f \in U$, $e \,\widetilde{\mathcal{L}}_U\, f$ if and only if $e \,\mathcal{L}\, f$. Hence if $U = E$ is a semilattice, there is at most one idempotent of E in each $\widetilde{\mathcal{L}}_E$-class. Dual remarks hold for the relation $\widetilde{\mathcal{R}}_U$. Thus if S is weakly E-abundant for a semilattice E, the elements a^* and a^+ for $a \in S$ are uniquely defined; such a semigroup is often referred to as *weakly E-adequate*. We can thus define the *ample* condition (A) on S by:

$$ae = (ae)^+a \text{ and } ea = a(ea)^*$$

for all $a \in S$ and $e \in E$.

Consider a weakly E-adequate semigroup S. If S satisfies (A) then it is clear that S satisfies (WIC). On the other hand, if S satisfies (WIC), $a \in S$ and $e \in E$, then first observe that $ae = a(a^*ea^*) = ta$ for some $t \in E$. It follows that $tae = ae$ and so $t(ae)^+ = (ae)^+$. Thus

$$(ae)^+a = t(ae)^+a = (ae)^+ta = (ae)^+ae = ae;$$

together with the dual argument we have that the ample identities hold. Thus we have shown:

Lemma 2.5. [21] *Let S be a weakly E-adequate semigroup. Then S satisfies (WIC) if and only if S satisfies (A).*

Any weakly E-adequate semigroup satisfying (A) and (C) is said to be *weakly E-ample.*

The following lemma is an easy extension of Lemma 2.4.

Lemma 2.6. [4, Lemma 3.2] *Let B be a band and let T be a B-full subsemigroup of a weakly B-abundant semigroup S. If S satisfies (WIC), then so does T.*

We end this section by showing that (WIC) is respected by strongly admissible morphisms.

Lemma 2.7. [4, Lemma 3.3] *Let B be a band and let S be a weakly B-abundant semigroup and let $\theta : S \to T$ be a strongly admissible morphism from S onto a semigroup T. Then S has (WIC) with respect to B if and only if T has (WIC) with respect to $B\theta$.*

3. Examples

The reader might well ask whether conditions (C) and (WIC) are independent for finite weakly abundant semigroups with band of idempotents. Indeed they are, for a finite weakly abundant semigroup with band of idempotents may satisfy neither, one, or both of conditions (C) and (WIC), as we now demonstrate.

First we comment that any finite regular semigroup with band of idempotents certainly satisfies (C) and (WIC); for a non-regular (indeed non-abundant) example we refer the reader to Example 6.4 of [4].

In [11] we argue that the subset $S = \{\alpha, \beta, \alpha^+, \beta^+, \beta\alpha, \epsilon\}$ of the partial transformation monoid $\mathcal{PT}_3$ on $\{1, 2, 3\}$ is a semigroup that is not abundant (but has further properties, which we need not mention here), where ϵ is the empty transformation and α, β are given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ \times & 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 \\ \times & \times & 2 \end{pmatrix}.$$

Here

$$\alpha^+ = \begin{pmatrix} 1 & 2 & 3 \\ \times & 2 & 3 \end{pmatrix} \text{ and } \beta^+ = (\beta\alpha)^+ = \begin{pmatrix} 1 & 2 & 3 \\ \times & \times & 3 \end{pmatrix}.$$

It is easy to check that the multiplication table of S is given by

	α^+	α	β^+	β	$\beta\alpha$	ϵ
α^+	α^+	α	β^+	β	$\beta\alpha$	ϵ
α	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
β^+	β^+	$\beta\alpha$	β^+	β	$\beta\alpha$	ϵ
β	β	$\beta\alpha$	ϵ	ϵ	ϵ	ϵ
$\beta\alpha$	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ
ϵ	ϵ	ϵ	ϵ	ϵ	ϵ	ϵ

The idempotents of $T = S^1$ form a semilattice $E(T) = \{1, \alpha^+, \beta^+, \epsilon\}$. It is clear that any *finite* monoid U with $E(U)$ a semilattice is weakly abundant, so that certainly T is weakly abundant. We see that $\alpha^* = 1 = 1^*$ so that $\alpha \,\widetilde{\mathcal{L}}\, 1$, but $\alpha^2 = \epsilon$ which is not $\widetilde{\mathcal{L}}$-related to α. Thus (C) fails to hold. We also have that $\beta^+ \leq \alpha^+$ and $\beta^+\alpha = \beta\alpha$, but

$\alpha(\beta^+\alpha)^* = \alpha(\beta\alpha)^* = \alpha 1 = \alpha$, so that condition (A) also fails. By Lemma 2.5, T^1 does not satisfy (WIC).

The semigroup of Example 2.2 of [7] is finite (weakly) abundant with semilattice of idempotents, and, as pointed out in [7], does not satisfy (A) and hence does not satisfy (WIC). On the other hand, as it is abundant, we automatically have that (C) holds.

It is easy to see that if S is the two element null semigroup with an identity adjoined, then S is weakly abundant with semilattice of idempotents, (C) fails, but (WIC) holds since S is commutative. For a slightly more sophisticated example, with band of idempotents *not* a semilattice, we have the following.

Example 3.1. *Let $S = \{a, b, u, 0\}$, where $\{a, b\}$ is a two element right zero semigroup, $\{u, 0\}$ is a two element null semigroup, and the binary operation is completed by insisting that $0x = 0 = x0$ for all $x \in S$, and*

$$au = bu = u,\ ua = ub = 0.$$

Then S is a semigroup, and $T = S^1$ is a finite weakly abundant semigroup with band of idempotents $E(T) = \{1, a, b, 0\}$ which has (WIC) but not (C).

Proof. It is easy to check that T is a semigroup,

$$a\,\widetilde{\mathcal{R}}\,b\,\widetilde{\mathcal{R}}\,u \text{ and } u\,\widetilde{\mathcal{L}}\,1,$$

so that T is weakly abundant with band of idempotents as given. ¿From $u\,\widetilde{\mathcal{L}}\,1$ we see (by multiplying with u on the right) that $\widetilde{\mathcal{L}}$ is not a right congruence and so (C) fails.

On the other hand (WIC) holds. Clearly $x1 = 1x$ and $x0 = 0x$ for any $x \in E(T)$. If $x \in E(T)$ and $x \le a^+$, then, taking $a^+ = a$, we have that $x = a$ or $x = 0$, which in either case commutes with a; dually for $x \le b^+$. If $y \in E(T)$ and $y \le u^+$, then taking $u^+ = a$ we again have that $y = a$ or $y = 0$. In the first case, $au = u = u1$ and in the second, $0u = 0 = u0$.

On the other hand, if $x \in E(T)$ and $x \le a^*$, then $x \in \{0, a\}$ so that again, x commutes with a; similarly if $x \le b^*$. Finally, *any* idempotent is below $u^* = 1$ in the natural partial order. We know that 1 and 0 commute with u, and

$$ua = ub = 0 = 0u,$$

completing our check that (WIC) holds. □

4. Finite weakly abundant semigroups with (C) and subsemigroup of regular elements

For any semigroup S we denote the set of regular elements by $\operatorname{Reg} S$. We will be interested in semigroups in which $\operatorname{Reg} S$ forms a subsemigroup.

Theorem 4.1. [13, Result 7] *In any semigroup S, $\operatorname{Reg} S$ forms a subsemigroup if and only if $\langle E(S)\rangle$ is regular.*

This section is devoted to the proof of the following result.

Theorem 4.2. *Let S be a finite weakly abundant semigroup with (C) such that $\operatorname{Reg} S$ is a subsemigroup. If $S \in \boldsymbol{A} \vee \boldsymbol{G}$, then $\mathcal{H} \subseteq \mu$.*

Proof. We begin with a subsidiary lemma. The reader will find it useful to bear in mind, that for a semigroup S and $E \subseteq E(S)$, if $a \,\widetilde{\mathcal{L}}\, be$ for some $a, b \in S$ and $e \in E$, then as certainly $(be)e = be$, we have that $ae = a$. Dually, if $a \,\widetilde{\mathcal{R}}\, eb$, then $ea = a$.

Lemma 4.3. *If S is a finite weakly abundant semigroup and $\varphi : S \to T$ is a strongly admissible morphism from S onto T, then T is finite weakly abundant. If $\operatorname{Reg} S$ is a subsemigroup of S, then $\operatorname{Reg} T$ is a subsemigroup of T.*

Proof. It is well known that if $e = a\varphi \in E(T)$, then as $e = a^n\varphi$ for all $n \in \mathbb{N}$, we have that $e = f\varphi$ for some $f \in E(S)$, so that $E(T) = E(S)\varphi$ and from Lemma 2.2, T is weakly abundant.

Suppose now that $\operatorname{Reg} S$ is a subsemigroup, and $a\varphi \in \operatorname{Reg} T$. As φ is onto, $a\varphi = a\varphi b\varphi a\varphi$ for some $a, b \in S$. Hence $a\varphi = a\varphi(ba)\varphi$, so that $a\varphi = a\varphi(ba)^n\varphi$ for all $n \in \mathbb{N}$. Choose n with $(ba)^n = g \in E(S)$. From $a\varphi = (ag)\varphi$ and the fact that φ is strongly admissible, we have $a \,\widetilde{\mathcal{H}}\, ag$ in S, and so $a = ag = a(ba)^n$. Thus a is regular. The lemma now follows easily. $\square$

To prove Theorem 4.2, let S be a finite weakly abundant semigroup with (C) such that $\operatorname{Reg} S$ is a subsemigroup, and suppose that $S \in \mathbf{A} \vee \mathbf{G}$. Let $T = S/\mu$, so that by Lemma 4.3 and Proposition 2.3, T is weakly abundant with (C), is fundamental, and $R = \operatorname{Reg} T$ is a subsemigroup; by Lemma 2.4, R is fundamental. As $\mathbf{A} \vee \mathbf{G}$ is a pseudovariety, R lies in $\mathbf{A} \vee \mathbf{G}$, and as R is certainly regular, Proposition 1.6 of of [17] gives that $\mathcal{H}$ is a congruence on $R = \operatorname{Reg} T$. Since R is fundamental we deduce that $\mathcal{H}$ is trivial on R and hence R and T have only trivial subgroups. But T is finite and so by Proposition 3.4.2 of [15], $\mathcal{H}$ is trivial on T. Thus if $a, b \in S$ and $a \,\mathcal{H}\, b$ in S, we have that $a\mu\,\mathcal{H}\,b\mu$ in T and so $a\mu = b\mu$, giving $a\,\mu\,b$. Hence $\mathcal{H} \subseteq \mu$ as required. □

Since an abundant semigroup is weakly abundant with (C) and in such a semigroup $\mathcal{H}^* = \widetilde{\mathcal{H}}$, we deduce the following.

Corollary 4.4. *Let S be a finite abundant semigroup such that* $\operatorname{Reg} S$ *is a subsemigroup. If $S \in \boldsymbol{A} \vee \boldsymbol{G}$, then $\mathcal{H} \subseteq \mu$.*

An abundant semigroup in which the idempotents generate a regular subsemigroup, and which satisfies an appropriately defined idempotent connected condition, is *concordant* [1]. From Theorem 4.1 $\operatorname{Reg} S$ is a subsemigroup in any concordant semigroup S.

Corollary 4.5. [8, Proposition 2.1] *If S is a finite concordant semigroup in $\boldsymbol{A} \vee \boldsymbol{G}$, then $\mathcal{H} \subseteq \mu$.*

5. The congruence δ_B

For convenience of reference we follow the pattern of established terminology and say that a semigroup S is *weakly bountiful* if it is weakly abundant, $E(S)$ is a band, and it satisfies (WIC). We recall that a semigroup S is *bountiful* if it is abundant, $E(S)$ is a band, and it is *idempotent connected*. As remarked in [4], an abundant semigroup with band of idempotents is idempotent connected if and only if it has (WIC). Thus a bountiful semigroup is weakly bountiful. The aim of this section is to investigate for weakly bountiful semigroups the analogue of the least inverse congruence on an orthodox semigroup. The approach in the *bountiful* case was initiated in [3], where the appropriate relation δ was introduced. It was later established in [25] and [12] that δ is always a congruence on a bountiful semigroup.

Throughout this section B denotes a band, and $E(e)$ the $\mathcal{D}$-class of $e \in B$; since B is a band we have that $\mathcal{D} = \mathcal{J}$, so that the sets $E(e)$ are partially ordered by the partial order associated with $\mathcal{J}$. We consider the relation δ_B for a weakly B-abundant semigroup S, soon specialising to the case where and S has (WIC), so that if $B = E(S)$, S is weakly bountiful. Some of our results concerning δ have also appeared in [19], where different terminology is employed. However, the authors of [19] claim that for a weakly B-abundant semigroup, conditions (IC) and (WIC) are equivalent - a counterexample is provided in [4].

The relation δ_B is defined on S as follows:

$$a\,\delta_B\,b \text{ if and only if } a = ebf,\ \ b = gah \text{ for some } e, f, g, h \in B.$$

Lemma 5.1. *(c.f* [19, Lemma 3.5]*) Let S be a weakly B-abundant semigroup. The following conditions are equivalent:*

(i) $a \,\delta_B\, b$;

(ii) $a = ebf$ *and* $b = gah$ *for some* $e \in E(b^+), f \in E(b^*), g \in E(a^+)$ *and* $h \in E(a^*)$;

(iii) $E(a^+)aE(a^*) = E(b^+)bE(b^*)$.

Moreover, if $a \,\delta_B\, b$*, then*

$$E(a^+) = E(b^+) \text{ and } E(a^*) = E(b^*).$$

Proof. We first remark that for any $a \in S$, if

$$a \,\widetilde{\mathcal{R}}_B\, a^+ \,\widetilde{\mathcal{R}}_B\, a^\dagger$$

where $a^+, a^\dagger \in B$, then certainly $a^+ \,\mathcal{R}\, a^\dagger$ in B, so that $E(a^+) = E(a^\dagger)$. Consequently, $E(a^+)$ and dually, $E(a^*)$, do not depend upon the choice of a^+, a^*.

(i) implies (ii). Since $a \,\delta_B\, b$, there exist $e, f, g, h \in B$ such that $a = ebf$ and $b = gah$. For any b^+ we have $a = eb^+bf$, so that $eb^+a = a$. It follows that $eb^+a^+ = a^+$ and so in the semilattice $B/\mathcal{D}$,

$$E(a^+) = E(eb^+a^+) \leq E(eb^+) \leq E(b^+).$$

Together with the dual, we obtain that

$$E(a^+) = E(eb^+) = E(b^+).$$

Thus we can assume that $e, g \in E(a^+) = E(b^+)$. Dually, we can take $f, h \in E(a^*) = E(b^*)$.

(ii) implies (iii). If a, b, e, f, g, h are given as in (ii), then as above we deduce $E(a^+) = E(b^+)$ and $E(a^*) = E(b^*)$. Hence

$$E(a^+)aE(a^*) = E(b^+)ebfE(b^*) \subseteq E(b^+)bE(b^*),$$

so that, together with the opposite inclusion, we have shown that (iii) holds.

(iii) implies (i). We have that

$$a = a^+aa^* \in E(a^+)aE(a^*) = E(b^+)bE(b^*),$$

so that $a = ebf$, and similarly $b = gah$, for some $e, f, g, h \in B$. □

In what follows the reader should bear in mind that for a band B, two elements are $\mathcal{D}$-related if and only if they are mutually inverse.

Corollary 5.2. *Let S be a weakly B-abundant semigroup. For any $e, f \in B$,*

$$e \,\delta_B\, f \text{ if and only if } e \,\mathcal{D}\, f.$$

Proof. Let $e, f \in B$. If $e \,\mathcal{D}\, f$, then $e = efe$ and $f = fef$, so that $e \,\delta_B\, f$. Conversely, if $e \,\delta_B\, f$, then by Lemma 5.1, $E(e^+) = E(f^+)$ and so $e \,\mathcal{D}\, f$ as required. □

Armed with alternative descriptions of δ_B, we now show that if S has (WIC), then it is a congruence, arguing as in [12]. Recall that a congruence ρ on a semigroup S is *idempotent pure* if $a\,\rho\,a^2$ implies that $a = a^2$ for all $a \in S$.

Lemma 5.3. *Let S be a weakly B-abundant semigroup with (WIC). Then the relation δ_B is an idempotent pure congruence on S such that $\delta_B \cap \widetilde{\mathcal{H}}_B = \iota$.*

Proof. From Lemma 5.1 (iii), it is clear that δ_B is an equivalence. Suppose now that $a, b, c \in S$ with $a\,\delta_B\,b$, and $e, f, g, h \in B$ have been chosen with $e, g \in E(a^+) = E(b^+), f, h \in E(a^*) = E(b^*)$ such that $a = ebf, b = gah$. Notice that for any b^+ we have that $eb^+\,\mathcal{D}\,b^+$ in B, and as $\mathcal{D}$ is a semilattice congruence on B, $c^*eb^+\mathcal{D}\,c^*b^+$ for any c^*. Consequently,

$$\begin{aligned}
ca &= cebf \\
&= cc^*eb^+bf \\
&= c(c^*eb^+)(c^*b^+)(c^*eb^+)bf \\
&= c(c^*eb^+c^*)(b^+c^*eb^+)bf \\
&= (xc)(by)f = x(cb)yf
\end{aligned}$$

for some $x, y \in B$, using (WIC). Similarly, $cb = u(ca)v$ for some $u, v \in B$. It follows that $ca\,\delta_B\,cb$ so that δ_B is a left congruence. Dually, δ_B is a right congruence.

To see that δ_B is idempotent pure, suppose now that $a\,\delta_B\,a^2$. Using Lemma 5.1 we know that $a^2 = uav$ for some $u \in E(a^+)$ and $v \in E(a^*)$. Now

$$\begin{aligned}
a^2 &= a^+a^2a^* \\
&= a^+(uav)a^* \\
&= (a^+ua^+)a(a^*va^*) \\
&= a^+aa^* \\
&= a
\end{aligned}$$

as required.

Finally, if $a\,(\widetilde{\mathcal{H}}_B \cap \delta_B)\,b$, then $a^+b = b$ and $ba^* = b$ since $a\widetilde{\mathcal{H}}_B b$ so that, with $e, f \in B$ as above, we have

$$\begin{aligned}
a &= ebf \\
&= a^+(ebf)a^* \\
&= (a^+ea^+)b(a^*fa^*)\ , \\
&= a^+ba^* \\
&= b
\end{aligned}$$

so that $\widetilde{\mathcal{H}}_B \cap \delta_B = \iota$. $\square$

A proof that δ_B is a congruence also appears in [19, Lemma 3.10].

Corollary 5.4. *Let S be a weakly B-abundant semigroup with (WIC). Then S is a subdirect product of S/μ_B and S/δ_B.*

Proposition 5.5. *(c.f* [19, Theorem 3.9]*) Let S be a weakly B-abundant semigroup with (WIC). The natural morphism*

$$\delta_B^\natural : S \to S/\delta_B$$

is admissible. Consequently, S/δ_B is weakly $\underline{B}$-adequate where $\underline{B} = B\delta_B^\natural$. Further, S/δ_B satisfies the ample identities (A). If S has (C) with respect to B, then S/δ_B has (C) with respect to $\underline{B}$ and is therefore weakly $\underline{B}$-ample.

Proof. We remark first that by Corollary 5.2, $\underline{B}$ is isomorphic to $B/\mathcal{D}$ and hence is a semilattice.

Let $a \in S$ and suppose that $a \,\widetilde{\mathcal{R}}_B\, e$ where $e \in B$. Then $ea = a$, so certainly in S/δ_B, $e\delta_B\, a\delta_B = a\delta_B$.

On the other hand suppose that $f \in B$ and $f\delta_B\, a\delta_B = a\delta_B$. Then $fa\,\delta_B\, a$ so that $a = gfah$ for some $g, h \in B$. It follows that $gfa = a$ and so $gfe = e$. Consequently, $e\,\mathcal{D}\, fe$ so that by Corollary 5.2, $e\,\delta_B\, fe$ and $f\delta_B\, e\delta_B = e\delta_B$. Thus $a\delta_B\, \widetilde{\mathcal{R}}_{\underline{B}}\, e\delta_B$.

It follows that if $a, b \in S$ and $a\,\widetilde{\mathcal{R}}_B\, b$, then (since there certainly exists an idempotent $e \in B$ with $a\,\widetilde{\mathcal{R}}_B\, e\,\widetilde{\mathcal{R}}_B\, b$), $a\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\, b\delta_B$ in S/δ_B. Together with the dual argument, we have shown that $\delta_B^\natural$ is admissible. It is then clear that S/δ_B is weakly $\underline{B}$-abundant, hence weakly $\underline{B}$-adequate as $\underline{B}$ is a semilattice.

We now argue that condition (A) holds for S/δ_B. Suppose now that $a\delta_B, e\delta_B \in S/\delta_B$, where $e \in B$. Since S has (WIC), $a(a^*ea^*) = fa$ for some $f \in B$. Now

$$\begin{aligned} a\delta_B\, e\delta_B &= (aa^*)\delta_B\, e\delta_B \\ &= a\delta_B\, a^*\delta_B\, e\delta_B\, a^*\delta_B \\ &= (a(a^*ea^*))\delta_B \\ &= (fa)\delta_B \\ &= f\delta_B\, a\delta_B. \end{aligned}$$

Together with the dual argument, we have shown that S/δ satisfies (WIC) with respect to the semilattice $\underline{B}$; then Lemma 2.5 gives that condition (A) holds.

Suppose now that S satisfies condition (C) with respect to B; we show that S/δ satisfies condition (C) with respect to $\underline{B}$. To this end, let $a, b, c \in S$ with $a\delta\,\widetilde{\mathcal{R}}_{\underline{B}}\, b\delta$. Choose a^+, b^+ and notice that as $\delta_B^\natural$ is admissible,

$$a^+\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\, a\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\, b\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\, b^+\delta_B.$$

Recalling that $\underline{B}$ is a semilattice, we have that $a^+\delta_B = b^+\delta_B$; Corollary 5.2 gives that $a^+ \mathcal{D}\, b^+$ in B. Consequently, there exists $e \in B$ with $a^+ \mathcal{R}\, e\, \mathcal{L}\, b^+$ in B. We know that $\widetilde{\mathcal{R}}_B$ is a left congruence on S, and $a^+ \widetilde{\mathcal{R}}_B\, e$, so that

$$ca\,\widetilde{\mathcal{R}}_B\, ca^+\,\widetilde{\mathcal{R}}_B\, ce = ceb^+\,\widetilde{\mathcal{R}}_B\, ceb.$$

Admissibility of $\delta_B^\natural$ gives that in S/δ_B,

$$(ca)\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,(ceb)\delta_B.$$

Examining $(ceb)\delta_B$, and making use of the fact that $\underline{B}$ is a semilattice, we have

$$\begin{aligned}(ceb)\delta_B &= c\delta_B\, e\delta_B\, b\delta_B\\ &= c\delta_B\, e\delta_B\,(b^+b)\delta_B\\ &= c\delta_B\, e\delta_B\, b^+\delta_B\, b\delta_B\\ &= c\delta_B\,(b^+\delta_B\, e\delta_B\, b^+\delta_B)\, b\delta_B\\ &= c\delta_B\,(b^+eb^+b)\delta_B.\end{aligned}$$

But S has (WIC), so that $(b^+eb^+)b = bf$ for some $f \in B$. We now have

$$(ca)\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\, c\delta_B(bf)\delta_B = (cb)\delta_B\, f\delta_B.$$

It follows that for any $g \in B$, if $g\delta_B\,(cb)\delta_B = (cb)\delta_B$, then $g\delta_B\,(ca)\delta_B = (ca)\delta_B$. Together with the argument reversing the roles of a and b, we have shown that

$$c\delta_B\, a\delta_B = (ca)\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,(cb)\delta_B = c\delta_B\, b\delta_B,$$

so that $\widetilde{\mathcal{R}}_{\underline{B}}$ is a left congruence. Dually, $\widetilde{\mathcal{L}}_{\underline{B}}$ is a right congruence so that S/δ_B has (C) (with respect to $\underline{B}$). □

At the beginning of this section we claimed that δ_B is the analogue of the least inverse congruence on an orthodox semigroup. That is, if γ is a congruence on a weakly B-abundant semigroup S with (WIC) such that S/γ is weakly $B\gamma^\natural$-ample, then $\delta_B \subseteq \gamma$. Indeed, we can show rather more than this.

Proposition 5.6. *(c.f [19, Theorem 3.9]) Let S be a weakly B-abundant semigroup with (WIC). Let ρ be a congruence on S such that $B\rho^\natural$ is a semilattice. Then $\delta_B \subseteq \rho$.*

Proof. Let $a, b \in S$ and suppose that $a\,\delta_B\, b$. By Lemma 5.1, $a = ebf$ for some $e \in E(a^+) = E(b^+), f \in E(a^*) = E(b^*)$. We have that

$b^+\rho, b^*\rho \in B\rho^\natural$, which is a semilattice, and so

$$\begin{array}{rcl} a\rho & = & (ebf)\rho \\ & = & (eb^+bb^*f)\rho \\ & = & e\rho\, b^+\rho\, b\rho\, b^*\rho\, f\rho \\ & = & (b^+\rho\, e\rho\, b^+\rho)b\rho(b^*\rho\, f\rho\, b^*\rho) \\ & = & (b^+eb^+)\rho\, b\rho\, (b^*fb^*)\rho \\ & = & b^+\rho\, b\rho\, b^*\rho \\ & = & (b^+bb^*)\rho \\ & = & b\rho, \end{array}$$

so that $\delta_B \subseteq \rho$ as claimed. □

The reader might ask when the morphism $\delta_B^\natural$ is *strongly* admissible, for a weakly B-abundant semigroup with (WIC). From Lemma 2.2, this is equivalent to $\delta_B = \ker \delta_B^\natural \subseteq \widetilde{\mathcal{H}}_B$ and so from Lemma 5.3, to $\delta_B = \iota$. The final result of this section follows immediately from Proposition 5.6 and Corollary 5.2.

Corollary 5.7. *Let S be a weakly B-abundant semigroup with (WIC). Then δ_B is trivial if and only if B is a semilattice.*

6. The least unipotent congruence

One ingredient in the proof of our main result, Theorem 8.13, is the congruence δ examined in Section 5. Another important constituent is the existence of finite covers of a special kind for finite weakly bountiful semigroups, given in Proposition 8.1. To explain their nature, we must now discuss the least unipotent monoid congruence on a weakly bountiful semigroup; of course, if our semigroup is finite, this will be the least group congruence.

To construct the cover in Proposition 8.1 we make use of the existence of covers for weakly adequate semigroups with (A), and to contruct *these*, must call upon existing results in the one-sided case. With this in mind, we are obliged here to have a rather general discussion concerning unipotent congruences on semigroups from classes wider than those we have so far considered.

Let S be a semigroup. We say that a congruence τ on S is a *unipotent (monoid) congruence* if S/τ is unipotent, that is, has exactly one idempotent (and S/τ is a monoid).

The proof of the next lemma is clear.

Lemma 6.1. *Let S be a semigroup with $E(S) \neq \emptyset$. Then a congruence ρ is unipotent if and only if for all $a, b \in S$*

$$a\,\rho\, a^2,\ b\,\rho\, b^2 \textit{ implies that } a\,\rho\, b.$$

Consequently, there exists a least unipotent congruence σ on S.

For a semigroup S with $E(S) \neq \emptyset$ we will always denote the least unipotent congruence on S by σ. Clearly, $e\,\sigma\,f$ for any $e, f \in E(S)$.

As hinted above, we must now introduce the one-sided versions of weakly adequate semigroups with (A). Let S be a semigroup with semilattice of idempotents. We say that S is *weakly left adequate* if every $\widetilde{\mathcal{R}}$-class contains a (necessarily unique) idempotent. We again denote the unique idempotent in the $\widetilde{\mathcal{R}}$-class of $a \in S$ by a^+. For a weakly left adequate semigroup S, we say that S satisfies *condition (AL)* if $ae = (ae)^+a$ for all $a \in S, e \in E(S)$. Weakly right adequate semigroups and condition (AR) are defined dually. Clearly a semigroup is both weakly left and weakly right adequate with (AL) and (AR) if and only if S is weakly adequate with (A).

Lemma 6.2. *Let S be a weakly left (or right) adequate semigroup with (AL) (or (AR). Then σ is the least monoid unipotent congruence on S.*

Proof. Fix an $e \in E(S)$, so that $e\sigma = f\sigma$ for all $f \in E(S)$. Let $a \in S$. From $a^+a = a$ we have that

$$e\sigma\, a\sigma = a^+\sigma\, a\sigma = (a^+a)\sigma = a\sigma$$

and using (AL),

$$a\sigma\, e\sigma = (ae)\sigma = ((ae)^+a)\sigma = (ae)^+\sigma\, a\sigma = a\sigma.$$

Consequently, σ is a unipotent monoid congruence, and clearly then is the least such. ☐

A weakly left adequate semigroup with (AL) is (in the terminology of [2]) wlqa, and so from Proposition 2.4 of that article we have the following closed form for σ.

Proposition 6.3. [2] *Let S be a weakly left adequate semigroup satisfying (AL). Then for any $a, b \in S$,*

$$a\,\sigma\,b \text{ if and only if } ea = eb \text{ for some } e \in E(S).$$

Dually, the relation σ on a weakly right adequate semigroup with (AR) is given by the rule that $a\,\sigma\,b$ if and only if $af = bf$ for some $f \in E(S)$. Thus we deduce:

Corollary 6.4. *Let S be a weakly adequate semigroup with (A). Then for any $a, b \in S$,*

$$a\,\sigma\,b \Leftrightarrow ea = eb \text{ for some } e \in E(S) \Leftrightarrow af = bf \text{ for some } f \in E(S).$$

Let S be a weakly B-abundant semigroup with (WIC) (so that if $E(S) = B$, then S is weakly bountiful). Since every $a \in S$ has a left and a right idempotent identity, it is clear that σ is the least unipotent monoid congruence on S. Following [21] in the bountiful case, we define the relation τ on S by the rule that for any $a, b \in S$,

$$a \,\tau\, b \text{ if and only if } ea = bf \text{ for some } e, f \in B.$$

Proposition 6.5. *Let S be weakly B-abundant with (WIC). Then the relation τ is a congruence on S. If $B = E(S)$, so that S is weakly bountiful, then $\tau = \sigma$.*

Proof. Let $a \in S$. Then for any a^+, a^* we have that $a^+a = aa^*$, so that τ is reflexive.

Suppose now that $a \,\tau\, b$, so that $ea = bf$ for some $e, f \in B$. Then $(a^+ea^+)a = a^+bf$, so that by (WIC) we have that $ag = a^+bf$ for some $g \in B$. Again by (WIC),

$$agb^* = a^+b(b^*fb^*) = a^+hb$$

for some $h \in B$. Thus $b \,\tau\, a$ and τ is symmetric.

If $a, b, c \in S$ and $a \,\tau\, b \,\tau\, c$, then

$$ea = bf \text{ and } gb = ch$$

for some $e, f, g, h \in B$. Now

$$(ge)a = g(ea) = g(bf) = (gb)f = (ch)f = c(hf)$$

so that $a \,\sigma\, c$ and τ is transitive.

It is clear that all idempotents of B are τ-related. We remark that for any $a \in S$ and $e \in B$ we have that

$$a^+(ea) = (a^+ea^+)a = af$$

for some $f \in B$, so that $a \,\tau\, ea$ and dually, $a \,\tau\, ae$.

To show that τ is a left congruence, suppose that $a, b, c \in S$ and $a \,\tau\, b$. By the remark above,

$$c^*a \,\tau\, a \,\tau\, b$$

and so $ec^*a = bf$ for some $e, f \in B$. Hence $cec^*a = cbf$, giving $c(c^*ec^*)a = cbf$ and so by (WIC), $hca = cbf$ for some $h \in B$. Dually, τ is a right congruence and hence a congruence.

From comments above, we have that $e\tau$ is the identity of S/τ, for any $e \in B$. Suppose now that $a \,\tau\, a^2$. Then $ea = a^2f$, for some $e, f \in B$. From $ea = a^2(a^*fa^*)$, we may assume that $f \leq a^*$, so that by (WIC) we have that $af = ya$ for some $y \in B$. We know that $yea\,\tau\,a$, moreover,

$$\begin{aligned}(yea)^2 &= (yea)(yea)\\ &= yea\,ya^2f\\ &= ye(aya)af\\ &= ye(a^2f)af\\ &= ye(ea)af\\ &= yea^2f\\ &= yeea\\ &= yea\end{aligned}$$

so that yea is idempotent. Consequently, if S is weakly bountiful, then $yea \in E = E(S)$ and so $a\tau = (yea)\tau$ is the identity of S/τ, and S/τ is unipotent.

Finally, if ρ is any congruence on S such that $e\,\rho\,f$ for all $e, f \in B$ and $e\rho$ is the identity of S/ρ, then if $a, b \in S$ and $ea = bf$ for some $e, f \in B$, we have

$$a\rho = (e\rho)(a\rho) = (ea)\rho = (bf)\rho = (b\rho)(f\rho) = b\rho,$$

so that $a\,\rho\,b$ and $\tau \subseteq \rho$. Thus if S is weakly bountiful, $\tau = \sigma$. □

We remark that Corollary 6.4 is also a consequence of Proposition 6.5.

7. Covers

A semigroup S is *E-unitary* if for any $a \in S$, $e, ea \in E(S)$ implies that $a \in E(S)$. The one-sidedness of this definition is only apparent, since in an E-unitary semigroup, if $e, ae \in E(S)$, then $a \in E(S)$. An *(E-unitary) cover* of a weakly bountiful semigroup S is an (E-unitary) weakly bountiful semigroup T together with a surjective admissible morphism $\psi : T \to S$ which maps $E(T)$ isomorphically onto $E(S)$; ψ is called a *covering morphism.* We show in the next section that any weakly bountiful semigroup has an E-unitary weakly bountiful cover T for which $\sigma \cap \mu = \iota$. The existence of T is the final cornerstone in the proof of Theorem 8.13. To construct T we must first prove the existence of finite proper covers of finite weakly adequate semigroups satisfying (A), where a weakly left adequate semigroup with (AR) is *left proper* if $\sigma \cap \widetilde{\mathcal{R}} = \iota$, a weakly right adequate semigroup with (AL) is *right proper* if $\sigma \cap \widetilde{\mathcal{L}} = \iota$ and weakly adequate semigroup with (A) is *proper* if it is both left and right proper. Our argument is inspired by the approach of [9] using (A, B)-categories - to make our argument self-contained we present the result using a direct approach.

Let S be a weakly left adequate semigroup. We may regard S as an algebra of type $(2, 1)$, where the unary operation is $a \mapsto a^+$.

Proposition 7.1. [2] *Let S be a finite weakly left adequate semigroup with (AL). Then there is a finite left proper weakly left adequate semigroup T with (AL) and an idempotent separating $(2, 1)$-morphism φ from T onto S.*

Proof. The cover T of S constructed in Theorem 3.8 of [2] has the property that if $E(S)$ is a semilattice, then so is $E(T)$. □

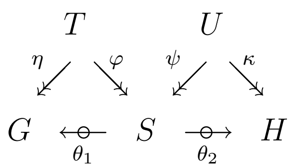
Suppose now that S is a finite weakly adequate semigroup with (A). Let T and φ be as in Proposition 7.1. Notice that if $e \in E(S)$, then as φ is onto,

$$e = a\varphi = (a\varphi)^+ = a^+\varphi$$

for some $a \in T$. Thus, φ maps $E(T)$ isomorphically onto $E(S)$. Of course, we can also invoke the dual of that proposition, to guarantee the existence of a right proper, weakly right adequate semigroup U with (AR) together with an idempotent separating $(2, 1)$-morphism ψ from U onto S, where in this case the unary operation is $a \mapsto a^*$.

Let $G = T/\sigma$ and $H = U/\sigma$, so that both G and H are finite unipotent monoids, that is, both G and H are finite groups.

We have the following diagram



where η and κ are the canonical epimorhisms.

Let $\theta_1 = \varphi^{-1}\eta$ and $\theta_2 = \psi^{-1}\kappa$. From the remark following Proposition 7.1, it is easy to see that θ_1 and θ_2 are relational morphisms from S to G and from S to H, respectively. That is, for $i = 1, 2$, for all $s, t \in S$ $s\theta_i \neq \emptyset$, $(s\theta_i)(t\theta_i) \subseteq (st)\theta_i$ and $1 \in e\theta_i$ for any $e \in E(S)$.

We now remark that any unipotent monoid is weakly adequate with (A), so that certainly any group is weakly adequate with (A). In a natural fashion we may regard weakly adequate semigroups as algebras of type $(2, 1, 1)$. With this signature, it is easy to write down quasi-identities axiomatising the class of weakly adequate semigroups with (A). Thus this class is a quasi-variety, and as such it is closed under subalgebra and direct product.

We now let

$$P = \{(g, s, h) : g \in s\theta_1, h \in s\theta_2\} \subseteq G \times S \times H.$$

Since θ_1 and θ_2 are relational morphisms, it is clear that P is a subsemigroup of $G \times S \times H$. Also, for any $(g, s, h) \in P$ we have that

$$(g, s, h)^+ = (g^+, s^+, h^+) = (1, s^+, 1) \in P$$

so that P is closed under $^+$. Dually, P is closed under *, so that P is a $(2, 1, 1)$-subalgebra of $G \times S \times H$ and is thus weakly adequate with (A). Clearly

$$E(P) = \{(1, e, 1) : e \in E(S)\},$$

so that if $\theta : P \to S$ is the projection onto the second co-ordinate, θ is an idempotent separating $(2, 1, 1)$-morphism onto S.

The semigroup P is proper. For, if $(g, s, h)\,(\sigma \cap \widetilde{\mathcal{R}})\,(k, t, \ell)$, then

$$(1, s^+, 1) = (g, s, h)^+ = (k, t, \ell)^+ = (1, t^+, 1)$$

and by Corollary 6.4,

$$(1, e, 1)(g, s, h) = (1, e, 1)(k, t, \ell)$$

for some $e \in E(S)$. Hence $s^+ = t^+, g = k, h = \ell$ and $es = et$. We also know that $g \in s\theta_1 \cap t\theta_1$. Thus there exists $p, q \in T$ such that $p\varphi = s, p\eta = q\eta = g$ and $q\varphi = t$. Since ker $\eta = \sigma$ we have that $p\,\sigma\,q$. Now

$$p^+\varphi = (p\varphi)^+ = s^+ = t^+ = (q\varphi)^+ = q^+\varphi,$$

as φ is a $(2, 1)$-morphism. Since φ is idempotent separating, we have that $p^+ = q^+$ and hence $p\,\widetilde{\mathcal{R}}\,q$ in T. But T is left proper, so we deduce that $p = q$ and hence $s = t$. Thus P is left proper. Together with the dual argument, we have shown that P is proper.

We say that a *cover* for a weakly adequate semigroup S with (A) is a weakly adequate semigroup U with (A), together with an idempotent separating $(2, 1, 1)$-morphism from U onto S (the reader may check that this definition is consistent with that for a cover of a weakly bountiful semigroup). We have thus proved the following result.

Theorem 7.2. *Let S be a finite weakly adequate semigroup with (A). Then S has a finite proper cover.*

The proof of Proposition 7.2 can also be invoked to prove the existence of a (finite) proper cover with (A) and (C) of a (finite) weakly adequate semigroup with (A) and (C). For this, we need the existence of suitable covers in the one-sided case, guaranteed by [11] and [10]. The gap is in the infinite case with condition (A) but without (C).

8. The main theorem

We are now in a position to prove our main result, which states that if S is a finite weakly bountiful semigroup with $\mathcal{H} \subseteq \mu$, then $S \in \mathbf{A} \vee \mathbf{G}$.

We show that any weakly bountiful semigroup has an E-unitary cover on which $\sigma \cap \mu = \iota$; the corresponding result for bountiful semigroups is proven in [8]. We know that $\sigma \cap \mu = \iota$ holds for all orthodox semigroups by [17, Lemma 2.2], and so our result is simply a generalisation of the existence of E-unitary orthodox covers for orthodox semigroups due independently to McAlister [17], Szendrei [22] and Takizawa [24].

The proof of the following theorem, follows that of [8], itself inspired by that in [17] for orthodox semigroups. But there are subtle differences, most notably that we are not assuming condition (C).

Theorem 8.1. *Let S be a finite weakly bountiful semigroup. Then S has a finite E-unitary cover T such that $\sigma \cap \mu = \iota$ on T.*

We start with the construction of T, and then in a series of lemmas show that T has the desired properties.

Let S be a finite weakly bountiful semigroup; we denote the band $E(S)$ by B. From Proposition 5.5 we have that S/δ is weakly $E(S)\delta^\natural$-adequate with (A); but S is finite, and so $E(S/\delta) = E(S)\delta^\natural$. Thus S/δ is weakly adequate with (A). Proposition 5.5 also says that that $\delta^\natural$ is admissible. Consequently, for any $s \in S$ and s^+, s^* since we have that

$$s^+ \, \widetilde{\mathcal{R}} \, s \, \widetilde{\mathcal{L}} \, s^*$$

it follows that

$$s^+\delta \, \widetilde{\mathcal{R}} \, s\delta \, \widetilde{\mathcal{L}} \, s^*\delta.$$

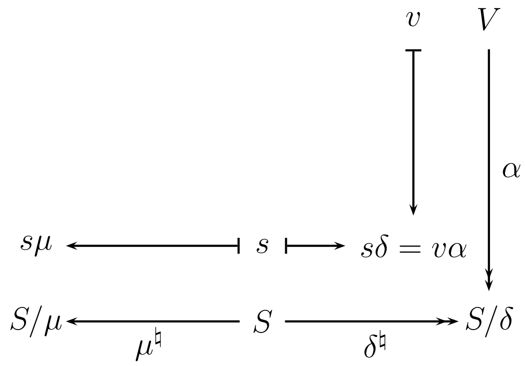
But S/δ is weakly adequate, and so

$$s^+\delta = (s\delta)^+ \text{ and } s^*\delta = (s\delta)^*.$$

By Theorem 7.2, S/δ has a finite proper cover V (where by definition V is weakly adequate with (A)) with $(2,1,1)$ covering morphism α. Let

$$T = \{(s\mu, v) \in S/\mu \times V : v\alpha = s\delta\}.$$

The following scheme may help the reader.



Be warned: to say that a pair $(s\mu, v)$ of $S \times V$ lies in T is to say that $s\mu = t\mu$ for some t with $t\delta = v\alpha$. It is easy to see that T is a subsemigroup of the direct product $S/\mu \times V$. We also note here that from Lemma 2.2 and Proposition 2.3, the morphism $\mu^\natural$ is strongly admissible and S/μ is weakly $\overline{B}$-abundant and $\overline{B}$-fundamental, where $\overline{B} = B\mu^\natural$ is isomorphic to B. But as S is finite, $E(S/\mu) = \overline{B}$, so that S/μ is fundamental weakly abundant with band of idempotents $\overline{B}$; moreover, by Lemma 2.7, S/μ has (WIC). Thus S/μ is a fundamental, weakly bountiful semigroup.

Lemma 8.2. *Let $(s\mu, v), (t\mu, w) \in T$, where $s\delta = v\alpha$ and $t\delta = w\alpha$. Then*

$$s\,\widetilde{\mathcal{R}}\,t \text{ if and only if } s\mu\,\widetilde{\mathcal{R}}\,t\mu$$

and

$$\text{if } s\,\widetilde{\mathcal{R}}\,t \text{ then } v\,\widetilde{\mathcal{R}}\,w.$$

The dual statement holds for $\widetilde{\mathcal{L}}$.

Proof. The first statement is clear from the fact that $\mu^\natural$ is strongly admissible.

Suppose now that $s\,\widetilde{\mathcal{R}}\,t$. Then as $\delta^\natural$ is admissible,

$$v\alpha = s\delta\,\widetilde{\mathcal{R}}\,t\delta = w\alpha.$$

Now as α is a $(2,1,1)$-morphism, we have that

$$v^+\alpha = (v\alpha)^+ = (w\alpha)^+ = w^+\alpha.$$

But α is idempotent separating, so we deduce that $v^+ = w^+$ and hence that $v\,\widetilde{\mathcal{R}}\,w$. □

Lemma 8.3. *The idempotents of T are given by*

$$E(T) = \{(e\mu, v) \in S/\mu \times E(V) : e \in B \text{ and } v\alpha = e\delta\}$$

and form a band isomorphic to B.

Proof. If $e \in B$ and $v \in E(V)$ with $v\alpha = e\delta$, then clearly, $(e\mu, v) \in E(T)$.

On the other hand, if $(s\mu, v) \in T$ is idempotent, where $s\delta = v\alpha$, then $(s\mu)^2 = s\mu$ and $v^2 = v$. From the latter we have $(s\delta)^2 = s\delta$ so that $s\,\delta\,s^2$ and by Lemma 5.3, s is idempotent.

Since B and $E(V)$ are subsemigroups of S and V respectively, it follows easily that $E(T)$ is a subsemigroup of T.

Define $\theta : E(T) \to B$ by $(e\mu, v)\theta = e$, where $e \in B, v \in E(V)$ and $e\delta = v\alpha$. Suppose now that $(e\mu, v), (f\mu, w) \in E(T)$ with $e, f \in B, v, w \in E(V)$, and $e\delta = v\alpha$, $f\delta = w\alpha$. If $(e\mu, v) = (f\mu, w)$, then $e\,(\mu \cap \delta)\,f$ so that $e = f$ by Lemma 5.3, and θ is well defined. On the other hand if $(e\mu, v)\theta = (f\mu, w)\theta$, that is, if $e = f$, then from Lemma 8.2, we certainly have that $v\,\widetilde{\mathcal{R}}\,w$ and so $v = w$, giving that θ is one-one. For any $e \in B$ we have that $e\delta$ is idempotent in S/δ, so that putting $e\delta = u\alpha$ we have that $e\delta = (e\delta)^+ = (u\alpha)^+ = u^+\alpha$. Hence $(e\mu, u^+) \in E(T)$ and $(e\mu, u^+)\theta = e$. It now follows easily that θ is an isomorphism. □

Lemma 8.4. *Let $(s\mu, u) \in T$ where $s\delta = u\alpha$. Then for any s^+, s^*,*

$$(s^+\mu, u^+)\,\widetilde{\mathcal{R}}\,(s\mu, u)\,\widetilde{\mathcal{L}}\,(s^*\mu, u^*).$$

Proof. We have that α is a $(2,1,1)$-morphism and so

$$u^+\alpha = (u\alpha)^+ = (s\delta)^+ = s^+\delta.$$

Hence $(s^+\mu, u^+)$ and dually, $(s^*\mu, u^*)$ lie in T.

Clearly

$$(s^+\mu, u^+)(s\mu, u) = (s\mu, u) = (s\mu, u)(s^*\mu, u^*).$$

Further, if $(e\mu, v)(s\mu, u) = (s\mu, u)$, where $e \in B, v \in E(V)$ and $e\delta = v\alpha$, then $e\mu\, s\mu = s\mu$ and $vu = u$. Consequently, $es\,\mu\,s$ and so as $\mu \subseteq \widetilde{\mathcal{H}}$ we have from a remark following the statement of Theorem 4.2 that $es = s$. Now $es^+ = s^+$ and $vu^+ = u^+$, from which

$$(e\mu, v)(s^+\mu, u^+) = (s^+\mu, u^+)$$

and so $(s\mu, u)\,\widetilde{\mathcal{R}}\,(s^+\mu, u^+)$. Dually, $(s\mu, u)\,\widetilde{\mathcal{L}}\,(s^*\mu, u^*)$. □

Corollary 8.5. *The semigroup T is weakly abundant.*

It is now easy to describe $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ in T.

Lemma 8.6. *Let $(s\mu, u), (t\mu, v) \in T$ where $s\delta = u\alpha$ and $t\delta = v\alpha$. Then*

$$(s\mu, u)\,\widetilde{\mathcal{R}}\,(t\mu, v) \text{ if and only if } s\,\widetilde{\mathcal{R}}\,t$$

and dually,

$$(s\mu, u)\,\widetilde{\mathcal{L}}\,(t\mu, v) \text{ if and only if } s\,\widetilde{\mathcal{L}}\,t.$$

Proof. From Lemma 8.4 we have that

$$(s\mu, u)\,\widetilde{\mathcal{R}}\,(t\mu, v) \text{ if and only } (s^+\mu, u^+)\,\widetilde{\mathcal{R}}\,(t^+\mu, v^+).$$

If $(s^+\mu, u^+)\,\widetilde{\mathcal{R}}\,(t^+\mu, v^+)$, then

$$((s^+t^+)\mu, u^+v^+) = (t^+\mu, v^+) \text{ and } ((t^+s^+)\mu, v^+u^+) = (s^+\mu, u^+),$$

and as μ is idempotent separating we deduce that $s^+t^+ = t^+$ and $t^+s^+ = s^+$, giving that $s^+\,\widetilde{\mathcal{R}}\,t^+$ and so $s\,\widetilde{\mathcal{R}}\,t$ as required.

Conversely, if $s\,\widetilde{\mathcal{R}}\,t$, then $s^+\,\widetilde{\mathcal{R}}\,t^+$, so that $s^+\mu\,\widetilde{\mathcal{R}}\,t^+\mu$ and by Lemma 8.2, $u^+\,\widetilde{\mathcal{R}}\,v^+$, whence $u^+ = v^+$. Therefore $(s^+\mu, u^+)\,\widetilde{\mathcal{R}}\,(t^+\mu, v^+)$, completing the proof of the lemma. □

Lemma 8.7. *The semigroup T is weakly bountiful.*

Proof. It remains only to show that T has condition (WIC). To this end, let $(s\mu, u) \in T$ with $s\delta = u\alpha$. By Lemma 8.2 and Lemma 8.6 we have that the idempotents in the $\widetilde{\mathcal{R}}$-class of $(s\mu, u)$ are precisely those elements $(s^+\mu, u^+)$ where $s\,\widetilde{\mathcal{R}}\,s^+$.

Now choose $(s^+\mu, u^+)$ and let $(e\mu, v) \in E(T)$ with $e \in B, v \in E(V)$ and $e\delta = v\alpha$, and suppose that $(e\mu, v) \leq (s^+\mu, u^+)$. Then

$$(s^+\mu, u^+)(e\mu, v) = (e\mu, v) = (e\mu, v)(s^+\mu, u^+).$$

Using the fact that μ is idempotent separating, we have

$$s^+e = e = es^+ \text{ and } u^+v = v.$$

We are assuming that S has (WIC), and so, as $e \leq s^+$, we know that $es = sf$ for some $f \in B$.

Since α is a $(2,1,1)$-morphism from V onto S/δ, there is an idempotent w in V with $w\alpha = f\delta$ so that $(f\mu, w) \in E(T)$.

Now

$$(vu)\alpha = v\alpha u\alpha = e\delta s\delta = (es)\delta = (sf)\delta = s\delta f\delta = u\alpha w\alpha = (uw)\alpha.$$

As α preserves $*$ and is one-one on $E(V)$, we get $(vu)^* = (uw)^*$. Hence, since V is weakly adequate with (A), and by Lemma 2.1, we have

$$vu = u(vu)^* = u(uw)^* = u(uw)^*w = uw(uw)^* = uw.$$

Consequently,

$$(e\mu, v)(s\mu, u) = (s\mu, u)(f\mu, w).$$

Together with the dual argument, we have shown that T has (WIC) and hence is weakly bountiful. □

We must now make an observation concerning the relationship between the properties of being proper and of being E-unitary, for a weakly adequate semigroup with (A).

Lemma 8.8. *Let W be a proper weakly adequate semigroup with (A). Then W is E-unitary.*

Proof. Observe that if $a \in W, e \in E(W)$ and $ae \in E(W)$, then $a\, e = (ae)e$, so that $a \,\sigma\, ae \,\sigma\, a^*$. Then $a\ (\sigma \cap \widetilde{\mathcal{L}})\ a^*$ and so $a = a^* \in E(W)$. □

The converse of Lemma 8.8 is not true, as demonstrated in Example 3 of [5].

Lemma 8.9. *The semigroup T is E-unitary.*

Proof. Let $(s\mu, u) \in T$ where $s\delta = u\alpha$, and let $(e\mu, v) \in E(T)$ where $e \in B, v \in E(V)$ and $e\delta = v\alpha$. Suppose that $(s\mu, u)(e\mu, v) \in E(T)$. Then $uv \in E(V)$ and so $u \in E(V)$ since V is proper and hence E-unitary by Lemma 8.8.

We now have that $s\delta$ is an idempotent of S/δ and so, by Lemma 5.3, s is idempotent. Thus $(s\mu, u) \in E(T)$ and it follows that T is E-unitary. □

For the next lemma we use the characterisation of μ as the largest congruence contained in $\widetilde{\mathcal{H}}$ as detailed in [14].

Lemma 8.10. [14, Proposition 1.5.10] *Let ρ be an equivalence relation on a semigroup S and let $\rho^\flat$ denote the largest congruence contained in ρ. Then for any $a, b \in S$,*

$$a\,\rho^\flat\, b \text{ if and only if for all } x, y \in S^1, xay\,\rho\, xby.$$

Lemma 8.11. *The semigroup T has the property that $\sigma \cap \mu = \iota$.*

Proof. Let $(s\mu, u), (t\mu, v) \in T$ with $s\delta = u\alpha$ and $t\delta = v\alpha$. Suppose that $(s\mu, u)\,\mu\,(t\mu, v)$ in T. Recalling that μ is the largest congruence contained in $\widetilde{\mathcal{H}}$ we have from Lemma 8.6 that $s\,\widetilde{\mathcal{H}}\,t$. Now pick any $c, d \in S$. Since α is onto, we can find $x, y \in V$ with $c\delta = x\alpha$ and $d\delta = y\alpha$, so that $(c\mu, x), (d\mu, y) \in T$. We have that in T,

$$(c\mu, x)(s\mu, u)(d\mu, y)\,\widetilde{\mathcal{H}}\,(c\mu, x)(t\mu, v)(d\mu, y),$$

and so

$$((csd)\mu, xuy)\,\widetilde{\mathcal{H}}\,((ctd)\mu, xvy)$$

whence from Lemma 8.6, $csd\,\widetilde{\mathcal{H}}\,ctd$ in S. Together with a similar argument in case c or d is an adjoined identity, we have shown that $s\,\mu\,t$ in S, and so $s\mu = t\mu$. It follows from Lemma 8.2 that $u\,\widetilde{\mathcal{H}}\,v$.

We now make the stronger assumption that $(s\mu, u)\,(\mu \cap \sigma)\,(t\mu, v)$ in T. Therefore

$$(e\mu, k)(s\mu, u) = (t\mu, v)(f\mu, h)$$

for some idempotents $(e\mu, k), (f\mu, h) \in T$. Hence $ku = vh = (vh)^{+}v$, whence (multiplying both sides of our equation on the left by $k(vh)^{+}$) $u\,\sigma\,v$ in V. But V is proper, so we may deduce that $u = v$. Thus $\sigma \cap \mu$ is trivial on T. □

The proof of Theorem 8.1 is completed by the next lemma, in which we extend the domain of the morphism θ in the proof of Lemma 8.3.

Lemma 8.12. *The mapping $\theta : T \to S$ given by $(s\mu, u)\theta = s$, where $s\delta = u\alpha$ is a well defined admissible morphism onto S which maps $E(T)$ isomorphically onto $E(S)$.*

Proof. Let $(s\mu, u), (t\mu, v) \in T$ with $s\delta = u\alpha$ and $t\delta = v\alpha$. If $(s\mu, u) = (t\mu, v)$, then from $s\,\mu\,t$ and

$$s\delta = u\alpha = v\alpha = t\delta$$

we have that $s\,(\widetilde{\mathcal{H}} \cap \delta)\,t$ and so $s = t$ by Lemma 5.3. Thus θ is well defined. It is clear that θ is a homomorphism. If $s \in S$, then, since α is surjective, there is an element $v \in V$ such that $v\alpha = s\delta$, so that $(s\mu, v) \in T$ and θ is surjective. We have seen in Lemma 8.3 that $\theta|_{E(T)} : E(T) \to B$ is an isomorphism. That θ is (strongly) admissible is immediate from Lemma 8.6. □

Having proved Theorem 8.1, it is now easy to prove our main result.

Theorem 8.13. *If S is a finite weakly bountiful semigroup with $\mathcal{H} \subseteq \mu$, then $S \in \mathbf{A} \vee \mathbf{G}$.*

Proof. Let T be a finite E-unitary cover of S with $\mu \cap \sigma = \iota$ and covering map $\theta : T \to S$, the existence of which is guaranteed by Theorem 8.1. Since $\mu \cap \sigma = \iota$, it follows that T can be embedded (as a subdirect product) in $T/\mu \times T/\sigma$. Since T/σ is unipotent and finite, we have $T/\sigma \in \mathbf{G}$.

From Proposition 3.6 of [15] we have that any subgroup G of S/μ is the image of a subgroup H of S. Since $H \subseteq H_e$ for some $e \in B$, and $\mathcal{H} \subseteq \mu$, it follows that S/μ has only trivial subgroups, and hence by Proposition 4.2 of [15], $S/\mu \in \mathbf{A}$.

We claim that $T/\mu \cong S/\mu$. To see this, suppose that $(s\mu, u), (t\mu, v) \in T$ with $s\delta = u\alpha$ and $t\delta = v\alpha$. If $(s\mu, u)\,\mu\,(t\mu, v)$ in T, then we have seen in the proof of Lemma 8.11 that $s\,\mu\,t$.

Conversely, suppose that $s\,\mu\,t$. Then $s\,\widetilde{\mathcal{H}}\,t$ so that by Lemma 8.6, $(s\mu, u)\,\widetilde{\mathcal{H}}\,(t\mu, v)$. Moreover, for any $(c\mu, x), (d\mu, y) \in T$, with $c\delta = x\alpha$ and $d\delta = y\alpha$, we note that $csd\,\widetilde{\mathcal{H}}\,ctd$ and so again by Lemma 8.6,

$$(c\mu, x)(s\mu, u)(d\mu, y)\,\widetilde{\mathcal{H}}\,(c\mu, x)(t\mu, v)(d\mu, y);$$

completing the argument with c or d being an adjoined identity gives that $(s\mu, u)\,\mu\,(t\mu, v)$ in T.

We have shown that for any $(s\mu, u), (t\mu, v) \in T$ with $s\delta = u\alpha$ and $t\delta = v\alpha$,

$$(s\mu, u)\,\mu\,(t\mu, v) \text{ in } T \text{ if and only if } s\,\mu\,t \text{ in } S.$$

Thus $\psi : T/\mu^T \to S/\mu^S$ given by $(s\mu^S, u)\mu^T\psi = s\mu^S$, where $s\delta = u\alpha$, is an isomorphism. Consequently, $T/\mu \in \mathbf{A}$ and thus $S \in \mathbf{A} \vee \mathbf{G}$, since S divides $T/\mu \times T/\sigma$. □

We now put together Theorems 4.2 and 8.13.

Corollary 8.14. *Let S be a finite weakly bountiful semigroup with (C). Then $S \in \boldsymbol{A} \vee \boldsymbol{G}$ if and only if $\mathcal{H} \subseteq \mu$.*

As commented at the beginning of Section 5, a bountiful semigroup is weakly bountiful, and by remarks in Section 4, $\widetilde{\mathcal{H}} = \mathcal{H}^*$ on such a semigroup. We can thus deduce the following result from [8].

Corollary 8.15. [8] *If S is a finite bountiful semigroup, then $S \in \boldsymbol{A} \vee \boldsymbol{G}$ if and only if $\mathcal{H} \subseteq \mu$.*

We remark that Corollary 8.14 is a stronger result than Corollary 8.15; the example from [4] alluded to in Section 3 is finite and weakly bountiful with (C) but not bountiful.

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