THE FREE AMPLE MONOID

JOHN FOUNTAIN, GRACINDA M.S. GOMES, AND VICTORIA GOULD

Abstract. We show that the free weakly E-ample monoid on a set X is a full submonoid of the free inverse monoid $FIM(X)$ on X. Consequently, it is ample, and so coincides with both the free weakly ample and the free ample monoid $FAM(X)$ on X. We introduce the notion of a semidirect product $Y * T$ of a monoid T acting *doubly* on a semilattice Y with identity. We argue that the free monoid X^* acts doubly on the semilattice $\mathcal{Y}$ of idempotents of $FIM(X)$ and that $FAM(X)$ is embedded in $\mathcal{Y} * X^*$. Finally we show that every weakly E-ample monoid has a proper *ample* cover.

1. Introduction

A monoid M is *left ample*[1] if it is isomorphic to a submonoid of a symmetric inverse monoid $\mathcal{I}_X$ which is closed under the unary operation $\alpha \mapsto \alpha^+$, where $\alpha^+ = \alpha\alpha^{-1} = I_{\operatorname{dom}\alpha}$, that is, the identity map on the domain $\operatorname{dom}\alpha$ of α. *Right ample* monoids are defined dually and we say that a monoid M is *ample* if it is both left and right ample. In Section 2 we recall that (left, right) ample monoids have abstract characterisations obtained from the generalisations $\mathcal{R}^*$ and $\mathcal{L}^*$ of Green's relations $\mathcal{R}$ and $\mathcal{L}$ respectively, and form quasi-varieties of algebras. Clearly inverse monoids are ample, but the latter class is much wider: ample monoids are not in general regular.

Since the classes of left ample and of ample monoids are non-trivial quasi-varieties [17], we know that the free left ample monoid and the free ample monoid exist on any non-empty set X. The natural question is then, what do these free algebras look like? The description of the

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[1]Formerly, *left type A*; see [6], [16]

free left ample monoid on X was provided by the first author in [8][2] and reformulated in [16]: it is a submonoid of the free inverse monoid $FIM(X)$ on X. What then, of the structure of the free ample monoid $FAM(X)$ on X? Perhaps surprisingly, results for ample monoids are harder to obtain than those in the one-sided case. For example, there is no embedding theorem for ample monoids into inverse monoids analogous to those that exist for left and right ample monoids (see, for example, Theorem 4.5 of [23]). The main result of this article shows that $FAM(X)$ embeds as a full submonoid of $FIM(X)$.

The classes of (left, right) ample monoids are contained in the classes of weakly (left, right) ample monoids and these are themselves contained in the yet wider classes of weakly (left, right) E-ample monoids. Whereas (left, right) ample monoids and weakly (left, right) ample monoids form quasi-varieties, weakly (left, right) E-ample monoids form *varieties* of algebras [17].

Weakly left E-ample monoids exist under a variety of names, and have recently attracted considerable attention. We believe their first occurrence to be as reducts of the embeddable *function systems* of Schweizer and Sklar [33], which were developed through a series of papers in the 1960s. Function systems were revisited by Schein in [31], correcting a misconception of [33]. A survey of this material, in the setting of relation algebras, was given by Schein in [32] and more recently by Jackson and Stokes [22]. Weakly left E-ample semigroups (under another name) appear for the first time as a class in their own right in the work of Trokhimenko [35]. They are the *type SL2 γ-semigroups* of the papers of Batbedat [1, 2] published in the early 1980s. In this decade they have arisen in the work of Jackson and Stokes [21] in the guise of *(left) twisted C-semigroups* and in that of Manes [25] as *guarded semigroups*, motivated by consideration of closure operators and categories, respectively. The work of Manes has a forerunner in the *restriction categories* of Cockett and Lack [4], who were influenced by considerations of theoretical computer science. Indeed the third author and Hollings refer in [18] to weakly E-ample semigroups as *two-sided restriction semigroups.*

A monoid M is *weakly left E-ample* if it is isomorphic to a submonoid of some partial transformation monoid $\mathcal{PT}_X$, closed under the unary operation $\alpha \mapsto \alpha^+ = I_{\operatorname{dom}\alpha}$; note that we no longer claim that $\alpha^+ = \alpha\alpha^{-1}$, since α^{-1} may not exist. Here 'E' is both a generic symbol, and refers to the specific set $\{\alpha^+ \mid \alpha \in M\}$ of local identities of $\mathcal{PT}_X$ contained in M. If $E = E(M)$ then M is said to be *weakly left ample.*

[2]In fact, [8] actually considers the left/right dual case for *semigroups.*

Weakly right (E)-ample monoids are defined dually, and a monoid is *weakly (E)-ample* if it is both weakly left and weakly right (E)-ample (with respect to the same set of idempotents). As in the ample case, weakly (left, right) (E)-ample monoids have axiomatic descriptions, this time involving the further generalisations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ ($\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$) of $\mathcal{R}$ and $\mathcal{L}$ respectively, and form quasi-varieties (indeed in the 'E' case, varieties), so that free algebras exist [17]. It is known [14] that the free weakly left ample monoid coincides with the free left ample monoid; we show here that the corresponding result holds in the two sided case. Indeed the rather stronger statement is true, namely that the free weakly E-ample monoid is $FAM(X)$.

The free inverse monoid is proper and thus by O'Carroll [28] embeds into a semidirect product of a group with a semilattice. Since $FAM(X)$ embeds into $FIM(X)$ the same must be true for $FAM(X)$. We show this directly, via an investigation of what we call double actions of monoids on semilattices. By saying that a monoid T, acting by morphisms on the left and right of a semilattice Y with identity, acts *doubly*, we mean that the two actions are related via the *compatibility conditions* (specified in Section 6). In such a case we show that the semidirect product $Y * T$ contains as a full subsemigroup a weakly E-ample monoid $Y *_m T$, where here $E \cong Y$, which we call the *monoid part* of $Y * T$. Further, $Y *_m T$ is weakly ample if T is unipotent (that is, the only idempotent of T is the identity) and ample if T is cancellative. We argue that X^* acts doubly on the semilattice $\mathcal{Y}$ of $FAM(X)$ and further, $FAM(X)$ is isomorphic to $\mathcal{Y} *_m X^*$.

Finally, we consider proper covers for weakly E-ample monoids. On any weakly E-ample monoid M we denote by σ_E the least monoid congruence such that E is contained in a congruence class; if $E = E(M)$, then $\sigma = \sigma_{E(M)}$ is the least unipotent monoid congruence on M. A *cover* of M is a weakly E-ample monoid $\widehat{M}$ together with a surjective E-separating morphism (that is, separating the idempotents of E), which respects $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. We say that M is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota = \widetilde{\mathcal{L}}_E \cap \sigma_E$, We show that if T is a monoid acting doubly on a semilattice Y with identity, then $Y *_m T$ is proper, and moreover any weakly E-ample monoid has a cover of the form $E *_m X^*$. It follows that any weakly E-ample monoid has a proper *ample* cover.

The structure of the paper is as follows. After Section 2 in which we give further details concerning ample and weakly (E-)ample monoids, we recall the structure of the free inverse monoid in Section 3. We are then in a position to show in Section 4 that the free weakly E-ample monoid on a set X embeds fully into $FIM(X)$, and consequently coincides with the free weakly ample monoid and the free ample monoid $FAM(X)$. Our embedding allows us to determine, in Section 5, the relations $\mathcal{R}^*, \mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*$ on $FAM(X)$ and to argue that $FAM(X)$ is residually finite.

In Section 6 we change tack and consider double actions of monoids on semilattices with identity. We show how to obtain such a double action from any weakly E-ample monoid with any given set of generators; this is followed by a brief Section 7 using these techniques to construct proper ample covers of weakly E-ample monoids. In Section 8 we revisit $FAM(X)$ and give the promised description in terms of semidirect products. We finish with a discussion of FA covers in Section 9; these are the analogue for weakly E-ample monoids of the notion of an F-inverse cover of an inverse monoid.

2. Weakly E-ample and ample monoids

In this section we remind the reader of the alternative approaches towards, and some salient facts concerning, the classes of monoids under consideration here. Further details and references may be found in [17].

We presented left ample monoids in the introduction via their representations as submonoids of symmetric inverse monoids. They have alternative descriptions via the relation $\mathcal{R}^*$ and as a quasi-variety of algebras of type $(2,1,0)$, which we now outline.

The relation $\mathcal{R}^*$ is defined on a monoid M by the rule that for any $a, b \in M$, $a \, \mathcal{R}^* \, b$ if and only if for all $x, y \in M$,

$$xa = ya \text{ if and only if } xb = yb.$$

It is easy to see that $\mathcal{R}^*$ is a left congruence, $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{R} = \mathcal{R}^*$ if M is regular. In general, however, the inclusion can be strict.

Let M be a monoid such that $E(M)$ is a semilattice. From the commutativity of idempotents it is clear that any $\mathcal{R}^*$-class contains at most one idempotent. Where it exists we denote the (unique) idempotent in the $\mathcal{R}^*$-class of a by a^+. If *every* $\mathcal{R}^*$-class contains an idempotent, $^+$ is then a unary operation on M and we may regard M as an algebra of type $(2,1,0)$; as such, morphisms must preserve the unary operation of $^+$ (and hence the relation $\mathcal{R}^*$). We may refer to such morphisms as '$(2,1,0)$-morphisms' if there is danger of ambiguity. Of course, any semigroup isomorphism must preserve all the additional operations. Similarly, if X is a set of generators of a left ample monoid as an algebra with the augmented signature, then we say that X is a set of $(2,1,0)$-*generators* and write $M = \langle X \rangle_{(2,1,0)}$ for emphasis. We remark here that if M is inverse, then $a^+ = aa^{-1}$ for all $a \in M$.

Proposition 2.1. *A monoid M is left ample if and only if $E(M)$ is a semilattice, every $\mathcal{R}^*$-class of M contains an idempotent and the* left ample *identity (AL) holds:*

$$ae = (ae)^+a \text{ for all } a \in M, e \in E(M) \quad \text{(AL)}.$$

As algebras of type $(2, 1, 0)$*, left ample monoids form a quasi-variety.*

The relation $\mathcal{L}^*$ is the dual of $\mathcal{R}^*$ and may be used to give an abstract characterisation of right ample monoids. We denote the unique idempotent in the $\mathcal{L}^*$-class of a, where it exists, by a^*. Observe that if M is inverse, then $a^* = a^{-1}a$ for all $a \in M$. The right ample identity (AR) states that $ea = a(ea)^*$ for all $a \in M, e \in E(M)$. Right ample monoids form a quasi-variety of algebras of type $(2, 1, 0)$ where now the unary operation is $a \mapsto a^*$. A monoid is *ample* if it is both left and right ample; ample monoids therefore form a quasi-variety of algebras of type $(2, 1, 1, 0)$. We remark that as any inverse monoid is certainly ample, any submonoid of an inverse monoid that is closed under $^+$ and * is ample. On the other hand it is undecidable whether a finite ample monoid embeds as a $(2, 1, 1, 0)$-algebra into an inverse monoid [19].

We now turn our attention to the 'weak' case. Let E be a set of idempotents contained in a monoid M; at this stage we do not insist that $E = E(M)$. The relation $\widetilde{\mathcal{R}}_E$ on M is defined by the rule that for any $a, b \in M$, $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b,$$

that is, a and b have the same set of left identities from E. It is easy to see that for any monoid M, we have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$, with both inclusions equalities if M is regular and $E = E(M)$; in general, however, these inclusions can be strict. The relation $\widetilde{\mathcal{R}}_E$ is certainly an equivalence; however, unlike the case for $\mathcal{R}$ and $\mathcal{R}^*$, it need not be left compatible, not even when $E = E(M)$.

Suppose now that E forms a commutative subsemigroup of M; we will say simply that E is a *semilattice*. It is clear that in this case any $\widetilde{\mathcal{R}}_E$-class contains at most one idempotent from E. If every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E, we again have a unary operation $a \mapsto a^+$, where a^+ is now the (unique) idempotent of E in the $\widetilde{\mathcal{R}}_E$-class of a. We may then consider M as an algebra of type $(2, 1, 0)$. In the case that $E = E(M)$, we drop the 'E' from notation and terminology, for example, we write $\widetilde{\mathcal{R}}_{E(M)}$ more simply as $\widetilde{\mathcal{R}}$.

Proposition 2.2. *Let M be a monoid and $E \subseteq E(M)$. Then M is weakly left E-ample if and only if E is a semilattice, every $\widetilde{\mathcal{R}}_E$-class of*

M contains an idempotent of E, the relation $\widetilde{\mathcal{R}}_E$ is a left congruence, and the left ample *identity (AL) holds:*

$$ae = (ae)^+a \text{ for all } a \in M \text{ and } e \in E \qquad (AL).$$

As algebras of type $(2,1,0)$, weakly left E-ample monoids form a variety, and weakly left ample monoids a quasi-variety.

It is worth making the remark that if M is a weakly left E-ample monoid, then $E = \{a^+ : a \in M\}$. Moreover, the identity of M must lie in E, for we must have that $1^+ = 1$.

The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}_E$ on a monoid M are the duals of $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{R}}_E$; *weakly right ($E$-)ample* monoids may be defined in terms of these relations. In each case, we denote the dual of the operation $^+$ by *. As stated in the introduction, a monoid is weakly (E-)ample if it is both left and right weakly (E-)ample where $E = \{a^+ : a \in M\} = \{a^* : a \in M\}$. The classes of ample, weakly ample and weakly E-ample monoids form quasi-varieties (in the weakly E-ample case, varieties) of algebras of type $(2,1,1,0)$.

We now give two technical results which will be useful in the subsequent sections. The first follows immediately from the fact that in a weakly left (right) E-ample monoid, $\widetilde{\mathcal{R}}_E$ ($\widetilde{\mathcal{L}}_E$) is a left (right) congruence; the relation $\leqslant$ appearing in its statement is the natural partial order on E.

Lemma 2.3. *Let M be a weakly E-ample monoid. Then for any $a, b \in M$ and $e \in E$:*
(i) $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$;
(ii) $(ea)^+ = ea^+$ and $(ae)^* = a^*e$;
(iii) $(ab)^+ \leqslant a^+$ and $(ab)^* \leqslant b^*$.

The following result is proven inductively, using the ample identities together with Lemma 2.3.

Lemma 2.4. *Let M be a weakly E-ample monoid, a an element of M and $e_1, \dots, e_n \in E$. Then*

$$e_1 \dots e_n a = a(e_1 a)^* \dots (e_n a)^* \text{ and } ae_1 \dots e_n = (ae_1)^+ \dots (ae_n)^+ a.$$

Consequently,

$$(e_1 \dots e_n a)^* = (e_1 a)^* \dots (e_n a)^* \text{ and } (ae_1 \dots e_n)^+ = (ae_1)^+ \dots (ae_n)^+.$$

Finally in this section we present a short discussion of the relation σ_E on a monoid M, where $E \subseteq E(M)$ is a subsemilattice of M. The relation σ_E is defined by the rule that for any $a, b \in M$,

$$a \,\sigma_E\, b \text{ if and only if } ea = eb$$

for some $e \in E$. It is clear that σ_E is a right congruence on M. If $E = E(M)$, we write σ for $\sigma_{E(M)}$. From [6, 14, 17, 20, 23] we have the following.

Proposition 2.5. *Let M be a monoid and $E \subseteq E(M)$ a subsemilattice:*

(i) if M is weakly left E-ample, then σ_E is the least congruence on M such that $e\,\sigma_E\,f$ for all $e, f \in E$;

(ii) if M is weakly left ample, then σ is the least unipotent congruence on M;

(iii) if M is left ample, then σ is the least right cancellative congruence on M;

(iv) if M is ample, then σ is the least cancellative congruence on M;

(v) if M is inverse, then σ is the least group congruence on M.

Considerations of duality now tell us that if M is (weakly) ample, then $a\,\sigma\,b$ if and only if $af = bf$ for some $f \in E(M)$, and if M is weakly E-ample, then $a\,\sigma_E\,b$ if and only if $af = bf$ for some $f \in E$.

It is well known that an inverse monoid is E-unitary if and only if it is proper, where here proper means that $\mathcal{R} \cap \sigma = \iota$ or equivalently, $\mathcal{L} \cap \sigma = \iota$. Analogously, we say that a left ample monoid is *proper* if $\mathcal{R}^* \cap \sigma = \iota$, a weakly left ample monoid is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$ and a weakly left E-ample monoid is *proper* if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$. Since $\mathcal{R}^* = \widetilde{\mathcal{R}}$ for a left ample monoid (and so certainly for an inverse monoid), there is little danger of ambiguity. In the two sided case (where in general we do not have the natural duality guaranteed by the existence of the involution ${}^{-1}$ in the inverse case), we say that an ample monoid is *proper* if $\mathcal{R}^* \cap \sigma = \mathcal{L}^* \cap \sigma = \iota$, with the obvious alterations in the weakly (E-)ample cases. Proper left ample monoids are E-unitary, but the converse is not true [6].

3. Free algebras and the free inverse monoid on X

Let $\mathcal{C}$ be a class of algebras with a common signature and let X be a set; if $\mathcal{C}$ has no nullary operations, we insist that X be non-empty. Recall that $A \in \mathcal{C}$ is *free* on X if there exists an embedding $\iota : X \to A$ such that for any $B \in \mathcal{C}$ and any map $\kappa : X \to B$, there is a unique morphism $\theta : A \to B$ such that $\iota\theta = \kappa$. It is well known and easy to see that if $\mathcal{C}$ is closed under taking of subalgebras, then the uniqueness of θ is equivalent to $X\iota$ being a generating set for A. Classic results of universal algebra [3, 26] tell us that if $\mathcal{C}$ is any non-trivial quasi-variety, then a free algebra on X exists, and is unique up to isomorphism. One of the first questions one therefore asks about a quasi-variety (or indeed a variety) of algebras is: what is the structure of the free algebra on a given set? Since every algebra in a quasi-variety is a homomorphic image of a free algebra, this question is of some importance.

For the purposes of this paper we first recall the construction of the free inverse monoid $FIM(X)$. Our account follows that in [20], the reader is also referred to the original texts of [29] and [27]. For clarity and completeness we first outline the construction of the free monoid and the free group on a set X.

First, the free monoid. Let X be a set. By a *word* w over X we mean a finite string $w = x_1 x_2 \dots x_n$, where $x_i \in X$, $1 \leqslant i \leqslant n$ and $n \geqslant 0$; the *length* of w is then n. We allow as a word the empty string, denoted here by 1, which has length 0. The free monoid $FM(X)$ on X is then given by

$$FM(X) = \{w \mid w \text{ is a word over } X\}$$

with binary operation of juxtaposition, and injection $\iota : X \to FM(X)$ which associates $x \in X$ with the corrresponding word of length 1 in $FM(X)$. It is standard to denote $FM(X)$ by X^* and identify X with $X\iota$.

Let Y be a set and let $v, w \in Y^*$. We say that v is a *prefix* of w if $w = vw'$ for some $w' \in Y^*$. The relation $\leqslant$ is then defined on Y^* by the rule that for any $v, w \in Y^*$,

$$w \leqslant v \text{ if and only if } v \text{ is a prefix of } w.$$

Clearly, $\leqslant$ is a partial order on Y^* that is compatible with multiplication on the left, and 1 is the greatest element of Y^*. For future convenience we define the notation $w^{\downarrow}$ for $w = x_1 \dots x_n \in Y^*$ by

$$w^{\downarrow} = \{1, x_1, x_1 x_2, \dots, x_1 \dots x_n\},$$

that is, $w^{\downarrow}$ is the set of prefixes of w.

Armed with the description of free monoids, we can progress to free groups. Again, let X be a set, and now let $X^{-1} = \{x^{-1} : x \in X\}$ be a set in bijective correspondence with X such that $X \cap X^{-1} = \emptyset$. Consider the free monoid $(X \cup X^{-1})^*$. A word $w \in (X \cup X^{-1})^*$ is *reduced* if w contains no sub-word of the form xx^{-1} or $x^{-1}x$. Words $w, v \in (X \cup X^{-1})^*$ are *equivalent* if v can be obtained from w by a process of insertion and deletion of factors of the form xx^{-1} and $x^{-1}x$, where $x \in X$. It is a fact that any word $w \in (X \cup X^{-1})^*$ is equivalent to a unique reduced word w^r. The free group $FG(X)$ on X is then given by

$$FG(X) = \{w \in (X \cup X^{-1})^* \mid w \text{ is reduced}\}$$

with binary operation $\cdot$ where

$$w \cdot v = (wv)^r.$$

Note that we may consider X^* as a submonoid of $FG(X)$ and identify elements of X with positive reduced words of length one. It is useful to note that for $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \in FG(X)$, where $x_i \in X$ and $\epsilon_i \in \{1, -1\}$, we have $w^{-1} = x_n^{-\epsilon_n} \dots x_1^{-\epsilon_1}$.

The free group is, in turn, used in the construction of the free inverse monoid, which we now describe. For any $w \in FG(X)$ we have, of course, that w is a word in the free monoid $(X \cup X^{-1})^*$, and we can therefore refer to prefixes of w. We say that a subset A of the free group $FG(X)$ is *prefix closed* if $w \in A$ implies that every prefix of w is in A. Let $\mathcal{Z}$ be the semilattice of finite subsets of $FG(X)$ under union. For $g \in FG(X)$ and $A \in \mathcal{Z}$ we put

$$g \cdot A = \{g \cdot h \mid h \in A\};$$

clearly $FG(X)$ then acts on $\mathcal{Z}$ by automorphisms. For future purposes we remark that if $w \in FG(X)$, then

$$w^{-1} \cdot w^{\downarrow} = (w^{-1})^{\downarrow}.$$

Let $\mathcal{Y}$ denote those elements of $\mathcal{Z}$ that are prefix closed, so that for any $A \in \mathcal{Z}$ we have that $A \in \mathcal{Y}$ if and only if

$$w \in A \Rightarrow w^{\downarrow} \subseteq A.$$

We note that if $A \in \mathcal{Y}$, then $1 \in A$. The free inverse monoid $FIM(X)$ on X is then given by

$$FIM(X) = \{(A, g) \mid A \in \mathcal{Y}, g \in A\}$$

with multiplication

$$(A, g)(B, h) = (A \cup g \cdot B, g \cdot h\},$$

and injection $\iota : X \to FIM(X)$ given by

$$x\iota = (\{1, x\}, x).$$

The identity of $FIM(X)$ is $(\{1\}, 1)$, and the semilattice of idempotents is

$$E(FIM(X)) = \{(A, 1) : A \in \mathcal{Y}\}.$$

For any $(A, g) \in FIM(X)$ we have that

$$(A, g)^{-1} = (g^{-1} \cdot A, g^{-1})$$

whence

$$(A, g)^+ = (A, g)(A, g)^{-1} = (A, 1)$$

and

$$(A, g)^* = (A, g)^{-1}(A, g) = (g^{-1} \cdot A, 1).$$

It is worth recording the following, which may be found in [24, 20].

Lemma 3.1. *For any* $(A, g), (B, h) \in FIM(X)$*:*
(i) $(A, g)\,\mathcal{R}\,(B, h)$ *if and only if* $A = B$*;*
(ii) $(A, g)\,\mathcal{L}\,(B, h)$ *if and only if* $g^{-1} \cdot A = h^{-1} \cdot B$*;*
(iii) $(A, g)\,\mathcal{D}\,(B, h)$ *if and only if* $A = a \cdot B$ *for some* $a \in A$*;*
(iv) $(A, g)\,\sigma\,(B, h)$ *if and only if* $g = h$*.*
Consequently, $FIM(X)$ *is proper.*

We now define a subset of $FIM(X)$, to which we give the temporary notation $A(X)$, by

$$A(X) = \{(A, g) \in FIM(X) \mid g \in X^*\}.$$

Clearly, $A(X)$ is a submonoid of $FIM(X)$ that is closed under both the unary operations $^+$ and *. Since ample monoids form a quasi-variety of algebras of type $(2, 1, 1, 0)$, we immediately deduce the following.

Corollary 3.2. *The monoid* $A(X)$ *is ample.*

Certainly $A(X)$ contains $X\iota$; we will see in the next section that $A(X)$ is the free weakly E-ample (and hence also the free ample, and the free weakly ample) monoid on X.

4. The free weakly E-ample monoid on X

We begin this section by showing that $A(X)$ is generated by $X\iota$.

Lemma 4.1. *For any set* X*,*

$$A(X) = \langle X\iota \rangle_{(2,1,1,0)} = \big\langle \big\{(\{1, x\}, x) \mid x \in X\big\}\big\rangle_{(2,1,1,0)}.$$

Proof. For convenience in this proof we drop the subscript indicating the signature. We remark first that since $(\{1\}, 1)$ is the image of the nullary operation, $(\{1\}, 1) \in \langle X\iota \rangle$. Let $(A, g) \in A(X)$, where $g \neq 1$. Since A is prefix closed and $g \in A$, it is certainly true that

$$(A, g) = (A, 1)(g^{\downarrow}, g).$$

We show that $(A, 1)$ and $(g^{\downarrow}, g) \in \langle X\iota \rangle$, whence the result follows. It is clear that if $g = x_1 \dots x_n$ where $x_i \in X$, then

$$(g^{\downarrow}, g) = (\{1, x_1\}, x_1) \dots (\{1, x_n\}, x_n) = x_1\iota \dots x_n\iota \in \langle X\iota \rangle,$$

so we must concentrate on proving that $(A, 1) \in X\iota$ for an arbitrary $A \in \mathcal{Y}$.

Since A is prefix closed, we have that

$$A = g_1^{\downarrow} \cup \dots \cup g_m^{\downarrow}$$

for some $g_1,\dots,g_m\in A$; from the description of $\leqslant$ given in Section 3, we may take $g_1,\dots,g_m$ to be the elements of A that are minimal with respect to the partial ordering $\leqslant$ on $(X\cup X^{-1})^*$. Now

$$(A,1)=(g_1^{\downarrow},1)\dots(g_m^{\downarrow},1),$$

so that our task is to show that $(g^{\downarrow},1)\in\langle X\iota\rangle$, for any non-identity reduced word $g\in(X\cup X^{-1})^*$. We proceed by induction on the length of such a g. If g has length 1, then $g=x$ or $g=x^{-1}$, for some $x\in X$. If $g=x$, then $(g^{\downarrow},1)=(\{1,x\},1)\in X\iota$. On the other hand, if $g=x^{-1}$, then $(g^{\downarrow},1)=(\{1,x\},x)^*\in\langle X\iota\rangle$.

Suppose that the length of g is $k>1$, so that $g=xh$ or $g=x^{-1}h$ for some $x\in X$ and reduced $h\in(X\cup X^{-1})^*$ with length $k-1$, and make the inductive assumption that $(h^{\downarrow},1)\in\langle X\iota\rangle$. If $g=xh$, then

$$(g^{\downarrow},x)=(\{1,x\},x)(h^{\downarrow},1)\in\langle X\iota\rangle,$$

so that

$$(g^{\downarrow},1)=(g^{\downarrow},x)^+\in\langle X\iota\rangle.$$

On the other hand, if $g=x^{-1}h$, then

$$\begin{aligned}\big((h^{\downarrow},1)(\{1,x\},x)\big)^* &= (h^{\downarrow}\cup\{x\},x)^*\\ &= (x^{-1}\cdot(h^{\downarrow}\cup\{x\}),1)\\ &= (g^{\downarrow},1),\end{aligned}$$

so that $(g^{\downarrow},1)\in\langle X\iota\rangle$ as required. $\square$

Suppose now that X is a set and $\theta:X\to M$ is a map from X to a weakly E-ample monoid M with $E\subseteq E(M)$ a semilattice containing 1. Certainly θ lifts to a $(2,0)$-morphism from X^* to M, which for convenience we also denote by θ. We now define $\theta':FG(X)\to E$ inductively as follows:

$$1\theta'=1$$

and for any $x\in X, g,h\in FG(X)$ with $xg,x^{-1}h$ reduced,

$$(xg)\theta'=(x\theta\, g\theta')^+ \text{ and } (x^{-1}h)\theta'=(h\theta' x\theta)^*.$$

In the next result, bear in mind that hk denotes juxtaposition in the free monoid $(X\cup X^{-1})^*$.

Lemma 4.2. *Let X,M,θ and θ' be defined as above. Then for any $g,h,k\in FG(X)$ and $w\in X^*$:*

(i) if $g=hk$ with hk reduced as written, then $g\theta'\leqslant h\theta'$ in E;
(ii) if $g=wk$ with wk reduced as written, then $g\theta'=(w\theta\, k\theta')^+$;
(iii) if $g=w^{-1}k$ with $w^{-1}k$ reduced as written, then $g\theta'=(k\theta'\, w\theta)^*$;
(iv) $w\theta'=(w\theta)^+$ and $w^{-1}\theta'=(w\theta)^*$;
(v) $(w\cdot g)\theta' w\theta=(w\theta\, g\theta')^+w\theta=w\theta g\theta'$;

(vi) $w\theta(w^{-1}\cdot g)\theta' = w\theta(g\theta'\, w\theta)^* = g\theta' w\theta$.

Proof. (i) Notice that if $e, f \in E$ with $e \leqslant f$, then for any $a \in M$, we have that $ae = a(fe) = (af)e$, so that by Lemma 2.3, $(ae)^+ \leqslant (af)^+$. Dually, $(ea)^* \leqslant (fa)^*$.

Let $g, h, k \in FG(X)$ be such that $g = hk$. We show by induction on the length of h that $g\theta' \leqslant h\theta'$. If $h = 1$, then

$$g\theta' \leqslant 1 = 1\theta' = h\theta'.$$

Suppose now that the length of h is $n > 1$ and the result is true for all h's of length $n-1$. We have that $h = xw$ or $h = x^{-1}w$ for some $x \in X$ and $w \in FG(X)$, where w has length $n-1$; our inductive assumption gives that $(wk)\theta' \leqslant w\theta'$.

In the first case $g = xwk$ and

$$g\theta' = (xwk)\theta' = (x\theta\,(wk)\theta')^+ \leqslant (x\theta\, w\theta')^+ = h\theta'.$$

On the other hand, if $h = x^{-1}w$, then

$$g\theta' = (x^{-1}wk)\theta' = ((wk)\theta'\, x\theta)^* \leqslant (w\theta'\, x\theta)^* = h\theta'.$$

The result follows by induction.

(ii) Suppose that $g = wk$. We proceed by induction on the length of w, the result being clear if $w = 1$. Suppose now that $w = xv$ where $x \in X$ and $v \in X^*$, and make the inductive assumption that $(vk)\theta' = (v\theta\, k\theta')^+$. Using Lemma 2.3 we have that

$$g\theta' = (xvk)\theta' = (x\theta\,(vk)\theta')^+ = (x\theta v\theta\, k\theta')^+ = ((xv)\theta\, k\theta')^+ = (w\theta\, k\theta')^+.$$

The result follows.

The argument for (iii) is dual to that for (ii), and (iv) follows immediately from (ii) and (iii).

(v) If wg is reduced, then $w\cdot g = wg$ and using (ii) and (AL),

$$(w\cdot g)\theta' w\theta = (wg)\theta' w\theta = (w\theta\, g\theta')^+ w\theta = w\theta g\theta'$$

as required.

Suppose now that $w = x_1 \dots x_n$ and let $g = x_n^{-1}\dots x_j^{-1}h$ where $h \neq x_{j-1}^{-1}k$ for $k \in FG(X)$. Observe that since g is reduced, we also have that h does not begin with x_j. Put $u = x_1 \dots x_{j-1}$ and $v = x_j \dots x_n$, so that $w = uv$. Then

$$\begin{array}{rcll}
(w\cdot g)\theta'\,w\theta & = & (uh)\theta'\,w\theta & \\
 & = & (u\theta\,h\theta')^{+}\,u\theta\,v\theta & \text{using (ii)}\\
 & = & u\theta\,h\theta'\,v\theta & \text{since } M \text{ is weakly } E\text{-ample}\\
 & = & u\theta\,v\theta(h\theta'\,v\theta)^{*} & \text{again, since } M \text{ is weakly } E\text{-ample}\\
 & = & w\theta(h\theta'\,v\theta)^{*} & \\
 & = & w\theta(v^{-1}h)\theta' & \text{using (iii)}\\
 & = & w\theta\,g\theta' & \\
 & = & (w\theta\,g\theta')^{+}\,w\theta. &
\end{array}$$

(vi) The argument is dual to that for (v).

□

Lemma 4.3. *Let X, M, θ and θ' be defined as above. Then the mapping $\overline{\theta} : A(X) \to M$ given by*

$$(\{g_1,\dots,g_n\},w)\overline{\theta} = g_1\theta'\dots g_n\theta' w\theta$$

is a $(2,1,1,0)$-morphism such that $\iota\overline{\theta} = \theta$.

Proof. Since $\theta' : FG(X) \to E$ and E is a semilattice, certainly $\overline{\theta}$ is well defined. For any $x \in X$ we have

$$x\iota\overline{\theta} = (\{1,x\},x)\overline{\theta} = 1\theta'\,x\theta'\,x\theta = (x\theta)^{+}x\theta = x\theta,$$

using the fact that $1\theta' = 1$ and the fact that $x\theta' = (x\theta)^{+}$ by definition of θ'. Clearly $\overline{\theta}$ also preserves the identity.

To see that $\overline{\theta}$ is a semigroup morphism, let $(A,w),(B,v) \in A(X)$ where

$$A = \{g_1,\dots,g_m\},\ B = \{h_1,\dots,h_n\} \subseteq FG(X).$$

Then, making use of the fact that E is a semilattice,

$$\begin{array}{rcll}
((A,w)(B,v))\overline{\theta} & = & (A \cup w\cdot B, wv)\overline{\theta} & \\
 & = & (\{g_1,\dots,g_m, w\cdot h_1,\dots,w\cdot h_n\}, wv)\overline{\theta} & \\
 & = & g_1\theta'\dots g_m\theta'\,(w\cdot h_1)\theta'\dots(w\cdot h_n)\theta'(wv)\theta & \\
 & = & g_1\theta'\dots g_m\theta'\,(w\cdot h_1)\theta'\dots(w\cdot h_n)\theta'\,w\theta\,v\theta & \\
 & = & g_1\theta'\dots g_m\theta'\,(w\theta\,h_1\theta')^{+}\dots(w\theta\,h_n\theta')^{+}\,w\theta\,v\theta & \text{by Lemma 4.2 (v)}\\
 & = & g_1\theta'\dots g_m\theta'\,w\theta\,h_1\theta'\dots h_n\theta'\,v\theta & \text{by Lemma 2.4}\\
 & = & (A,w)\overline{\theta}(B,v)\overline{\theta} &
\end{array}$$

as required.

From remarks preceding Lemma 3.1 we have

$$(A,w)^{+}\overline{\theta} = (A,1)\overline{\theta} = g_1\theta'\dots g_m\theta'\,1\theta = g_1\theta'\dots g_m\theta'.$$

But $w \in A \cap X^{*}$; without loss of generality we can assume $w = g_m$. From Lemma 4.2 (iv), $g_m\theta' = (g_m\theta)^{+}$, and so

$$(A,w)^{+}\overline{\theta} = g_1\theta'\dots g_m\theta'(w\theta)^{+} = (g_1\theta'\dots g_m\theta'\,w\theta)^{+} = \big((A,w)\overline{\theta}\big)^{+},$$

so that $\overline{\theta}$ preserves $^+$.

It remains to show that $\overline{\theta}$ preserves *. With (A, w) as above,

$$\begin{array}{rcll}
(A,w)^*\overline{\theta} & = & (w^{-1}\cdot A, 1)\overline{\theta} & \\
& = & (\{w^{-1}\cdot g_1, \dots, w^{-1}\cdot g_m\}, 1)\overline{\theta} & \\
& = & (w^{-1}\cdot g_1)\theta' \dots (w^{-1}\cdot g_m)\theta' & \\
& = & (w^{-1})\theta'(w^{-1}\cdot g_1)\theta' \dots (w^{-1}\cdot g_m)\theta' & \text{since } 1 \in A \\
& = & (w\theta)^*(w^{-1}\cdot g_1)\theta' \dots (w^{-1}\cdot g_m)\theta' & \text{by Lemma 4.2 (iv)} \\
& = & (w\theta(w^{-1}\cdot g_1)\theta' \dots (w^{-1}\cdot g_m)\theta')^* & \text{by Lemma 2.3} \\
& = & (g_1\theta' \dots g_m\theta' w\theta)^* & \text{by Lemma 4.2 (vi)} \\
& = & \big((A,w)\overline{\theta}\big)^*. &
\end{array}$$

Thus $\overline{\theta} : A(X) \to M$ is a $(2, 1, 1, 0)$-morphism as claimed. □

From Lemmas 4.1 and 4.3 we can now deduce our main result.

Theorem 4.4. *For any set X the submonoid of $FIM(X)$ given by*

$$A(X) = \{(A, g) \in FIM(X) \mid g \in X^*\}$$

is the free weakly E-ample monoid on X. Moreover, since $A(X)$ is ample, $A(X)$ is both the free weakly ample monoid and the free ample monoid $FAM(X)$ on X.

We remark that we have concentrated for convenience on *monoids*. Deleting all references to a multiplicative identity in the definition of ample and weakly (E-)ample monoids, we obtain the quasi-varieties of of ample and weakly (E-)ample *semigroups*. A cursory examination of the proof of Theorem 4.4 shows us that the free (weakly (E-)) ample semigroup on X is $FAM(X) \setminus \{1\}$.

5. Properties of $FAM(X)$

Theorem 4.4 identifies $FAM(X)$ as the submonoid $A(X)$ of $FIM(X)$ introduced at the end of Section 3. We remark that $FAM(X)$ is a *full* submonoid of $FIM(X)$, that is, it contains all the idempotents of $FIM(X)$.

From the fact that $FAM(X)$ is a $(2, 1, 1, 0)$-subalgebra of $FIM(X)$ we deduce that for any $(A, g), (B, h) \in FAM(X)$,

$$\begin{array}{rcl}
(A,g)\,\mathcal{R}^*\,(B,h) \text{ in } FAM(X) & \Leftrightarrow & (A,g)^+ = (B,h)^+ \\
& \Leftrightarrow & (A,g)\,\mathcal{R}\,(B,h) \text{ in } FIM(X);
\end{array}$$

the dual comment holding for $\mathcal{L}^*$. The following result is now immediate from Lemma 3.1.

Proposition 5.1. *Let X be a non-empty set. Then for any elements $(A,g),(B,h)\in FAM(X)$:*

(i) $(A,g)\,\mathcal{R}^*\,(B,h)$ if and only if $A=B$;
(ii) $(A,g)\,\mathcal{L}^*\,(B,h)$ if and only if $g^{-1}\cdot A=h^{-1}\cdot B$;
(iii) $(A,g)\,\sigma\,(B,h)$ if and only if $g=h$.

Corollary 5.2. *The monoid $FAM(X)$ is proper.*

Proof. From Proposition 5.1 we see that $\mathcal{R}^*\cap\sigma=\iota=\mathcal{L}^*\cap\sigma$. $\square$

To show that the relation $\mathcal{D}^*$ is the restriction to $FAM(X)$ of the relation $\mathcal{D}$ in $FIM(X)$ requires a little more work.

First, we observe that $\mathcal{R}^*\circ\mathcal{L}^*\neq\mathcal{L}^*\circ\mathcal{R}^*$ in $FAM(X)$. To see this, take $x\in X$ and notice that from Proposition 5.1,

$$(\{1,x\},1\}\,\mathcal{R}^*\,(\{1,x\},x)\,\mathcal{L}^*\,(\{1,x^{-1}\},1).$$

If $(\{1,x\},1)\,\mathcal{L}^*\,(B,h)\,\mathcal{R}^*\,(\{1,x^{-1}\},1)$, then from $(B,h)\,\mathcal{R}^*\,(\{1,x^{-1}\},1)$ we would be forced to have $B=\{1,x^{-1}\}$ and $h=1$, but it is not true that $(\{1,x\},1)\,\mathcal{L}^*\,(\{1,x^{-1}\},1)$.

From Proposition 5.1 it is immediate that for any $(A,g),(B,h)\in FAM(X)$,

$$(A,g)\,\mathcal{R}^*\circ\mathcal{L}^*(B,h) \text{ if and only if } k^{-1}\cdot A=h^{-1}\cdot B$$

for some $k\in A\cap X^*$. If this holds, then $B=h\cdot k^{-1}\cdot A=w\cdot A$ with $w=h\cdot k^{-1}$. Notice that as $A=w^{-1}\cdot B$ and $1\in B$, we must have that $w^{-1}\in A$. We now build on this observation.

Proposition 5.3. *For any $(A,g),(B,h)\in FAM(X)$,*

$$(A,g)\,\mathcal{D}^*\,(B,h) \textit{ if and only if } B=w\cdot A \textit{ for some } w\in FG(X).$$

Proof. Suppose that $(A,g)\,\mathcal{D}^*\,(B,h)$ so that, as $\mathcal{D}^*=\mathcal{R}^*\vee\mathcal{L}^*$, we have $(A,g)\,(\mathcal{R}^*\circ\mathcal{L}^*)^m(B,h)$ for some $m\in\mathbb{N}$ [20]. We show by induction on m that there exists $w\in FG(X)$ with $B=w\cdot A$.

The case $m=1$ is immediate from remarks preceding this proposition. For $m>1$ we have that

$$(A,g)\,(\mathcal{R}^*\circ\mathcal{L}^*)^{m-1}\,(C,k)\,\mathcal{R}^*\circ\mathcal{L}^*\,(B,h)$$

for some $(C,k)\in FAM(X)$. We make the inductive assumption that there exists an element $w_1\in FG(X)$ with $C=w_1\cdot A$. On the other hand, from the case for $m=1$ we know that there exists $w_2\in FG(X)$ with $B=w_2\cdot C$. Clearly then $B=(w_2\cdot w_1)\cdot A$.

Conversely, suppose that $B=w\cdot A$ for some $w\in FG(X)$. Observe that, as $1\in B$ we must have that $w^{-1}\in A$, and as $1\in A$, we must have $w\in B$. From the construction of $FG(X)$, we can write

$$w=u_1\cdot v_1^{-1}\cdot u_2\cdot v_2^{-1}\cdot\ldots\cdot u_n\cdot v_n^{-1}=u_1v_1^{-1}u_2v_2^{-1}\ldots u_nv_n^{-1}$$

for some $u_1, v_1, \ldots, u_n, v_n \in X^*$. Again we proceed by induction.

If $n = 1$ then as both A and B are prefix closed, we have that $u_1 \in B$ and $v_1 \in A$, so that from Proposition 5.1 we have that

$$(A, g)\,\mathcal{R}^*\,(A, v_1)\,\mathcal{L}^*\,(B, u_1)\,\mathcal{R}^*\,(B, h),$$

so that $(A, g)\,\mathcal{D}^*\,(B, h)$ as required.

Now let $n \geqslant 2$ and make the inductive assumption that our claim is true for $n-1$. Let $k = u_1 v_1^{-1} \ldots u_{n-1} v_{n-1}^{-1}$. From $w = k u_n v_n^{-1}$ we obtain that $v_n, v_n u_n^{-1} \in A$, since $w^{-1} \in A$ and A is prefix closed. This tells us that $(A, v_n u_n^{-1}) \in FIM(X)$ and as $FIM(X)$ is certainly closed under *, we have that $u_n v_n^{-1} \cdot A \in \mathcal{Y}$ and hence $(u_n v_n^{-1} \cdot A, u_n) \in FAM(X)$. As $B = k \cdot (u_n v_n^{-1} \cdot A)$ our induction hypothesis, together with the case for $n = 1$, gives

$$(A, g)\,\mathcal{D}^*\,(u_n v_n^{-1} \cdot A, u_n)\,\mathcal{D}^*\,(B, h)$$

as required. $\square$

An ample monoid M is *residually finite* (in the class of ample monoids) if for any $a, b \in M$ with $a \neq b$, there is a *finite* ample monoid N and a $(2,1,1,0)$-morphism $\varphi : M \to N$ such that $a\varphi \neq b\varphi$.

Proposition 5.4. *The free ample monoid $FAM(X)$ is residually finite.*

Proof. Let $(A, g), (B, h) \in FAM(X)$ with $(A, g) \neq (B, h)$. From Theorem 3.6 of [27] we know that $FIM(X)$ is residually finite as an (inverse) monoid. Hence there exists a finite inverse monoid N and a monoid morphism $\theta : FIM(X) \to N$ such that $(A, g)\theta \neq (B, h)\theta$. Since morphisms between inverse semigroups preserve inverses we certainly have that for any $(C, k) \in FIM(X)$,

$$(C, k)^+\theta = ((C, k)(C, k)^{-1})\theta = (C, k)\theta(C, k)\theta^{-1} = (C, k)\theta^+$$

and dually, θ preserves *. Hence $\varphi = \theta|_{FAM(X)} : FAM(X) \to N$ is a $(2,1,1,0)$-morphism and so $\mathrm{Im}\ \varphi$ is a $(2,1,1,0)$-subalgebra of N. It follows that $\mathrm{Im}\ \varphi$ is a finite ample monoid and clearly $(A, g)\varphi \neq (B, h)\varphi$. $\square$

We remark that consequently, $FAM(X)$ is certainly residually finite in both the class of weakly ample and the class of weakly E-ample monoids.

An algebra A is *hopfian* if the identity congruence is the only congruence ρ on A such that $A/\rho \cong A$. As remarked in [7], a result of Evans [5] that says that in any variety the finitely generated residually finite algebras are hopfian, is also valid for quasi-varieties.

Corollary 5.5. *If X is a finite set, then the monoid $FAM(X)$ is hopfian.*

6. Semidirect products and weakly E-ample monoids

After considering semidirect products of semilattices by monoids, we introduce the notion of a double action of a monoid on a semilattice with identity. We then show how to construct proper weakly E-ample monoids from semidirect products of monoids acting doubly on a semilattice with identity. In the following sections we demonstrate that $FAM(X)$ may be constructed in this manner and show how our technique provides a transparent method of obtaining covers for (weakly (E-))ample monoids.

Let T be a monoid, acting on the left of a semilattice Y by morphisms. That is, there is a map from $T \times Y$ to Y, given by $(t, y) \mapsto t \cdot y$, such that for all $s, t \in T$ and for all $e, f \in Y$:

$$1 \cdot e = e,\ (st) \cdot e = s \cdot (t \cdot e) \text{ and } s \cdot (ef) = (s \cdot e)(s \cdot f).$$

The *semidirect product* $Y * T$ is the set $Y \times T$ under a binary operation given by

$$(e, s)(f, t) = (e(s \cdot f), st).$$

The proof of the following lemma is straightforward, and follows in a similar way to that of [13, Proposition 3.1] in the unipotent case.

Lemma 6.1. *Let $S = Y * T$ be the semidirect product of a monoid T with a semilattice Y.*

Then:

(i) S is a proper weakly left $\overline{Y}$-ample semigroup, where

$$\overline{Y} = \{(e, 1) \mid e \in Y\} \cong Y;$$

(iii) for all $(e, t) \in S$, $(e, t)^+ = (e, 1)$;

(iv) for all $(e, s), (f, t) \in S$,

$$(e, s)\, \widetilde{\mathcal{R}}_{\overline{Y}}\, (f, t) \text{ if and only if } e = f;$$

(v) for all $(e, s), (f, t) \in S$,

$$(e, s)\, \sigma_{\overline{Y}}\, (f, t) \text{ if and only if } s = t;$$

(vi) $S/\sigma_{\overline{Y}} \simeq T$.

Suppose now that T is a monoid acting by morphisms on the left of a semilattice Y *with identity*.

We now define the *monoid part* $Y *_m T$ of $Y * T$ to be

$$Y *_m T = \{(e, t) : e \leqslant t \cdot 1\} \subseteq Y * T.$$

Lemma 6.2. *Let T be a monoid acting by morphisms on a semilattice Y with identity. Then $Y *_m T$ as defined above is a subsemigroup of $Y * T$ containing $\overline{Y}$. Moreover, $Y *_m T$ is a proper weakly left $\overline{Y}$-ample monoid, with identity $(1,1)$.*

Proof. Clearly $1 \cdot 1 = 1$, so that $\overline{Y} \subseteq Y *_m T$.

Let $(e,s),(f,t) \in Y *_m T$; we have

$$(e,s)(f,t) = (e(s \cdot f), st).$$

From $(f,t) \in Y *_m T$ we have that $f \leqslant t \cdot 1$ and since the action $\cdot$ preserves order,

$$s \cdot f \leqslant s \cdot (t \cdot 1) = st \cdot 1.$$

Hence

$$e(s \cdot f) \leqslant st \cdot 1$$

so that $Y *_m T$ is closed.

We now verify that $(1,1)$ is the identity of $Y *_m T$. Let $(e,s) \in Y *_m T$; then

$$(1,1)(e,s) = (1(1 \cdot e), 1\,s) = (e,s)$$

and on the other hand,

$$(e,s)(1,1) = (e(s \cdot 1), s\,1) = (e,s)$$

since $e \leqslant s \cdot 1$.

By Lemma 6.1, $Y * T$ is a proper weakly left $\overline{Y}$-ample semigroup such that $(e,s)^+ = (e,1)$. Since $\overline{Y} \subseteq Y *_m T$ it is clear that $Y *_m T$ is closed under $^+$ and consequently, $Y *_m T$ is a weakly left $\overline{Y}$-ample monoid which *as such* is proper, that is, $\sigma_{\overline{Y}} \cap \widetilde{\mathcal{R}}_{\overline{Y}} = \iota$. □

We denote a right action of a monoid T on a semilattice Y by

$$(e,t) \mapsto e \circ t.$$

We say that a monoid T *acts doubly* on a semilattice Y with identity, if T acts by morphisms on the left and right of Y and the *compatibility conditions* hold, that is

$$(t \cdot e) \circ t = (1 \circ t)e \text{ and } t \cdot (e \circ t) = e(t \cdot 1)$$

for all $t \in T, e \in Y$.

Proposition 6.3. *Let T be a monoid acting doubly on a semilattice Y with identity. Then*

$$Y *_m T = \{(e,t) : e \leqslant t \cdot 1\} \subseteq Y * T$$

is a proper weakly $\overline{Y}$-ample monoid with identity $(1,1)$ such that

$$(e,t)^+ = (e,1) \text{ and } (e,t)^* = (e \circ t, 1).$$

If T is unipotent, then $Y *_m T$ is weakly ample. If T is left (right) cancellative, then $Y *_m T$ is right (left) ample.

Proof. From Lemma 6.2, we know that $Y *_m T$ is weakly left $\overline{Y}$-ample, and as such is proper.

We now show that $Y *_m T$ is weakly right $\overline{Y}$-ample. Let (e, s) be an element of $Y *_m T$. Observe that

$$(e, s)(e \circ s, 1) = (e(s \cdot (e \circ s), s) = (e\, e(s \cdot 1), s)$$

using the second of the compatibility conditions. But $e \leqslant s \cdot 1$ so that $(e, s)(e \circ s, 1) = (e, s)$.

Suppose now that $(e, s)(f, 1) = (e, s)$. Then $e(s \cdot f) = e$ hence

$$\begin{array}{rrcl} & (e(s \cdot f)) \circ s & = & e \circ s \\ \Rightarrow & (e \circ s)((s \cdot f) \circ s) & = & e \circ s \\ \Rightarrow & (e \circ s)(1 \circ s)f & = & e \circ s \\ \Rightarrow & e \circ s & \leqslant & f \end{array}$$

so that $(e \circ s, 1) \leqslant (f, 1)$ and $(e, s)\, \widetilde{\mathcal{L}}_{\overline{Y}}, (e \circ s, 1)$.

To show that $\widetilde{\mathcal{L}}_{\overline{Y}}$ is right compatible, suppose that $(e, s)\, \widetilde{\mathcal{L}}_{\overline{Y}}\, (f, t)$, and $(g, u) \in Y *_m T$. Then $e \circ s = f \circ t$ and calculating,

$$\begin{array}{rcll} \big((e, s)(g, u)\big)^* & = & (e(s \cdot g), su)^* & \\ & = & \big((e(s \cdot g)) \circ su, 1\big) & \\ & = & \big((e(s \cdot g)) \circ s) \circ u, 1\big) & \\ & = & \big(((e \circ s)((s \cdot g) \circ s)) \circ u, 1\big) & \\ & = & \big(((e \circ s)(1 \circ s)g) \circ u, 1\big) & \\ & = & \big(((e \circ s)g) \circ u, 1\big) & \text{as } e \circ s \leqslant 1 \circ s. \end{array}$$

But $e \circ s = f \circ t$ so that our argument gives

$$(e, s)(g, u)\, \widetilde{\mathcal{L}}_{\overline{Y}}\, (f, t)(g, u)$$

and $\widetilde{\mathcal{L}}_{\overline{Y}}$ is a right congruence as required.

Let $(e, s) \in Y *_m T, (f, 1) \in \overline{Y}$. Then

$$\begin{array}{rcl} (e, s)((f, 1)(e, s))^* & = & (e, s)(fe, s)^* \\ & = & (e, s)((fe) \circ s, 1) \\ & = & (e\,(s \cdot ((fe) \circ s)), s) \\ & = & (e\,(s \cdot ((f \circ s)(e \circ s))), s) \\ & = & (e((s \cdot (f \circ s))(s \cdot (e \circ s))), s) \\ & = & (e\, f(s \cdot 1)\, e(s \cdot 1), s) \\ & = & (ef, s) \\ & = & (f, 1)(e, s), \end{array}$$

thus completing the proof that $Y *_m T$ is weakly right $\overline{Y}$-ample.

To argue that $Y *_m T$ is proper *as a weakly $\overline{Y}$-ample* monoid, all that remains is to show that $\widetilde{\mathcal{L}}_{\overline{Y}} \cap \sigma_{\overline{Y}} = \iota$. To this end, suppose that $(e,s),(f,t) \in Y *_m T$ are related by $\widetilde{\mathcal{L}}_{\overline{Y}} \cap \sigma_{\overline{Y}}$, so that $s = t$ and $e \circ s = f \circ t$. Now

$$e = e(s \cdot 1) = s \cdot (e \circ s) = t \cdot (f \circ t) = f(t \cdot 1) = f$$

so that $(e,s) = (f,t)$ as required.

If T is unipotent then it is clear that $\overline{Y} = E(Y *_m T)$, so that $Y *_m T$ is weakly ample. Suppose now that T is right cancellative. To show that $Y *_m T$ is left ample, it remains only to show that $(e,s)\,\mathcal{R}^*\,(e,s)^+ = (e,1)$, for any $(e,s) \in Y *_m T$. To this end, let $(e,s),(f,t)$ and (g,u) be elements of $Y *_m T$ with

$$(f,t)(e,s) = (g,u)(e,s).$$

From $(f(t \cdot e), ts) = (g(u \cdot e), us)$ we obtain $f(t \cdot e) = g(u \cdot e)$ and $ts = us$. Right cancellativity of T yields that $t = u$ and consequently,

$$(f,t)(e,1) = (g,u)(e,1)$$

and $(e,s)\,\mathcal{R}^*\,(e,1)$.

Finally, suppose that T is left cancellative. Let $(e,s) \in Y *_m T$; we argue that $(e,s)\,\mathcal{L}^*\,(e \circ s, 1) = (e,s)^*$. If $(f,t),(g,u) \in Y *_m T$ with

$$(e,s)(f,t) = (e,s)(g,u),$$

then we have $st = su$, so that from left cancellation in T, $t = u$. Further, $e(s \cdot f) = e(s \cdot g)$ and so

$$(e \circ s)((s \cdot f) \circ s) = (e \circ s)((s \cdot g) \circ s),$$

giving by compatibility,

$$(e \circ s)(1 \circ s)f = (e \circ s)(1 \circ s)g.$$

As $e \circ s \leqslant 1 \circ s$ we deduce that $(e \circ s)f = (e \circ s)g$ and hence

$$(e \circ s, 1)(f,t) = ((e \circ s)f, t) = ((e \circ s)g, u) = (e \circ s, 1)(g,u).$$

□

Consider again a monoid T acting doubly on a semilattice Y with identity. It may be thought that the construction provided in Proposition 6.3 is somewhat one-sided. However, letting

$$T *_m Y = \{(t,e) : e \leqslant 1 \circ t\}$$

and defining $\theta : Y *_m T \to T *_m Y$ by

$$(e,t)\theta = (t, e \circ t)$$

we claim that θ is an isomorphism.

To this end, let $t \in T$ and $e \in Y$. As $e \circ t \leqslant 1 \circ t$, we have that $(t, e \circ t) \in T *_m Y$. By the same token, $(t \cdot e, t) \in Y *_m T$. If $e \leqslant 1 \circ t$ then

$$\begin{aligned} (t \cdot e, t)\theta &= (t, (t \cdot e) \circ t) \\ &= (t, (1 \circ t)e) \\ &= (t, e) \end{aligned}$$

so that θ is onto. Suppose now that $(e, s), (f, t) \in Y *_m T$. Then

$$\begin{aligned} ((e, s)(f, t))\theta &= (e(s \cdot f), st)\theta \\ &= (st, (e(s \cdot f)) \circ st) \\ &= (st, ((e(s \cdot f)) \circ s) \circ t) \\ &= (st, ((e \circ s)(1 \circ s)f) \circ t) \\ &= (st, ((e \circ s)f) \circ t) \\ &= (st, ((e \circ s) \circ t)(f \circ t)) \\ &= (s, e \circ s)(t, f \circ t) \\ &= (e, s)\theta(f, t)\theta \end{aligned}$$

so that θ is a morphism.

To see that θ is one-one, again let $(e, s), (f, t) \in Y *_m T$. Then

$$\begin{aligned} (e, s)\theta = (f, t)\theta &\Rightarrow (s, e \circ s) = (t, f \circ t) \\ &\Rightarrow e \circ s = f \circ t \text{ and } s = t \\ &\Rightarrow s \cdot (e \circ s) = s \cdot (f \circ s) \text{ and } s = t \\ &= (s \cdot 1)e = (s \cdot 1)f \text{ and } s = t \\ &\Rightarrow e = f \text{ and } s = t \\ &\Rightarrow (e, s) = (f, t) \end{aligned}$$

as $e, f \leqslant s \cdot 1$.

Clearly $(e, 1)\theta = (1, e)$, for any $e \in E$, so that θ is an isomorphism preserving the distinguished subsemilattices of $Y *_m T$ and $T *_m Y$.

We now give a natural construction of monoids of the form $Y *_m T$, starting with any weakly E-ample monoid M.

Let T be any *submonoid* of M. Define actions of T on the left and right of E by the rule

$$t \cdot e = (te)^+ \text{ and } e \circ t = (et)^*$$

for all $t \in T$ and $e \in E$. Notice that for any $e, f \in E$ and $s, t \in T$ we have $1 \cdot e = (1e)^+ = e$,

$$s \cdot (t \cdot e) = s \cdot (te)^+ = (s(te)^+)^+ = (ste)^+ = st \cdot e,$$

using Lemma 2.3, and

$$t \cdot (ef) = (tef)^+ = (te)^+(tf)^+ = (t \cdot e)(t \cdot f)$$

by Lemma 2.4. We have thus verified that $(t,e)\mapsto t\cdot e$ *is* an action by morphisms, the proof for $\circ$ being dual.

Lemma 6.4. *Let T act on both sides of E as above. Then T acts doubly on E.*

Proof. It remains only to show that the compatibility conditions hold. Let $e\in E$ and $t\in T$. Then

$$t\cdot(e\circ t)=t\cdot(et)^*=(t(et)^*)^+=(et)^+=et^+=e(t\cdot 1),$$

using the ample condition. Dually, $(t\cdot e)\circ t=e(1\circ t)$. $\square$

Proposition 6.5. *Let M be a weakly E-ample monoid and let T be a submonoid of M, acting on E as above. Then the map $\theta: E*_m T\to M$ given by*

$$(e,t)\theta=et$$

is an $\overline{E}$-separating $(2,1,1,0)$-morphism, where $\overline{E}=\{(e,1):e\in E\}$. If $M=ET$, then $E*_m T$ is a proper cover for M.

Proof. From Proposition 6.3, $E*_m T$ is a proper weakly $\overline{E}$-ample monoid where $\overline{E}\cong E$.

For any $(e,s),(f,t)\in E*_m T$,

$$\begin{aligned}((e,s)(f,t))\theta &= (e(s\cdot f),st)\theta\\ &= e(s\cdot f)st\\ &= e(sf)^+st\\ &= e(sf)t\\ &= (es)(ft)\\ &= (e,s)\theta(f,t)\theta.\end{aligned}$$

Moreover, using Proposition 6.3 again, and the fact that $e\leqslant s\cdot 1=s^+$,

$$(e,s)^+\theta=(e,1)\theta=e=es^+=(es)^+=((e,s)\theta)^+$$

and further,

$$(e,s)^*\theta=(e\circ s,1)\theta=e\circ s=(es)^*=((e,s)\theta)^*.$$

Since $(1,1)\theta=1$, we have that θ is a $(2,1,1,0)$-morphism which is clearly an isomorphism from $\overline{E}$ onto E.

Suppose now that $M=ET$. Let $m\in M$; by assumption, $m=es=(es^+)s$ for some $e\in E, s\in T$. Now $es^+\leqslant s^+$ so that $(es^+,s)\in E*_m T$ and clearly $m=(es^+,s)\theta$. Thus θ is onto, hence completing the proof that $E*_m T$, together with θ, form a proper cover for M. $\square$

As an example to illustrate Proposition 6.5, let M be a weakly E-ample monoid, with set of generators X, where M is regarded as a $(2,1,1,0)$–algebra. We write

$$M = \langle X\rangle_{(2,1,1,0)}.$$

Clearly M contains a submonoid T generated by X; we write

$$T = \langle X\rangle_{(2,0)}.$$

Lemma 6.6. *Let M, X and T be as above. Then $M = ET$ and $E *_m T$ is a cover of M.*

Proof. Notice that as $1 \in E \cap T$, we have that $E \cup T \subseteq ET$, and so ET is closed under the nullary and both unary operations. If $es, ft \in ET$, where $e, f \in E$ and $s, t \in T$, then

$$(es)(ft) = e(sf)t = e(sf)^+st \in ET.$$

Consequently, as $X \subseteq ET$ and ET is closed under all the basic operations, $M \subseteq ET$ as required. □

In the next section we strengthen this result by showing that a weakly E-ample monoid has a proper cover which is *ample*.

7. A covering theorem

Lawson showed in [23] that every ample semigroup has a proper ample cover, from which the corresponding result for monoids is immediate. Lawson's result is, in fact, a consequence of a more general result of Simmons [34]. The first two authors used different techniques to demonstrate in [10] that an ample monoid S has a proper cover, which may be taken to be finite, if S is finite. In [12] we explain how the results of [13] and [15] may be used to prove that a (finite) weakly ample monoid has a (finite) proper cover. Further, Proposition 3.3 of [15] tells us that finite proper weakly ample monoids are, in fact, ample, so that finite weakly ample monoids have finite ample covers.

Our aim in this section is to give a simple and direct proof of the following result. In Section 9 we follow this with an alternative approach inspired by [8] and [30] which consider the right ample and inverse cases respectively.

Theorem 7.1. *Let M be a weakly E-ample monoid. Then M has a proper ample cover.*

Proof. Let M be a weakly E-ample monoid; pick any set X of generators of M as a $(2,1,1,0)$-algebra and let $T = \langle X\rangle_{(2,0)}$, so that T acts doubly on E via morphisms via

$$t \cdot e = (te)^+ \text{ and } e \circ t = (et)^*,$$

where $t \in T$ and $e \in E$. Clearly then X^* acts by morphisms on E if we define

$$w \cdot e = \overline{w} \cdot e \text{ and } e \circ w = e \circ \overline{w}$$

where $\overline{w}$ is the image of $w \in X^*$ in T. It is easy to check that in this way X^* acts doubly on E. From Theorem 6.3 we know that

$$E *_m X^* = \{(e, w) : e \leqslant w^+\} \subseteq E * X^*$$

is a proper ample monoid.

Define $\varphi : E *_m X^* \to E *_m T$ by

$$(e, w)\varphi = (e, \overline{w}).$$

It is easy to check that φ is a surjective idempotent separating $(2, 1, 1, 0)$-morphism. From Proposition 6.5 and Lemma 6.6, $E *_m T$, together with $\theta : E *_m T \to M$ given by $(e, m)\theta = em$, form a proper cover. Hence $E *_m X^*$, together with $\varphi\theta$, form the required proper *ample* cover for M. $\square$

8. The free ample monoid revisited

Recall from Section 3 that for a non-empty set X, $\mathcal{Y}$ denotes the semilattice of finite prefix closed subsets of $FG(X)$ under union. In this section we use our results concerning double actions of monoids on semilattices with identity, to show that $FAM(X)$ is isomorphic to $\mathcal{Y} *_m X^*$, with actions given below, and hence embedded in $\mathcal{Y} * X^*$.

Let X be a non-empty set. We define a left and right 'actions' of X^* on $\mathcal{Y}$ by

$$u \bullet A = u^{\downarrow} \cup u \cdot A \text{ and } A \circ u = (u^{-1})^{\downarrow} \cup u^{-1} \cdot A$$

for any $u \in X^*$ and $A \in \mathcal{Y}$.

Lemma 8.1. *With the definitions as given above, X^* acts doubly on $\mathcal{Y}$.*

Proof. Let $A, B \in \mathcal{Y}$ and $u, v \in X^*$.

Certainly $1 \bullet A = 1^{\downarrow} \cup 1 \cdot A = A$, since $1 \in A$. Notice also that if

$$w = u \cdot a = u_0 a_2$$

where $u = u_0 a_1$ and $a = a_1^{-1} a_2 \in A$ and $u_0 a_2$ is reduced, then any prefix of w is either a prefix of u, hence lying in $u^{\downarrow}$, or else of the form $u_0 a_{21}$ where $a_2 = a_{21} a_{22}$ say. In this case $a_1^{-1} a_{21} \in A$ and

$$u_0 a_{21} = u \cdot a_1^{-1} a_{21} \in u \cdot A.$$

Thus $u^{\downarrow} \cup u \cdot A \in \mathcal{Y}$.

To see that $\bullet$ is an action, notice that

$$\begin{aligned} u \bullet (v \bullet A) &= u \bullet (v^{\downarrow} \cup v \cdot A) \\ &= u^{\downarrow} \cup u \cdot v^{\downarrow} \cup u \cdot (v \cdot A) \\ &= (uv)^{\downarrow} \cup uv \cdot A \\ &= uv \bullet A. \end{aligned}$$

To see that X^* acts by morphisms on the left of $\mathcal{Y}$, observe that

$$\begin{aligned} u \bullet (A \cup B) &= u^{\downarrow} \cup u \cdot (A \cup B) \\ &= u^{\downarrow} \cup u \cdot A \cup u \cdot B \\ &= (u^{\downarrow} \cup u \cdot A) \cup (u^{\downarrow} \cup u \cdot B) \\ &= u \bullet A \cup u \bullet B. \end{aligned}$$

The proof that $\circ$ is an action by morphisms is dual.

Finally, we check one of the compatibility conditions, the proof of the other being dual. Calculating, we have

$$\begin{aligned} (u \bullet A) \circ u &= (u^{\downarrow} \cup u \cdot A) \circ u \\ &= (u^{-1})^{\downarrow} \cup u^{-1} \cdot (u^{\downarrow} \cup u \cdot A) \\ &= (u^{-1})^{\downarrow} \cup u^{-1} \cdot u^{\downarrow} \cup u^{-1} \cdot (u \cdot A) \\ &= (u^{-1})^{\downarrow} \cup A \\ &= (\{1\} \circ u) \cup A \end{aligned}$$

since $u^{-1} \cdot u^{\downarrow} = (u^{-1})^{\downarrow}$.

$\square$

¿From Proposition 6.3, $\mathcal{Y} *_m X^*$ is a proper ample monoid; we claim it is $FAM(X)$.

Proposition 8.2. *The free ample monoid $FAM(X)$ coincides with $\mathcal{Y} *_m X^*$.*

Proof. We must show that as a set, $A(X) = \mathcal{Y} *_m X^*$, and that multiplication in $A(X)$ coincides with that in $\mathcal{Y} *_m X^*$.

We first note that for $A \in \mathcal{Y}$ and $w \in X^*$,

$$\begin{aligned} (A, w) \in \mathcal{Y} *_m X^* &\Leftrightarrow A \leqslant w \bullet \{1\} \\ &\Leftrightarrow w \bullet \{1\} \subseteq A \\ &\Leftrightarrow w^{\downarrow} \subseteq A \\ &\Leftrightarrow w \in A \\ &\Leftrightarrow (A, w) \in A(X). \end{aligned}$$

Now for $(A, w), (B, v) \in \mathcal{Y} *_m X^*$, multiplying in that monoid we have

$$\begin{aligned} (A, w)(B, v) &= (A \cup w \bullet B, wv) \\ &= (A \cup w^{\downarrow} \cup w \cdot B, wv) \\ &= (A \cup w \cdot B, wv), \end{aligned}$$

as $w^{\downarrow} \subseteq A$, coinciding with multiplication in $A(X)$ as required. $\square$

9. FA covers

Recall that on a weakly E-ample monoid M there is a partial order $\leqslant$ called the *natural partial order* defined by

$$a \leqslant b \text{ if and only if } a = eb \text{ for some } e \in E.$$

Of course, from the ample conditions, it follows that $a \leqslant b$ if and only if $a = bf$ for some $f \in E$.

For an inverse monoid M we recall from Proposition 2.5 that $\sigma = \sigma_{E(M)}$ is the least group congruence. An *F-inverse* monoid is an inverse monoid in which every σ-class has a maximum element (under the natural partial order). An F-inverse monoid is necessarily proper, and every inverse monoid has an F-inverse cover (see [30, 24]).

Our final goal is to give an analogue of this result for weakly E-ample monoids. First, we note that the definition of a weakly left FA-monoid in [11] applies to weakly left E-ample monoids; thus a weakly left E-ample monoid M is *weakly left FA* if every σ_E-class of M contains a maximum element, and for all $a, b \in M$,

$$m(a)^{+}m(ab)^{+} = (m(a)m(b))^{+} \qquad \text{(FL)}$$

where $m(a)$ is the maximum element in the σ_E-class of a. It is shown in [11] that if a weakly left ample monoid M is weakly left FA, then M is (left) proper, and (FL) is equivalent to

$$m(a)^{+}m(ab) = m(a)m(b)$$

for all $a, b \in M$. The same arguments apply to a weakly left E-ample monoid. *Weakly right FA* monoids are defined similarly with (FL) replaced by its dual:

$$m(ab)^{*}m(b)^{*} = (m(a)m(b))^{*}$$

for all elements a, b in the monoid.

If M is a left ample monoid which is weakly left FA, then following [9] we say that it is *left FA*. For our present purposes, we can restrict attention to ample monoids which are FA, that is, both left and right FA. The free ample monoid $FAM(X)$ on a set X is an example of such a monoid. For, if $(A, g) \in FAM(X)$, then, using Proposition 5.1 and the definition of the natural order, it is readily verified that $(g^{\downarrow}, g)$ is the maximum element in the σ-class of (A, g). Routine calculations now establish that (FL) and its dual hold, and so we have the following lemma.

Lemma 9.1. *The free ample monoid $FAM(X)$ on a set X is an FA monoid.*

Next, let ρ be a $(2,1,0)$-congruence on a right ample monoid M. As in [8] we define the relation $\rho_{\min}$ on M by

$$a\,\rho_{\min} b \text{ if and only if } ae = be \text{ for some } e \in E(M) \text{ with } e\,\rho\, a^*\rho\, b^*.$$

The following result is essentially Proposition 1.5 of [8].

Proposition 9.2. *Let ρ be a $(2,1,0)$-congruence on a right ample monoid M. Then $\rho_{\min}$ is a $(2,1,0)$-congruence on M, $\rho_{\min}|_{E(M)} = \rho|_{E(M)}$ and $\rho_{\min} \subseteq \tau$ for any semigroup congruence τ on M with $\tau|_{E(M)} = \rho|_{(E(M)}$. Furthermore, $M/\rho_{\min}$ is right ample and*

$$E(M/\rho_{\min}) = \{e\rho_{\min} : e \in E(M)\}.$$

We remark that the statement of Proposition 1.5 in [8] refers to τ being a $(2,1,0)$-congruence, but the proof given there does not require this.

Of course, there is a dual result for left ample monoids involving a congruence $\rho'_{\min}$ (of the appropriate kind) defined by

$$a\,\rho'_{\min} b \text{ if and only if } fa = fb \text{ for some } f \in E(M) \text{ with } f\,\rho\, a^+\rho\, b^+.$$

Thus if M is an ample monoid, $\rho_{\min}$ and $\rho'_{\min}$ are both defined on M, but it follows from Proposition 9.2 and its dual that $\rho_{\min} = \rho'_{\min}$ so that in this case $\rho_{\min}$ is a $(2,1,1,0)$-congruence. In fact, it is easy to see directly that the two congruences are equal using the ample conditions. The significance of $\rho_{\min}$ becomes apparent in the next lemma.

Lemma 9.3. *Let ρ be a $(2,1,1,0)$-congruence on an FA monoid M. Then $M/\rho_{\min}$ is an FA monoid.*

Proof. It is clear that $\rho_{\min} \subseteq \sigma$ and so the least cancellative congruence on $M/\rho_{\min}$ is $\sigma/\rho_{\min}$. Hence if $m(a)$ is the maximum element in the σ-class of $a \in M$, then $m(a)\rho_{\min}$ is the maximum element in the $(\sigma/\rho_{\min})$-class of $a\rho_{\min}$ in $M/\rho_{\min}$. Using this and the fact that $\rho_{\min}$ is a $(2,1,1,0)$-congruence, we see that condition (FL) and its dual hold for $M/\rho_{\min}$. □

It is now easy to prove our final result.

Theorem 9.4. *Let M be a weakly E-ample monoid. Then M is the image of an FA monoid under an idempotent separating $(2,1,1,0)$-morphism.*

Proof. Every weakly E-ample monoid is an image of a free ample monoid under a $(2,1,1,0)$-morphism. Thus $M \cong FAM(X)/\rho$ for some set X and $(2,1,1,0)$-congruence ρ. Now $\rho_{\min} \subseteq \rho$ so that writing F for $FAM(X)$ we have, from the homomorphism theorem for universal algebras, that $F/\rho \cong (F/\rho_{\min})/(\rho/\rho_{\min})$. By Lemma 9.1, F is an FA monoid and hence, by Lemma 9.3, so is $F/\rho_{\min}$. By Proposition 9.2, $\rho|_{E(F)} = \rho_{\min}|_{E(F)}$ so that the $(2,1,1,0)$-congruence $\rho/\rho_{\min}$ is idempotent separating and the result follows. $\square$

Since an FA monoid is proper, the above also offers an alternative proof of Theorem 7.1.

As at the end of Section 4, we consider the case of weakly E-ample *semigroups*. If S is a weakly E-ample semigroup, then S^1 is a weakly E-ample monoid, so that by Theorem 7.1 there is a proper ample monoid U and an idempotent separating $(2,1,1,0)$-morphism $\theta : U \to S^1$. It is easy to see that $S\theta^{-1}$ together with $\theta|_{S\theta^{-1}} : S\theta^{-1} \to S$ is an idempotent separating $(2,1,1)$-morphism onto S, so that $S\theta^{-1}$ is a proper ample cover of S.

We can also obtain a semigroup analogue of Theorem 9.4 although it should be noted that we have defined FA monoids but not FA semigroups. At the end of Section 4 we observed that $FAM(X) \setminus \{1\}$ is the free weakly E-ample semigroup on the set X. Writing $FAS(X)$ for this semigroup, we say (following the terminology of [1, 30, 8]) that an ample semigroup (or monoid) S is *quasi-free* if $S \cong FAS(X)/\tau$ for some $(2,1,1)$-congruence τ contained in the congruence σ. Thus Theorem 9.4 tells us that every weakly E-ample monoid has a quasi-free cover, and it is easy to adapt its proof to show that every weakly E-ample semigroup has a quasi-free ample cover.

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Department of Mathematics, University of York, Heslington, York YO10 5DD, UK

E-mail address: jbf1@york.ac.uk, varg1@york.ac.uk

Centro de Álgebra da Universidade de Lisboa, Av. Prof Gama Pinto, 2, 1649-003 Lisboa, Portugal, *and* Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1746–016 Lisboa, Portugal

E-mail address: ggomes@cii.fc.ul.pt