GRAPH PRODUCTS OF SEMIGROUPS

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Abstract. The graph product of semigroups is an essentially different construction from that for monoids. The second and the third authors showed in a recent paper that the classes of left abundant monoids and left Fountain monoids are closed under graph products of monoids. As a corollary, via a specific embedding of a graph product of semigroups into a graph product of monoids, one can deduce the same result for semigroups. The main aim of this current paper is two-fold. First, we give a direct and relatively simple proof of the aforementioned corollary, which avoids the somewhat involved calculations in the monoid case. Second, we give the characterisation of $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ in the graph product of semigroups, a question left open for monoids. We hope that our work here will inform a corresponding approach in the monoid case.

This paper is dedicated to the memory and achievements of Professor Guo Yuqi.

1. Introduction

The notion of a *graph product of groups* was introduced by Green [19] and extensively studied in various contexts, for example, [1, 2]. It generalises the concept of graph groups, (also known as right-angled Artin groups) [3, 13], by replacing the free groups in the construction by arbitrary groups. *Graph products of monoids* are defined in essentially the same way as for groups [7], and, as for groups, generalise notions of free product, restricted direct product, free (commutative) monoids and graph monoids. Here, analogously to the case for groups, graph monoids are graph products of free monogenic monoids, introduced in [5]. Such monoids are also known as free partially commutative monoids, right-angle Artin monoids and trace monoids, and they have broad applications in computer science, for example, concurrent processes [11, 10].

Much of the existing work in graph products of monoids and groups has been to show that various algorithmic or algebraic properties are preserved under graph products e.g. [20, 12, 8, 24]. The recent work [9] by the second and third authors of the current paper also follows this stream. The algebraic properties we are concerned with in [9] are *abundancy* and *Fountainicity* and their one-sided versions. These notions may be thought of as weakening that of regularity, and arise from many sources, for example, that of abundancy from

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projectivity of monogenic acts, and that of Fountainicity (also known as *weak abundancy*) from connections with ordered categories. Specifically, we show in [9] that the classes of left abundant and left Fountain monoids are closed under monoid graph product. A brief introduction to abundancy and Fountainicity may be found in Section 2 and [9]; see [27, 28, 15, 16, 25] for more details. It is worth noting that every element in the graph product of monoids can be represented by certain kind of normal form, a *left Foata normal form*. Such forms were originally established for graph monoids, via arguments using cancellativity, which cannot be called upon in the general case.

The notion of *graph product of semigroups* was introduced in [9]. It is a natural analogue of the notion of graph product of monoids within the class of semigroups. However, we stress this construction is different from that for monoids. In both the semigroup and monoid case the construction is designed so that the vertex semigroups or monoids embed as subsemigroups or submonoids of the graph product, respectively. In the monoid case, this requirement results in a significant effect on the combinatorics. Nevertheless, as stated in [9], a graph products of semigroups can always be embedded into a graph product of monoids. One of the applications of the main result in [9] shows that graph products of left abundant and left Fountain semigroups are left abundant and left Fountain, respectively ([9, Corollaries 7.4, 7.5]). The aim of this current paper is to give a direct and simple proof of these results, avoiding the heavy machinery of [9] which arises from the complexity of the structure of graph products of monoids. We hope that this will illustrate the concepts involved. Further, we give the characterisations of $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ in the graph product of semigroups. The question in the corresponding case for monoids was left open in [9].

This paper is organised as follows. In Section 2 we recall the notion of graph products of semigroups and describe the universal nature of this construction. Further, we give a brief introduction to the relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$, as well as to the notions of abundancy and Fountainicity. In Section 3 we show that every element in the graph product of semigroups may be represented by a reduced word and further by a left Foata normal form. Such forms are crucial to whole analysis of this work. With these preparations, we begin in Section 4 by describing the idempotents of the graph product of semigroups and exhibiting a decomposition of elements to show that the class of left abundant semigroups is closed under graph product. Further, we characterise the relation $\mathcal{R}^*$ on graph products of semigroups. The structure of Section 5 is similar to that of Section 4, but our concern here is changed to Fountainicity and the relation $\widetilde{\mathcal{R}}$. At the end of Section 5 we give necessary and sufficient conditions for the relation $\widetilde{\mathcal{R}}$ to be a left congruence on a graph product of semigroups.

2. Preliminaries

The aim of this section is to present the technicalities necessary to follow this article, to establish notation, and to give some preliminary results. In particular we recall the notion of graph products of semigroups from [9] and explain the universal nature of such semigroups. We define the properties of left abundancy and left Fountainicity for semigroups, and give some relevant facts. We assume the reader has a working knowledge of algebraic semigroup theory, as may be found in [21].

We start with the notion of free semigroups. Let X be a set. The *free semigroup* X^+ on X consists of all non-empty words over X with operation of juxtaposition. We denote a word by $x_1 \circ \cdots \circ x_n$ where $x_i \in X$ for $1 \leq i \leq n$; we also use $\circ$ for juxtaposition of words. Where possible we use x (or y, etc.) to denote a word with letters x_i (or y_i, etc.). Throughout, our convention is that if we say $x_1 \circ \cdots \circ x_n \in X^+$, then we mean that $x_i \in X$ for all $1 \leq i \leq n$, unless we explicitly state otherwise.

Let $\Gamma = \Gamma(V, E)$ be a simple, undirected, graph with no loops. Here V is a non-empty set of *vertices* and $E \subseteq V_2$ is the set of *edges* of Γ, where V_2 is the set of 2-element subsets of V. We think of $\{\alpha, \beta\} \in E$ as joining the vertices $\alpha, \beta \in V$. For notational ease we denote an edge $\{\alpha, \beta\}$ as (α, β) or (β, α); since our graph is undirected we are identifying (α, β) with (β, α). Let $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ be a set of semigroups indexed by V, called *vertex semigroups*, such that $S_\beta \cap S_\gamma = \emptyset$ for all $\beta \neq \gamma \in V$.

Definition 2.1. [9] The *graph product* $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ of $\mathcal{S}$ with respect to Γ is defined by the presentation

$$\mathscr{GP} = \langle X \mid R \rangle$$

where $X = \bigcup_{\alpha \in V} S_\alpha$ and $R = R_v \cup R_e$ is given by:

$$\begin{aligned} R_v &= \{x \circ y = xy : \ x, y \in S_\alpha, \alpha \in V\}, \\ R_e &= \{x \circ y = y \circ x : x \in S_\alpha, \, y \in S_\beta, (\alpha, \beta) \in E\}. \end{aligned}$$

The reader should note that the graph product of semigroups is a structure defined by a *semigroup* presentation. The construction of graph products of semigroups is fundamentally different to that of graph products of monoids, since semigroups are an algebra with a different signature to that of monoids. We will see in Remark 2.4 that each vertex semigroup embeds into the graph product. A graph product of monoids or, indeed, of groups, is defined by a monoid presentation, and it is constructed in such a way that the vertex monoids embed as submonoids. To effect the latter, one identifies the identities of the individual vertex monoids, leading to more complicated combinatorics.

Throughout we assume $|V| \geq 2$, as otherwise $\mathscr{GP}$ is isomorphic to the single vertex semigroup. We denote the $R^\sharp$-class of $w \in X^+$ in $\mathscr{GP}$ by $[w]$, where $R^\sharp$ is the congruence on $\mathscr{GP}$ generated by R. Clearly, for all $u, v \in X^+$, $[u][v] = [u \circ v]$.

With notation as above, the *free product* $\mathscr{FP} = \mathscr{FP}(\mathcal{S})$ of $\mathcal{S}$ with respect to Γ is exactly the presentation

$$\mathscr{FP} = \langle X \mid R_v \rangle.$$

The following is an application of the third isomorphism theorem for semigroups, and adjusted to explicitly mention generators; see, for example, [21, Theorem 1.5.4].

Lemma 2.2. *The semigroup* $\mathscr{GP}(\Gamma, S)$ *is the quotient semigroup of* $\mathscr{FP}(\mathcal{S})$ *by the congruence generated by the binary relations corresponding to the relator* R_e.

Notice that by taking $\Gamma_\emptyset = \Gamma(V, \emptyset)$ we have $\mathscr{FP}(\mathcal{S}) = \mathscr{GP}(\Gamma_\emptyset, S)$.

Lemma 2.3. *Let $V' \subseteq V$ and let $\Gamma' = \Gamma(V', E')$ be the resulting full subgraph of Γ. Let $\mathscr{GP}'$ be the corresponding graph product of the semigroups $\mathcal{S}' = \{S_\alpha : \alpha \in V'\}$. Then $\mathscr{GP}'$ is a retract of $\mathscr{GP}$.*

Proof. The proof is similar to that of [9, Proposition 2.3]. □

Remark 2.4. Let $\alpha \in V$. By taking $V' = \{\alpha\}$ in Lemma 2.3, we immediately see that S_α is naturally embedded in $\mathscr{GP}$ via $\iota_\alpha : S_\alpha \to \mathscr{GP}$, where for $x \in S_\alpha$ we have $x\iota_\alpha = [x]$.

We now explain the universal nature of graph products of semigroups.

Definition 2.5. Suppose that S is a semigroup and we have a collection of morphisms

$$\theta = \{\theta_\alpha : S_\alpha \to S \mid \alpha \in V\}.$$

We say that θ *satisfies the* Γ*-condition* if for all $x \in S_\alpha, y \in S_\beta$ with $(\alpha, \beta) \in E$ we have

$$(x\theta_\alpha)(y\theta_\beta) = (y\theta_\beta)(x\theta_\alpha).$$

The next result is analogous to [17, Proposition 1.6].

Proposition 2.6. *The collection of embeddings*

$$\iota = \{\iota_\alpha : S_\alpha \to \mathscr{GP} \mid \alpha \in V\}$$

satisfies the Γ*-condition. Further, $\mathscr{GP}$ is generated by $\{[s] : s \in S_\alpha, \alpha \in V\}$.*

Suppose that S is a semigroup and we have a collection of morphisms

$$\zeta = \{\zeta_\alpha : S_\alpha \to S \mid \alpha \in V\}$$

satisfying the Γ*-condition. Then there is a unique morphism*

$$\overline{\zeta} : \mathscr{GP} \to S$$

such that $\iota_\alpha\overline{\zeta} = \zeta_\alpha$ for all $\alpha \in V$.

Proof. Let $s \in S_\alpha, t \in S_\beta$ with $(\alpha, \beta) \in E$. Then

$$(s\iota_\alpha)(t\iota_\beta) = [s][t] = [s \circ t] = [t \circ s] = [t][s] = (t\iota_\beta)(s\iota_\alpha).$$

It is clear that $\mathscr{GP}$ is generated by $\{[s] : s \in S_\alpha, \alpha \in V\}$.

Let S be as given. Define a map

$$\xi : X^+ \longrightarrow S$$

by $s\xi = s\zeta_\alpha$ where $s \in S_\alpha$. It is easy to see from the definition of the Γ-condition that $R \subseteq \ker \xi$ and hence $R^\sharp \subseteq \ker \xi$. It follows that

$$\overline{\zeta} : \mathscr{GP} \to S,\ [w] \mapsto w\xi$$

is a well defined morphism. Further, for any $\alpha \in V$ and $s \in S_\alpha$,

$$s\iota_\alpha\overline{\zeta} = [s]\overline{\zeta} = s\xi = s\zeta_\alpha$$

so that $\iota_\alpha\overline{\zeta} = \zeta_\alpha$.

Suppose that there is another morphism $\zeta' : \mathscr{GP} \to S$ such that $\iota_\alpha\zeta' = \zeta_\alpha$ for all $\alpha \in V$. Since $\{[s] : s \in S_\alpha, \alpha \in V\}$ generates $\mathscr{GP}$ it follows that $\overline{\zeta} = \zeta'$. □

We now show that the conditions of Proposition 2.6 characterise $\mathscr{GP}$.

Proposition 2.7. *Suppose that U is a semigroup and we have a collection of embeddings*

$$\nu = \{\nu_\alpha : S_\alpha \to U\}$$

satisfying the Γ-condition, such that U is generated by $\{s\nu_\alpha : \alpha \in V, s \in S_\alpha\}$. Suppose also that U satisfies the condition that for any semigroup S and collection of morphisms

$$\theta = \{\theta_\alpha : S_\alpha \to S \mid \alpha \in V\}$$

satisfying the Γ-condition there is a unique morphism $\beta : U \to S$ such that $\nu_\alpha\beta = \theta_\alpha$ for all $\alpha \in V$. Then there is an isomorphism

$$\overline{\nu} : \mathscr{GP} \to U$$

such that $\iota_\alpha\overline{\nu} = \nu_v$ for all $\alpha \in V$.

Proof. From Proposition 2.6, the collection of embeddings

$$\iota = \{\iota_\alpha : S_\alpha \to \mathscr{GP}\}$$

satisfies the Γ-condition. So there is a unique morphism $\beta : U \to \mathscr{GP}$ such that $\nu_\alpha\beta = \iota_\alpha$ for each $\alpha \in V$. On the other hand, again by Proposition 2.6, there is a unique morphism $\overline{\nu} : \mathscr{GP} \to U$ such that $\iota_\alpha\overline{\nu} = \nu_\alpha$ for each $\alpha \in V$.

For any $\alpha \in V$ and $s \in S_\alpha$, we have

$$[s]\overline{\nu}\beta = s\iota_\alpha\overline{\nu}\beta = s\nu_\alpha\beta = s\iota_\alpha = [s]$$

and as $\mathscr{GP}$ is generated by $\{[s] : s \in S_\alpha, \alpha \in V\}$, we have that $\overline{\nu}\beta$ is the identity on $\mathscr{GP}$. Dually, we may show that $\beta\overline{\nu}$ is the identity on U, and we conclude that β and $\overline{\nu}$ are isomorphisms. $\square$

A similar result to Proposition 2.7 may be obtained for graph products of monoids, this was omitted from [9] due to considerations of paper length.

The rest of this section is to briefly recall the equivalence relations $\mathcal{R}$, $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ on a semigroup S and their left/right duals $\mathcal{L}$, $\mathcal{L}^*$ and $\widetilde{\mathcal{L}}$. These lead to notions of regular, (left) abundant and (left) Fountain semigroups. For details, we refer the readers to [27, 28, 15, 16, 25].

Let S be a semigroup; we denote by $E = E(S)$ the set of all idempotents of S. The relation $\mathcal{R}$ is defined by the rule that for any $a, b \in S$

$$a \mathrel{\mathcal{R}} b \Leftrightarrow aS^1 = bS^1.$$

The relation $\mathcal{L}$ is defined dually. Clearly, both $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on S. It is known that a semigroup S is regular if and only if each $\mathcal{R}$-class of S contains an idempotent if and only if each $\mathcal{L}$-class of S contains an idempotent. Regularity is often not preserved by algebraic constructions e.g. [22]. It is easy to see that graph products of regular semigroups with underlying graphs not complete will not be regular; indeed this is true for free products. So, it is natural to consider some relations on S larger than $\mathcal{R}$ and $\mathcal{L}$ and ask whether they contain idempotents.

The relation $\mathcal{R}^*$ is defined by the rule that for any $a, b \in S$ we have $a \mathrel{\mathcal{R}^*} b$ if and only if $a \mathrel{\mathcal{R}} b$ in some oversemigroup T of S. Equivalently,

$$a \mathrel{\mathcal{R}^*} b \Leftrightarrow (\forall x, y \in S^1)(xa = ya \Leftrightarrow xb = yb).$$

The relation $\mathcal{L}^*$ is defined dually. It is easy to see that $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{L} \subseteq \mathcal{L}^*$, and that we have $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{L} = \mathcal{L}^*$ whenever S is regular. Further, both $\mathcal{R}$ and $\mathcal{R}^*$ are left congruences, while both $\mathcal{L}$ and $\mathcal{L}^*$ are right congruences. See [27, 28, 18] for further details of $\mathcal{R}^*$ and $\mathcal{L}^*$.

Definition 2.8. A semigroup S is said to be *left abundant* if each $\mathcal{R}^*$-class of S contains an idempotent. *Right abundant semigroups* are define dually and we say S is *abundant* if it is both left and right abundant.

Note that a semigroup may be left but not right abundant, as is easily seen by considering a right but not left cancellative monoid. We have the following useful result to determine when an element is $\mathcal{R}^*$-related to an idempotent.

Lemma 2.9. [16] *Let S be a semigroup with $a \in S$ and $e \in E$. Then the following are statements are equivalent:*
(i) $a \mathrel{\mathcal{R}^*} e$;
(ii) $ea = a$ and for all $x, y \in S^1$, $xa = ya$ implies $xe = ye$.

Many semigroups (for example, left restriction semigroups) are not left abundant but have idempotents in classes of a relation $\widetilde{\mathcal{R}}$ that is a further extension of $\mathcal{R}^*$, and the same is true in the two-sided case (for example, semigroups of binary relations). The relation $\widetilde{\mathcal{R}}$ is defined by the rule that for any $a, b \in S$ we have

$$a \mathrel{\widetilde{\mathcal{R}}} b \Leftrightarrow (\forall e \in E)(ea = a \Leftrightarrow eb = b).$$

The relation $\widetilde{\mathcal{L}}$ is defined dually. Clearly $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ and $\mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ with equality if S is abundant. The relations $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ are introduced in [14] and were further developed in [25]. They have been the topic of extensive studies, particularly to understand to what extent the theory of regular and inverse semigroups might have 'non-regular' analogues; see, for example, [6, 23, 29]. Unlike $\mathcal{R}$ and $\mathcal{R}^*$, here we have that $\widetilde{\mathcal{R}}$ is not necessarily a left congruence. Similarly, $\widetilde{\mathcal{L}}$ is not necessarily a right congruence.

Definition 2.10. A semigroup S is said to be *left Fountain* if each $\widetilde{\mathcal{R}}$-class of S contains an idempotent. Dually, we may define *right Fountain semigroup*, and S is *Fountain* if it is both left and right Fountain.

Formerly, left Fountain was referred to as *weakly left abundant*, but in view of the perceived significance the notion was renamed by Margolis and Steinberg in [26]. Corresponding to Lemma 2.9 we have the well known:

Lemma 2.11. [18, Lemma 2.9] *Let S be a semigroup with $a \in S$ and $e \in E$. Then the following are statements are equivalent:*
(i) $a \mathrel{\widetilde{\mathcal{R}}} e$;
(ii) $ea = a$ and for all $f \in E$, $fa = a$ implies $fe = e$.

3. (left) Foata normal forms

Throughout this section $\mathscr{GP}$ denotes a graph product $\mathscr{GP}(\Gamma, \mathcal{S})$ of semigroups under the notation established in Section 2. We show that every element of $\mathscr{GP}$ may be represented by a reduced word, which is unique up to shuffle equivalence. Further, every reduced word is equivalent to a left Foata normal form, which is also unique, up to a certain sense described in Theorem 3.14. We remark that such forms were originally established for graph monoids and developed in [9] for graph products of monoids.

Definition 3.1. Let $s : X \to V$ be a map defined by $s(a) = \alpha$ if $a \in S_\alpha$. The *support* $s(x)$ of $x = x_1 \circ \cdots \circ x_n \in X^+$ is defined by

$$s(x) = \{s(x_i) : 1 \le i \le n\}.$$

When $s(x) = \{\alpha\}$ is a singleton, we write simply $s(x) = \alpha$. Notice that for any $x, y \in X^+$, if $[x] = [y]$ then $s(x) = s(y)$.

Definition 3.2. Let $x_1 \circ \cdots \circ x_n \in X^+$. A *reduction* is a step:

(v) $x_1 \circ \cdots \circ x_n \to x_1 \circ \cdots x_{i-1} \circ x_i x_{i+1} \circ x_{i+2} \circ \cdots \circ x_n$ where $x_i, x_{i+1} \in S_\alpha$ for some $\alpha \in V$.

A *shuffle* is a step:

(e) $x_1 \circ \cdots \circ x_n \to x_1 \circ \cdots \circ x_{i-1} \circ x_{i+1} \circ x_i \circ x_{i+2} \circ \cdots \circ x_n$ where $(s(x_i), s(x_{i+1})) \in E$.

Definition 3.3. Two words in X^+ are *shuffle equivalent* if one can be obtained from the other by applying relations in R_e, that is, by shuffles.

The next result captures how we may shuffle a word to re-order it.

Lemma 3.4. *Let $x = x_1 \circ \cdots \circ x_n \in X^+$. Then we can shuffle x to $x' = x_{i_1} \circ \cdots \circ x_{i_n}$ if and only if for all $1 \le j < k \le n$, if $i_k < i_j$ then $(s(x_{i_j}), s(x_{i_k})) \in E$.*

Proof. Suppose that we can shuffle x to x'. If $1 \le j < k \le n$ and $i_k < i_j$, then in the process we must have changed the order of x_{i_k} and x_{i_j}, so that by the definition of R_e we must have $(s(x_{i_j}), s(x_{i_k})) \in E$.

Conversely, let x' have the property that for all $1 \le j < k \le n$, if $i_k < i_j$ then $(s(x_{i_j}), s(x_{i_k})) \in E$. If $n = 1$ the result is immediate. Suppose for induction the result is true for $n - 1$. Then for $1 \le j < i_1$ we have $x_j = x_{i_{k(j)}}$ where $1 < k(j)$ but $i_{k(j)} < i_1$. By assumption $(s(x_{i_1}), s(x_{i_{k(j)}})) \in E$ so we may shuffle x_{i_1} in

$$x = x_1 \circ x_2 \cdots \circ x_{i_1 - 1} \circ x_{i_1} \circ x_{i_1+1} \circ \cdots x_n$$

to the left to obtain

$$x'' = x_{i_1} \circ x_1 \circ x_2 \cdots \circ x_{i_1-1} \circ x_{i_1+1} \circ \cdots x_n.$$

Considering now the word $x_1 \circ x_2 \cdots \circ x_{i_1-1} \circ x_{i_1+1} \circ \cdots x_n$ and applying our inductive hypothesis (with suitable relabelling) we obtain that x shuffles to x'. $\square$

Definition 3.5. A word $x = x_1 \circ \cdots \circ x_n \in X^+$ is *reduced* if for all $1 \le i < j \le n$ with $s(x_i) = s(x_j)$, there exists some $i < k < j$ with $(s(x_i), s(x_k)) \notin E$.

Clearly, if $s(x)$ is a complete subgraph, then x is reduced if and only if $s(x_i) \neq s(x_j)$ for all $1 \leq i < j \leq n$.

Remark 3.6. Let $x = x_1 \circ \cdots \circ x_m$, $y = y_1 \circ \cdots \circ y_n \in X^+$ be reduced. Then $x \circ y$ is *not* reduced exactly if there exists i, j with $1 \leq i \leq m, 1 \leq j \leq n$ such that $s(x_i) = s(y_j)$ and for all h, k with $i < h \leq m, 1 \leq k < j$ we have $(s(x_i), s(z)) \in E$ where $z = x_h$ or $z = y_k$.

The proof of the following result is similar to that of [9, Lemma 3.7], so omitted.

Lemma 3.7. *Let $w \in X^+$. Applying reductions and shuffles leads in a finite number of steps to a reduced word $\overline{w}$ with $[w] = [\overline{w}]$.*

The next result was originally proven for graph products of monoids in [19] and oft quoted. The argument for semigroups is much simpler, and worth stating.

Proposition 3.8. *Every element of the graph product $\mathscr{GP}$ is represented by a reduced word. Two reduced words represent the same element of $\mathscr{GP}$ if and only if they are shuffle equivalent. An element $w \in [x]$ is of minimal length in $[x]$ if and only if it is reduced.*

Proof. It follows from Lemma 3.7 that for any $[x] \in \mathscr{GP}$ we have $[x] = [\overline{x}]$ for some reduced word $\overline{x}$.

Next, we show that the set of all shuffle equivalence classes forms a confluent rewriting system; details for rewriting systems may be found in [4]. For convenience we denote by (x) the shuffle equivalence class of $x \in X^+$ and write $(x) \longrightarrow (y)$ if y is obtained from $x' \in (x)$ by applying a reduction.

Let $x = x_1 \circ \cdots \circ x_n \in X^+$ and pick $x' = x_{i_1} \circ \cdots \circ x_{i_n}$ and $x'' = x_{j_1} \circ \cdots \circ x_{j_n}$ in (x). Suppose that $s(x_{i_k}) = s(x_{i_{k+1}})$ so that we may perform a reduction to obtain

$$y' = x_{i_1} \circ \cdots \circ x_{i_{k-1}} \circ x_{i_k} x_{i_{k+1}} \circ x_{i_{k+2}} \circ \cdots \circ x_{i_n}.$$

Then by Lemma 3.4, y' is shuffle equivalent to

$$y = x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_n$$

where $p = i_k$ and $q = i_{k+1}$; notice we must have that $p < q$. Applying the same process to x'' results in a word

$$z = x_1 \circ \cdots \circ x_{r-1} \circ x_r x_t \circ x_{r+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n$$

where $r < t$.

Therefore, $(x) \longrightarrow (y)$ and $(x) \longrightarrow (z)$. We now need show that $(y) \xrightarrow{*} (v)$ and $(z) \xrightarrow{*} (v)$ for some $v \in X^+$, as depicted by the following picture

$$\begin{tikzcd}
 & (x) \arrow[dl] \arrow[dr] & \\
(y) \arrow[dr, "*"'] & & (z) \arrow[dl, "*"] \\
 & (v) &
\end{tikzcd}$$

Without loss of generality we may assume that $p \leq r$. If $p = r$ then from Lemma 3.4 (bearing in mind our graphs have no loops), we cannot have $p = r < q < t$ or $p = r < t < q$; we deduce that in this case $q = t$ so that $(y) = (z)$. If $p < r$, then again we cannot have that $r < q$, so that either $q = r$ or $q < r$.

If $q = r$, then $(y) = (y'')$ where y'' is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_t \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n$$

and then $(y'') \longrightarrow (v)$ where v is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q x_t \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n.$$

Similarly, $(z) \longrightarrow (v)$.

If $q < r$, then by shuffling and apply a reduction in each case we have $(y) \longrightarrow (u)$ and $(z) \longrightarrow (u)$ where u is the word

$$x_1 \circ \cdots \circ x_{p-1} \circ x_p x_q \circ x_{p+1} \circ \cdots \circ x_{q-1} \circ x_{q+1} \circ \cdots x_{r-1} \circ x_r x_t \circ x_{r+1} \circ \cdots \circ x_{t-1} \circ x_{t+1} \circ \cdots \circ x_n.$$

We have shown that the set of all shuffle equivalence classes forms a confluent rewriting system. It follows that any two reduced forms represent the same element of $\mathscr{GP}$ if and only if they are shuffle equivalent.

Let $w \in [x]$ for some words $w, x \in X^+$. It is clear that if w is of minimal length in $[x]$, then it must be reduced. Finally, if w is reduced then as certainly $[w] = [z]$ for some word z of minimal length in $[x]$, then z is also reduced, giving that w and z are shuffle equivalent, so that they have the same length. $\square$

Definition 3.9. If $x \in X^+$ and $[x] = [w]$ for a reduced word $w \in X^+$, then we say that w is a *reduced form* of x.

The following result will be used frequently in the rest of this work.

Lemma 3.10. *Let $[x] = [y]$ where $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_n$ are reduced and let $1 \leq m \leq n$. Then $[x_1 \circ \cdots \circ x_m] = [y_1 \circ \cdots \circ y_m]$ if and only if $[x_{m+1} \circ \cdots \circ x_n] = [y_{m+1} \circ \cdots \circ y_n]$.*

Proof. The proof is similar to that of [9, Lemma 3.13]. $\square$

Definition 3.11. A word $w \in X^+$ is a *complete block* if it is reduced, and $s(w)$ forms a complete subgraph of $\Gamma = \Gamma(V, E)$.

We now show that any reduced word in X^+ may be shuffled into a word that is a product of complete blocks.

Definition 3.12. Let $w \in X^+$. Then w is a *left Foata normal form* with *block length* k and *blocks* $w_i \in X^+$, $1 \leq i \leq k$, if:

(i) $w = w_1 \circ \cdots \circ w_k \in X^+$ is a reduced word;

(ii) $s(w_i)$ is a complete subgraph for all $1 \leq i \leq k$;

(iii) for any $1 \leq i < k$ and $\alpha \in s(w_{i+1})$, there is some $\beta \in s(w_i)$ such that $(\alpha, \beta) \notin E$.

If $[x] = [w]$ where w is a left Foata normal form, then w is a *left Foata normal form for* x.

Remark 3.13. (i) A complete block is precisely a word in left Foata normal form with block length 1.

(ii) If $w = w_1 \circ \cdots \circ w_k \in X^+$ is in left Foata normal form with blocks w_i, $1 \leq i \leq k$, then for any $1 \leq j \leq j' \leq k$ we have $w_j \circ w_{j+1} \circ \cdots \circ w_{j'}$ is also in left Foata normal form, with blocks w_h, $j \leq h \leq j'$.

(iii) If $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_m \in X^+$ are complete blocks, then $[x] = [y]$ if and only if x and y are shuffle equivalent if and only if $y_i = x_{i\sigma}$, $1 \leq i \leq n$, for some permutation σ of $\{1, \cdots, n\}$; in particular, $n = m$ and $s(x) = s(y)$.

(iv) A word $x = x_1 \circ \cdots \circ x_n$ is a complete block if and only if $s(x)$ is complete and $s(x_i) \neq s(x_j)$ for all $1 \leq i < j \leq n$.

The arguments in [9, Proposition 3.17 and Theorem 3.18], arguing for the existence and uniqueness of left Foata normal forms of elements of graph products of monoids, only involve shuffling of reduced words. The same arguments may be taken to show the corresponding results for $\mathscr{GP}$.

Theorem 3.14. *Every element in $\mathscr{GP}$ may be represented by a left Foata normal form. Let $w \in X^+$ and let $w_1 \circ w_2 \circ \cdots \circ w_k$ and $w'_1 \circ w'_2 \circ \cdots \circ w'_h$ be left Foata normal forms of w with blocks w_i, w'_j for $1 \leq i \leq k, 1 \leq j \leq h$. Then $k = h$ and $[w_i] = [w'_i]$ for $1 \leq i \leq k$.*

Clearly, we may define the notion of a *right Foata normal form* of an element in X^+, and the dual arguments to those for left Foata normal form hold.

4. Idempotents, abundancy and the relation $\mathcal{R}^*$ on $\mathscr{GP}$

In this section we first give a description of idempotents in $\mathscr{GP}$. With this in hand, we show that the graph product of left abundant semigroups is left abundant. Further, we give a characterisation of the relation $\mathcal{R}^*$ on $\mathscr{GP}$.

Lemma 4.1. *Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_n \in X^+$, where $s(x_i) = s(y_i)$ for all $1 \leq i \leq n$ and $s(x_i) \neq s(x_j)$ (and so $s(y_i) \neq s(y_j)$) for all $1 \leq i, j \leq n$ with $i \neq j$. Then*

$$[x] = [y] \Longleftrightarrow x_i = y_i \text{ for all } 1 \leq i \leq n.$$

Proof. For each $\alpha \in Y$, let $S_\alpha^{\underline{1}_\alpha}$ be the semigroup S_α with an identity $\underline{1}_\alpha$ adjoined *whether or not S_α is a monoid.* For any $\alpha \in V$ we define a morphism

$$\phi_\alpha : X^+ \longrightarrow S_\alpha^{\underline{1}_\alpha}$$

by its action on generators, where

$$z\phi_\alpha = \begin{cases} z & z \in S_\alpha \\ \underline{1}_\alpha & \text{else.} \end{cases}$$

We claim that $R^\sharp \subseteq \ker \phi_\alpha$.

To see $R_v \subseteq \ker \phi_\alpha$, let $\beta \in V$ and $g, h \in S_\beta$. If $\beta = \alpha$, then

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = gh = (gh)\phi_\alpha.$$

If $\beta \neq \alpha$, then

$$(g \circ h)\phi_\alpha = (g\phi_\alpha)(h\phi_\alpha) = \underline{1}_\alpha \underline{1}_\alpha = \underline{1}_\alpha = (gh)\phi_\alpha.$$

Now consider $a \in S_\beta, b \in S_\gamma$ with $\beta \neq \gamma, (\beta,\gamma) \in E$. If $\beta \neq \gamma = \alpha$, then

$$(a \circ b)\phi_\alpha = (a\phi_\alpha)(b\phi_\alpha) = \underline{1}_\alpha b = b\underline{1}_\alpha = (b\phi_\alpha)(a\phi_\alpha) = (b \circ a)\phi_\alpha.$$

Dual arguments hold for the case $\alpha = \beta \neq \gamma$. If $\alpha \neq \beta \neq \gamma \neq \alpha$, then

$$(a \circ b)\phi_\alpha = (a\phi_\alpha)(b\phi_\alpha) = \underline{1}_\alpha\underline{1}_\alpha = (b\phi_\alpha)(a\phi_\alpha) = (b \circ a)\phi_\alpha.$$

Thus $R_e \subseteq \ker\phi_\alpha$.

It follows that $R^\sharp \subseteq \ker\phi_\alpha$ and so $\overline{\phi}_\alpha : \mathscr{GP} \longrightarrow S_\alpha^{\underline{1}_\alpha}$ give by $[x]\overline{\phi}_\alpha = x\phi_\alpha$ is a well defined morphism. For each $1 \leq i \leq n$,

$$x_i = [x]\overline{\phi}_{s(x_i)} = [y]\overline{\phi}_{s(y_i)} = y_i.$$

The converse of the statement is clear. $\square$

Lemma 4.2. *Let $x = x_1 \circ \cdots \circ x_n \in X^+$ be a reduced word. Then $[x]$ is an idempotent if and only if $s(x)$ is complete and $x_i = x_i^2$ for all $1 \leq i \leq n$.*

Proof. Let x be as given. The sufficiency is clear. To show the necessity, suppose that $[x]$ is idempotent and let $s(x_i) = \alpha_i$ for all $1 \leq i \leq n$. If $s(x)$ is not a complete subgraph of Γ, then there must exist $1 \leq i < j \leq n$, such that $\alpha_i \neq \alpha_j$ and $(\alpha_i, \alpha_j) \notin E$. Let $(\{i\} * \{j\})^1$ be the free product on the trivial semigroups $\{i\}$ and $\{j\}$, with identity adjoined. We define a map

$$\psi : X^+ \longrightarrow (\{i\} * \{j\})^1$$

by its action on generators, where

$$z\psi = \begin{cases} i & z \in S_{\alpha_i} \\ j & z \in S_{\alpha_j} \\ 1 & \text{else.} \end{cases}$$

We now show that $R^\sharp \subseteq \ker\alpha$. Let $g, h \in S_\beta, \beta \in V$. If $\beta = \alpha_i$, then

$$(g \circ h)\psi = (g\psi)(h\psi) = ii = i = (gh)\psi.$$

Similar arguments hold for the case $\beta = \alpha_j$. If $\beta \notin \{\alpha_i, \alpha_j\}$, then

$$(g \circ h)\psi = (g\psi)(h\psi) = 11 = 1 = (gh)\psi.$$

Now consider $a \in S_\beta, b \in S_\gamma$ with $\beta \neq \gamma, (\beta,\gamma) \in E$. If $\alpha_i = \beta \neq \gamma$, then

$$(a \circ b)\psi = (a\psi)(b\psi) = i1 = 1i = (b\psi)(a\psi) = (b \circ a)\psi.$$

Similar arguments hold for the case $\alpha_j = \beta \neq \gamma$. If $\beta, \gamma \notin \{\alpha_i, \alpha_j\}$, then

$$(a \circ b)\psi = (a\psi)(b\psi) = 11 = (b\psi)(a\psi) = (b \circ a)\psi.$$

Since $(\alpha_i, \alpha_j) \notin E$, these are the only cases to consider. Hence $R^\sharp \subseteq \ker\psi$, giving a morphism

$$\mathscr{GP} \longrightarrow (\{i\} * \{j\})^1,\ [x] \mapsto x\psi.$$

By our assumption, $[x] = [x^2]$, and it follows that $x\psi = (x\psi)(x\psi)$. Notice that $x\psi$ must contain letters i and j, so that, if the length of the reduced form of $x\psi$ is l, then $l \geq 2$, so that the length of the reduced form of $(x\psi)(x\psi)$ is either $2l - 1$ or $2l$. By the uniqueness of the length of reduced form of $x\psi = (x\psi)(x\psi)$, we must have $l = 2l$ or $l = 2l - 1$, a contradiction. We deduce that $s(x)$ is a complete subgraph of Γ. It follows that

$$[x_1 \circ \cdots \circ x_n] = [x_1^2 \circ \cdots \circ x_n^2].$$

Since x is reduced and $s(x)$ is complete, $s(x_i) \neq s(x_j)$ for all $1 \leq i < j \leq n$. It follows from Lemma 4.1 that $x_i = x_i^2$ for all $1 \leq i \leq n$. $\square$

Next, we construct three maps in Lemmas 4.3 and 4.6, which are the key for the proof of the abundancy of $\mathscr{G}\mathscr{P}$. The reader should note that these maps are *not* morphisms. We begin by setting up some notation. For each $(\alpha, \beta) \notin E$, where $\alpha \neq \beta$, and for any $x \in X^+$, we obtain the word $x(\alpha, \beta)$ by deleting certain x_i from x, where $s(x_i) = \alpha$, by the rule that starting from the right we delete x_i as long as:
(1) there is at least one x_j with $j < i$ such that $s(x_j) = \beta$;
(2) there are no x_k with $i < k$ such that $s(x_k) = \beta$.
Let $\mathscr{L}$ be the binary relation on X^+ defined by

$$\mathscr{L} = \{(x \circ u \circ y, x \circ v \circ y) : x, y \in X^+, (u, v) \in R\}.$$

Notice that $R^\sharp$ is the transitive closure of $\mathscr{L}$.

Lemma 4.3. *For each $(\alpha, \beta) \notin E$, where $\alpha \neq \beta$, we define the map*

$$\theta_{\alpha,\beta} : X^+ \to \mathscr{G}\mathscr{P},\ x \mapsto x\theta_{\alpha,\beta} = [x(\alpha, \beta)].$$

Then

$$\overline{\theta}_{\alpha,\beta} : \mathscr{G}\mathscr{P} \to \mathscr{G}\mathscr{P},\ [w] \mapsto w\theta_{\alpha,\beta}$$

is well defined.

Proof. We need to show that $R^\sharp \subseteq \ker \theta_{\alpha,\beta}$. Since $R^\sharp$ is the transitive closure of $\mathscr{L}$, to show $R^\sharp \subseteq \ker \theta_{\alpha,\beta}$, we just need show that $\mathscr{L} \subseteq \ker \theta_{\alpha,\beta}$. It is sufficient to consider the following cases.

Case (i) $(u, v) = (s \circ t, st)$ where $s, t \in S_\alpha$. If $\beta \in s(y)$, then clearly

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u](y\theta_{\alpha,\beta}) = [x \circ v](y\theta_{\alpha,\beta}) = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

If β is in neither $s(y)$ nor $s(x)$, then

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u \circ y] = [x \circ v \circ y] = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

If $\beta \notin s(y)$ but $\beta \in s(x)$, then

$$(x \circ u \circ y)\theta_{\alpha,\beta} = (x \circ y)\theta_{\alpha,\beta} = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

Case (ii) $(u, v) = (s \circ t, st)$ where $s, t \in S_\beta$. We have

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u](y\theta_{\alpha,\beta}) = [x \circ v](y\theta_{\alpha,\beta}) = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

Case (iii) $(u, v) = (s \circ t, st)$ where $s, t \in S_\gamma$ and $\gamma \neq \alpha, \beta$. It is clear that

$$(x \circ u \circ y)\theta_{\alpha,\beta} = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

Case (iv) $(u, v) = (s \circ t, t \circ s)$ where $s \in S_\alpha$, $t \in S_\gamma$, $\gamma \neq \beta$ and $(\alpha, \gamma) \in E$. If $\beta \in s(y)$, then

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u](y\theta_{\alpha,\beta}) = [x \circ v](y\theta_{\alpha,\beta}) = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

If β is neither in $s(y)$ nor $s(x)$, then

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u \circ y] = [x \circ v \circ y] = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

If $\beta \notin s(y)$ but $\beta \in s(x)$, then

$$(x \circ u \circ y)\theta_{\alpha,\beta} = (x \circ t \circ y)\theta_{\alpha,\beta} = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

Case (v) $(u, v) = (s \circ t, t \circ s)$ where $s \in S_\beta, t \in S_\gamma$, $\gamma \neq \alpha$ and $(\beta, \gamma) \in E$. We have

$$(x \circ u \circ y)\theta_{\alpha,\beta} = [x \circ u](y\theta_{\alpha,\beta}) = [x \circ v](y\theta_{\alpha,\beta}) = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

Case (vi) $(u, v) = (s \circ t, t \circ s)$ where $s \in S_\mu, t \in S_\gamma, (\mu, \gamma) \in E$, $\mu, \gamma \notin \{\alpha, \beta\}$. It is clear

$$(x \circ u \circ y)\theta_{\alpha,\beta} = (x \circ v \circ y)\theta_{\alpha,\beta}.$$

The above arguments show that $R^\sharp \subseteq \ker \theta_{\alpha,\beta}$, so that $\overline{\theta}_{\alpha,\beta}$ exists as claimed. $\square$

Definition 4.4. For each $\alpha \in V$ and each $x = x_1 \circ \cdots \circ x_n \in X^+$, we define a set

$$N_\alpha(x) = \{k \in \{1, \cdots, n\} : s(x_k) = \alpha \text{ and for all } j > k, \text{ either } s(x_j) = \alpha \text{ or } (\alpha, s(x_j)) \in E\}.$$

Of course, $N_\alpha(x)$ may be empty. The proof of the following lemma is straightforward.

Lemma 4.5. *Let $\alpha \in V$ and $x = x_1 \circ \cdots \circ x_n \in X^+$. Suppose that $N_\alpha(x) = \{l_1, \cdots, l_r\}$ with $1 \leq l_1 < \cdots < l_r \leq n$. Then*

$$[x] = [x'][x_{l_1} \circ \cdots \circ x_{l_r}]$$

where x' is obtained by deleting all x_{l_i}, $1 \leq i \leq r$, from x.

Lemma 4.6. *For each $\alpha \in V$, the maps*

$$\phi_\alpha : X^+ \longrightarrow \mathscr{GP}^1 \text{ and } \psi_\alpha : X^+ \longrightarrow \mathscr{GP}^1$$

defined by

$$z\phi_\alpha = [z_{l_1} \circ \cdots \circ z_{l_r}] \text{ and } z\psi_\alpha = [z'],$$

where $z = z_1 \circ \cdots \circ z_m \in X^+$ and $N_\alpha(z) = \{l_1, \cdots, l_r\}$ with $l_1 < \cdots < l_r$, and z' is the obtained by deleting $z_{l_1}, \cdots, z_{l_r}$ from z, induce maps

$$\overline{\phi}_\alpha : \mathscr{GP} \longrightarrow \mathscr{GP}^1 \text{ and } \overline{\psi}_\alpha : \mathscr{GP} \longrightarrow \mathscr{GP}^1$$

defined by

$$[z]\overline{\phi}_\alpha = z\phi_\alpha \text{ and } [z]\overline{\psi}_\alpha = z\psi_\alpha.$$

Further, $[z] = (z\psi_\alpha)(z\phi_\alpha) = ([z]\overline{\psi}_\alpha)([z]\overline{\phi}_\alpha)$.

Proof. We first show that $R^\sharp \subseteq \ker\phi_\alpha$ and $R^\sharp \subseteq \ker\psi_\alpha$. Since $R^\sharp$ is the transitive closure of $\mathscr{L}$, it is sufficient to show that $\mathscr{L} \subseteq \ker\phi_\alpha$ and $\mathscr{L} \subseteq \ker\psi_\alpha$.

Let $x = x_1 \circ \cdots \circ x_p, y = y_1 \circ \cdots \circ y_q \in X^+$ and $(u, v) \in R$. We consider the following cases.

Case (i) $(u, v) = (s \circ t, t \circ s)$, where $s \in S_\beta, t \in S_\gamma$ with $(\beta,\gamma) \in E$ and $\beta,\gamma \neq \alpha$. It is easy to see that $N_\alpha(x \circ u \circ y) = N_\alpha(x \circ v \circ y)$ and $p+1, p+2$ are neither in $N_\alpha(x \circ u \circ y)$ nor $N_\alpha(x \circ v \circ y)$, so that

$$(x \circ u \circ y)\phi_\alpha = (x \circ v \circ y)\phi_\alpha \text{ and } (x \circ u \circ y)\psi_\alpha = (x \circ v \circ y)\psi_\alpha. \tag{1}$$

Case (ii) $(u, v) = (s\circ t, t\circ s)$ where $s \in S_\beta, t \in S_\alpha$ with $(\beta, \alpha) \in E$. We have the following 2 subcases.

Subcase (ii)(a) $N_\alpha(x \circ u \circ y) = \emptyset$. If $\alpha \notin s(y)$, then there exists $1 \leq j \leq q$ such that $(s(y_j), \alpha) \notin E$, giving $N_\alpha(x\circ v\circ y) = \emptyset$. If $\alpha \in s(y)$, then we pick j to be the greatest such that $s(y_j) = \alpha$. As $N_\alpha(x\circ u\circ y) = \emptyset$, there exists k with $j < k \leq q$ such that $(\alpha, s(y_k)) \notin E$, so that $N_\alpha(x \circ v \circ y) = \emptyset$. Therefore

$$(x \circ u \circ y)\phi_\alpha = 1 = (x \circ v \circ y)\phi_\alpha$$

and

$$(x \circ u \circ y)\psi_\alpha = [x \circ u \circ y] = [x \circ v \circ y] = (x \circ v \circ y)\psi_\alpha$$

so that Equation (1) holds.

Subcase (ii)(b) $N_\alpha(x \circ u \circ y) = \{l_1, \cdots, l_r\}$ where $1 \leq l_1 < \cdots < l_r \leq p + 2 + q$. If $p + 2 < l_1$, then we have $\{l_1, \cdots, l_r\} \subseteq N_\alpha(x \circ v \circ y)$; as $t \in S_\alpha$, there exists k with $1 \leq k < l'$ where $l'_1 = l_1 - (p + 2)$ such that $s(y_k) \neq \alpha$ and $(s(y_k), \alpha) \notin E$. In either case $N_\alpha(x \circ v \circ y) = \{l_1, \cdots, l_r\}$, and hence Equation (1) holds.

If $l_1 = p + 2$ (similarly if $l_1 = p + 1$), then $p + 1 \in N_i(x \circ v \circ y)$ and by the definition of $N_\alpha(x\circ u\circ y)$, we deduce that, for any $1 \leq j \leq p$ with $s(x_j) = \alpha$, there exists k with $j < k \leq p$ such that $s(x_k) \neq \alpha$ and $(s(x_k), \alpha) \notin E$. It follows that $N_\alpha(x \circ v \circ y) = \{p+1, l_2, \cdots, l_r\}$, and hence Equation (1) holds.

If $1 \leq l_1 \leq p$, then $p + 2 \in N_\alpha(x \circ u \circ y)$ and $p + 1 \in N_\alpha(x \circ v \circ y)$. Also, for any $1 \leq j < l_1$ with $s(x_j) = \alpha$, there must exist k with $j < k \leq l_1$ such that $(s(x_k), \alpha) \notin E$, so that $N_\alpha(x\circ v\circ y) = (\{l_1, l_2, \cdots, l_r\}\backslash\{p+2\}) \cup \{p+1\}$, and and again Equation (1) holds.

Case (iii) $(u, v) = (s \circ t, st)$ with $s, t \in S_\beta$. Whether β equals α or not, it is clear that Equation (1) holds.

The above arguments show that $R^\sharp \subseteq \ker\phi$ and $R^\sharp \subseteq \ker\psi$, so that $\overline{\phi}_\alpha$ and $\overline{\psi}_\alpha$ are maps as stated. Finally, it follows from Lemma 4.5 that $[z] = (z\psi_\alpha)(z\phi_\alpha)$. $\square$

Our next step is to show that the $\mathcal{R}^*$-class of an element of $\mathscr{GP}$ is determined by the left-most block of its left Foata normal form.

Lemma 4.7. *Let $w = w_1 \circ \cdots \circ w_n \in X^+$ be a left Foata normal form with blocks w_i for all $1 \leq i \leq n$. Then $[w]\ \mathcal{R}^*\ [w_1]$.*

Proof. Let $[x], [y] \in \mathscr{GP}$ be such that $[x \circ w] = [y \circ w]$. The idea of our proof is to delete letters from the end of w, in the expression $[x \circ w] = [y \circ w]$, until we end with $[x \circ w_1] = [y \circ w_1]$, by using maps defined in Lemma 4.3.

If $[w] = [w_1]$ we are done. Otherwise, if $1 < n$, pick an arbitrary $\alpha = s(w_n)$. Since w_n is a complete block, there must be exactly one letter contained in w_n with support α. Also, there exists $\beta \in s(w_{n-1})$ such that $(\alpha, \beta) \notin E$. Since $\beta \notin s(w_n)$,

$$[x \circ w_1 \circ \cdots \circ w_{n-1} \circ w'_n] = [x \circ w]\overline{\theta}_{\alpha,\beta} = [y \circ w]\overline{\theta}_{\alpha,\beta} = [y \circ w_1 \circ \cdots \circ w_{n-1} \circ w'_n]$$

where w'_n is obtained from w_n by deleting the element with support α. Notice that $w_1 \circ \cdots \circ w_{n-1} \circ w'_n$ is also a left Foata normal form. Repeating the above process, we may delete all the remaining letters of w_n one by one to obtain

$$[x \circ w_1 \circ \cdots \circ w_{n-1}] = [y \circ w_1 \circ \cdots \circ w_{n-1}].$$

Finite induction yields $[x \circ w_1] = [y \circ w_1]$, as required. The same argument applies to the case $[x][w] = [w]$. □

We now establish a connection with the relation $\mathcal{R}^*$ in $\mathscr{G}\mathscr{P}$ and the relation $\mathcal{R}^*$ in the vertex semigroups.

Lemma 4.8. *Let $z = z_1 \circ \cdots \circ z_n \in X^+$ be a complete block. Suppose that $z_k \,\mathcal{R}^*\, z'_k$ in $S_{s(z_k)}$ for $1 \leq k \leq n$ and put $z' = z'_1 \circ \cdots \circ z'_n$. Then $[z] \,\mathcal{R}^*\, [z']$ in $\mathscr{G}\mathscr{P}$.*

Proof. Let $x = x_1 \circ \cdots \circ x_m$, $y = y_1 \circ \cdots \circ y_k \in X^+$ be such that $[x][z] = [y][z]$. We now claim that $[x][z'] = [y][z']$. Let $s(z_1) = \alpha$. It follows from Lemma 4.6 that

$$[x_1 \circ \cdots \circ x_m \circ z_1 \circ \cdots z_n]\overline{\phi}_\alpha = [y_1 \circ \cdots \circ y_k \circ z_1 \cdots \circ z_n]\overline{\phi}_\alpha$$

and

$$[x_1 \circ \cdots \circ x_m \circ z_1 \circ \cdots z_n]\overline{\psi}_\alpha = [y_1 \circ \cdots \circ y_k \circ z_1 \cdots \circ z_n]\overline{\psi}_\alpha.$$

Suppose that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1 \circ \cdots \circ z_n) = \{r_1, \cdots, r_l\}, N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1 \circ \cdots \circ z_n) = \{s_1, \cdots, s_t\}.$$

Then we must have $r_l = m + 1$, $s_t = k + 1$ and

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1] = [y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1].$$

By Remark 2.4 we have $x_{r_1} \cdots x_{r_{l-1}} z_1 = y_{s_1} \cdots y_{s_{t-1}} z_1$ and then since $z_1 \;\mathcal{R}^*\; z'_1$, we deduce $x_{r_1} \cdots x_{r_{l-1}} z'_1 = y_{s_1} \cdots y_{s_{t-1}} z'_1$, so that

$$[x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z'_1] = [y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z'_1].$$

By using $\overline{\psi}_\alpha$, we obtain

$$[x' \circ z_2 \circ \cdots \circ z_n] = [y' \circ z_2 \circ \cdots \circ z_n]$$

where x' is obtained by deleting all x_{r_j} from x, where $1 \leq j \leq l-1$ and y' is obtained by deleting all y_{s_j} from y, where $1 \leq j \leq t-1$. Using the final part of Lemma 4.6 we have

$$[x' \circ z_2 \circ \cdots \circ z_n][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z'_1] = [y' \circ z_2 \circ \cdots \circ z_n][y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z'_1].$$

Notice that

$$N_\alpha(x_1 \circ \cdots \circ x_m \circ z'_1 \circ z_2 \circ \cdots \circ z_n) = N_\alpha(x_1 \circ \cdots \circ x_m \circ z_1 \circ z_2 \circ \cdots \circ z_n)$$

and

$$N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1' \circ z_2 \circ \cdots \circ z_n) = N_\alpha(y_1 \circ \cdots \circ y_k \circ z_1 \circ z_2 \circ \cdots \circ z_n)$$

so that, by Lemma 4.5,

$$[x' \circ z_2 \circ \cdots \circ z_n][x_{r_1} \circ \cdots \circ x_{r_{l-1}} \circ z_1'] = [x][z_1' \circ z_2 \circ \cdots \circ z_n]$$

Similarly,

$$[y' \circ z_2 \circ \cdots \circ z_n][y_{s_1} \circ \cdots \circ y_{s_{t-1}} \circ z_1'] = [y][z_1' \circ z_2 \circ \cdots \circ z_k],$$

so that

$$[x][z_1' \circ z_2 \circ \cdots \circ z_n] = [y][z_1' \circ z_2 \circ \cdots \circ z_n].$$

We repeat the same process with $z_2, \cdots, z_n$ to obtain $[x][z'] = [y][z']$. The same process works if we start with $[x][z] = [z]$. $\square$

Lemma 4.9. *Let $z = z_1 \circ \cdots \circ z_n \in X^+$ be a complete block. Suppose that for each $1 \leq k \leq n$ there exists an idempotent $z_k \in S_{s(z_k)}$ such that $z_k \,\mathcal{R}^*\, z_k^+$ in $S_{s(z_k)}$. Put $z^+ = z_1^+ \circ \cdots \circ z_n^+$. Then $[z^+]$ is an idempotent and $[z] \,\mathcal{R}^*\, [z^+]$ in $\mathscr{GP}$.*

Proof. It is easy to check that $[z^+]$ is an idempotent and the rest follows from Lemma 4.8. $\square$

We now at the position where we can state one of the main results of this section.

Theorem 4.10. *The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ of left abundant semigroups $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ with respect to Γ is left abundant.*

Proof. The result follows from Lemmas 4.7 and 4.9. $\square$

Of course, the left-right dual of Theorem 4.10 holds, and hence we have the following.

Corollary 4.11. *The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ of abundant semigroups $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ with respect to Γ is abundant.*

The rest of this section is devoted to giving a characterisation for $\mathcal{R}^*$ of $\mathscr{GP}$. In view of Lemma 4.7 we just need to find sufficient and necessity conditions for two complete blocks to be $\mathcal{R}^*$-related. Note that if we say $a \in S_\alpha$ is right cancellative, then we mean a is right cancellative in S_α. In any case, the next lemma shows that no ambiguity can arise.

Lemma 4.12. *Let $a \in S_\alpha$. Then a is right cancellative (a does not have a left identity) if and only if $[a]$ is right cancellative ($[a]$ does not have a left identity) in $\mathscr{GP}$.*

Proof. If $[a]$ is right cancellative ($[a]$ does not have identities) in $\mathscr{GP}$ then the fact that a is right cancellative (a does not have identities) follows from Remark 2.4.

Suppose now that a is right cancellative. Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be such that $[x][a] = [y][a]$. Suppose that

$$N_\alpha(x \circ a) = \{l_1, \cdots, l_r\} \text{ and } N_\alpha(y \circ a) = \{k_1, \cdots, k_t\}.$$

Then $l_r = n+1$, $k_t = m+1$,

$$N_\alpha(x) = \{l_1, \cdots, l_{r-1}\} \text{ and } N_\alpha(y) = \{k_1, \cdots, k_{t-1}\}.$$

By Lemma 4.6,

$$[x_{l_1} \circ \cdots \circ x_{l_{r-1}} \circ a] = [x \circ a]\overline{\phi}_\alpha = [y \circ a]\overline{\phi}_\alpha = [y_{k_1} \circ \cdots \circ y_{k_{t-1}} \circ a].$$

By Remark 2.4 we have $x_{l_1} \cdots x_{l_{r-1}} a = y_{k_1} \cdots y_{k_{t-1}} a$ in S_α, so that $x_{l_1} \cdots x_{l_{r-1}} = y_{k_1} \cdots y_{k_{t-1}}$ by the right cancellativity of a. On the other hand,

$$[x'] = [x \circ a]\overline{\psi}_\alpha = [y \circ a]\overline{\psi}_\alpha = [y'],$$

where x' is the word obtained by deleting all x_{l_j}, $1 \leq j \leq r-1$ and y' is the word obtained by deleting all y_{k_p}, $1 \leq p \leq t-1$. It follows again from Lemma 4.6 that

$$[x] = [x'][x_{l_1} \circ \cdots \circ x_{l_{r-1}}] = [y'][y_{k_1} \circ \cdots \circ y_{k_{t-1}}] = [y].$$

Suppose now that a does not have left identities but $[a]$ has in $\mathscr{GP}$. Then there exists some $[z] \in \mathscr{GP}$ such that $[z][a] = [a]$. Without loss of generality, we assume that z is reduced. Clearly, $s(z) \subseteq s(a)$, giving that z is a single letter in $S_{s(a)}$, and hence $[za] = [a]$. Therefore, $za = a$ by Remark 2.4, contradiction. $\square$

Corollaries 4.14 and 4.15 follow immediately from Lemma 4.12.

At this point it helps to define a variation on right cancellativity.

Definition 4.13. An element x of a semigroup S is *i-right cancellative* if it is right cancellative and there is no $u \in S$ such that $ux = x$.

Corollary 4.14. *Let $x = x_1 \circ \cdots \circ x_n \in X^+$ be a complete block. Suppose that $x_{k+1}, \cdots, x_n$ are i-right cancellative elements, for some $0 \leq k \leq n$. Then $[x]\ \mathcal{R}^*\ [x_1 \circ \cdots \circ x_k]$, where if $k = 0$ then we interpret this as saying $[x]$ is i-right cancellative.*

Corollary 4.15. *Let $a \in S_\alpha, b \in S_\beta$ such that $[a]\,\mathcal{R}^*\,[b]$ in $\mathscr{GP}$. Then a is i-right cancellative if and only if b is i-right cancellative.*

Remark 4.16. For a complete block $x_1 \circ \cdots \circ x_n \in X^+$, as $s(x)$ is complete, $[x] = [x_{1\sigma} \circ \cdots \circ x_{n\sigma}]$ for any permutation σ of $\{1, \cdots, n\}$. So, in what follows, without loss of generality we may always assume the i-right cancellative elements succeed the non-i-right cancellative elements in a complete block.

Lemma 4.17. *Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be complete blocks. Suppose that $x_{k+1}, \cdots, x_n, y_{l+1}, \cdots, y_m$ are i-right cancellative, for some $0 \leq k \leq n, 0 \leq l \leq m$, and $x_1, \cdots, x_k, y_1, \cdots, y_l$ are not. Then $[x]\,\mathcal{R}^*\,[y]$ in $\mathscr{GP}$ implies $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$.*

Proof. Suppose that $[x]\ \mathcal{R}^*\ [y]$. Then

$$[x_1 \circ \cdots \circ x_k]\ \mathcal{R}^*\ [y_1 \circ \cdots \circ y_l]$$

by Corollary 4.14. Suppose that $s(x_1 \circ \cdots \circ x_k) \neq s(y_1 \circ \cdots \circ y_l)$. Without loss of generality there exists $1 \leq j \leq k$ such that $s(x_j) = \gamma \notin s(y_1 \circ \cdots \circ y_l)$. By assumption, we have that either x_j is not right cancellative or x_j is right cancellative and has a left identity.

If x_j is not right cancellative, then there must exists $u, v \in S_\gamma$ with $u \neq v$ but $ux_j = vx_j$, giving $[u][x_j] = [v][x_j]$, and so $[u][x] = [v][x]$. Since $[x]\ \mathcal{R}^*\ [y]$, we have $[u][y] = [v][y]$, so that

$$[u][y_1 \circ \cdots \circ y_l] = [v][y_1 \circ \cdots \circ y_l]$$

by Corollary 4.14. As $y_1 \circ \cdots \circ y_l$ is reduced and $s(u) = s(v) \notin s(y_1 \circ \cdots \circ y_l)$, we deduce that $u \circ y_1 \circ \cdots \circ y_l$ and $v \circ y_1 \circ \cdots \circ y_l$ are reduced by Remark 3.6. It follows from Lemma 3.10 that $[u] = [v]$ and so $u = v$ by Lemma 2.4, contradiction.

If x_j is right cancellative and there exists $z \in S_{s(x_j)}$ such that $zx_j = x_j$, then $[z][x_j] = [x_j]$, and so $[z][x] = [x]$. Therefore, $[z][y] = [y]$, implying that $s(z) \subseteq s(y)$. As $s(z) = \gamma \notin s(y_1 \circ \cdots \circ y_l)$, $z \circ y$ reduces to $y_1 \circ \cdots \circ y_{i-1} \circ zy_i \circ y_{i+1} \circ \cdots \circ y_n$ for some $l < i \leq m$. It follows from Remark 3.13(iii) that $zy_i = y_i$, and so y_i has a left identity, contradiction.

Therefore, $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$. □

Remark 4.18. Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be complete blocks such that no letters of x or of y are i-right cancellative. If $[x] \ \mathcal{R}^* \ [y]$, then $s(x) = s(y)$ by Lemma 4.17, and so $n = m$. Since $s(y)$ is complete, $[y] = [y']$ for any y' obtained by permuting the letters of y. Thus, in what follows, without loss of generality we may assume $s(x_i) = s(y_i)$ for all $1 \leq i \leq n$.

Lemma 4.19. *Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be complete blocks such that no letters contained in x and y are i-right cancellative in the corresponding vertex semigroups. Then $[x] \ \mathcal{R}^* \ [y]$ if and only if $s(x) = s(y), n = m$ and $x_i \ \mathcal{R}^* \ y_i$ for all $1 \leq i \leq n$.*

Proof. Suppose that $[x] \ \mathcal{R}^* \ [y]$. Then $s(x) = s(y)$, $n = m$ and $s(x_i) = s(y_i)$ for all $1 \leq i \leq n$, by Lemma 4.17 and Remark 4.18. Let $1 \leq i \leq n$ and $a, b \in S_{s(x_i)}$ be such that $ax_i = bx_i$. Then

$$[a][x] = [x_1 \circ \cdots \circ x_{i-1} \circ ax_i \circ x_{i+1} \circ \cdots \circ x_n] = [x_1 \circ \cdots \circ x_{i-1} \circ bx_i \circ x_{i+1} \circ \cdots \circ x_n] = [b][x]$$

implying $[a][y] = [b][y]$, so that

$$[y_1 \circ \cdots \circ y_{i-1} \circ ay_i \circ y_{i+1} \circ \cdots \circ y_n] = [y_1 \circ \cdots \circ y_{i-1} \circ by_i \circ y_{i+1} \circ \cdots \circ y_n].$$

By Lemma 3.10, $[ay_i] = [by_i]$ and so $ay_i = by_i$ by Lemma 2.4. Since we cannot have $ax_i = x_i$ for any $a \in S_{S(x_i)}$, it follows that $x_i \ \mathcal{R}^* \ y_i$.

The converse is a direct application of Lemma 4.8. □

We can now state the second main result of this section.

Theorem 4.20. *Let $[u], [v] \in \mathscr{GP}$. Let u, v have left Foata normal forms with first blocks $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_m \in X^+$, respectively. Suppose that $x_{k+1}, \cdots, x_n$ and $y_{l+1}, \cdots, y_m$ are i-right cancellative, for some $0 \leq k \leq n, 0 \leq l \leq m$, but $x_1, \cdots, x_k$ and $y_1, \cdots, y_l$ are not. Then $[u] \ \mathcal{R}^* \ [v]$ if and only if $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$, $l = k$ and $x_i \ \mathcal{R}^* \ y_i$ for all $1 \leq i \leq k$.*

Proof. This follows immediately from Lemma 4.7, Corollary 4.14 and Lemma 4.19. □

5. Fountainicity and the relation $\widetilde{\mathcal{R}}$ on $\mathscr{GP}$

In this section we explore the generalised Green's relation $\widetilde{\mathcal{R}}$ on $\mathscr{GP}$. We show that the graph product of left Fountain semigroups is left Fountain.

The following result follows immediately from Lemma 4.7 and the fact that $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$.

Corollary 5.1. *Let $w = w_1 \circ \cdots \circ w_n \in X^+$ be a left Foata normal form with blocks w_i for $1 \leq i \leq n$. Then $[w]\ \widetilde{\mathcal{R}}\ [w_1]$.*

Lemma 5.2. *Let $z = z_1 \circ \cdots \circ z_n \in X^+$ be a complete block. Suppose that $z_k \,\widetilde{\mathcal{R}}\, z'_k$ in $S_{s(z_k)}$ for all $1 \leq k \leq n$ and put $z' = z'_1 \circ \cdots \circ z'_n$. Then $[z] \,\widetilde{\mathcal{R}}\, [z']$ in $\mathscr{GP}$.*

Proof. Let $e = e_1 \circ \cdots \circ e_m$ be a reduced word such that $[e]$ is an idempotent in $\mathscr{GP}$. It follows from Lemma 4.2 that $s(e)$ is complete and $e_i^2 = e_i$ for all $1 \leq i \leq m$. Suppose that $[e][z] = [z]$. Then $s(e) \subseteq s(z)$. Without loss of generality, suppose that $s(e_1) = s(z_1), \cdots, s(e_m) = s(z_m)$. Then

$$[e_1 z_1 \circ \cdots \circ e_m z_m \circ z_{m+1} \circ \cdots \circ z_n] = [z_1 \circ \cdots \circ z_m \circ z_{m+1} \circ \cdots \circ z_n].$$

By Remark 3.13 (iii), for all $1 \leq i \leq m$, $e_i z_i = z_i$, implying that $e_i z'_i = z'_i$. Thus

$$[e][z'] = [e_1 z'_1 \circ \cdots \circ e_m z'_m \circ z'_{m+1} \circ \cdots \circ z'_n] = [z'_1 \circ \cdots \circ z'_m \circ z'_{m+1} \circ \cdots \circ z'_n] = [z'].$$

Therefore, $[z] \,\widetilde{\mathcal{R}}\, [z']$ in $\mathscr{GP}$. □

Lemma 5.3. *Let $z = z_1 \circ \cdots \circ z_n \in X^+$ be a complete block. Suppose that for each $1 \leq k \leq n$ there exists an idempotent $z_k^+ \in S_{s(z_k)}$ such that $z_k \,\widetilde{\mathcal{R}}\, z_k^+$ in $S_{s(z_k)}$. Put $z^+ = z_1^+ \circ \cdots \circ z_n^+$. Then $[z^+]$ is an idempotent and $[z] \,\widetilde{\mathcal{R}}\, [z^+]$ in $\mathscr{GP}$.*

Proof. It is easy to check that $[z^+]$ is an idempotent in $\mathscr{GP}$ and $[z] \,\widetilde{\mathcal{R}}\, [z^+]$ follows from Lemma 5.2. □

Therefore, when all vertex semigroups are left Fountain, we have the following.

Theorem 5.4. *The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ of left Fountain semigroups $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ with respect to Γ is left Fountain.*

Clearly, the left-right dual of Theorem 5.4 holds, resulting in the following.

Corollary 5.5. *The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ of Fountain semigroups $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ with respect to Γ is Fountain.*

It follows from Corollary 5.1 that if $u = u_1 \circ \cdots \circ u_n, v = v_1 \circ \cdots \circ v_m$ are two left Foata normal forms with blocks u_i, v_j where $1 \leq i \leq n, 1 \leq j \leq m$, then $[u]\ \widetilde{\mathcal{R}}\ [v]$ if and only if $[u_1]\ \widetilde{\mathcal{R}}\ [v_1]$. Therefore, to characterise $\widetilde{\mathcal{R}}$ in $\mathscr{GP}$, we just need consider the question of when two complete blocks are $\widetilde{\mathcal{R}}$-related.

If $k = 0$ in our next result, we interpret this result as saying that $[x]$ has no idempotent left identity.

Lemma 5.6. *Let $x = x_1 \circ \cdots \circ x_n \in X^+$ be a complete block. Suppose that $x_1, \cdots, x_k$ have idempotent left identities in the corresponding vertex semigroups but $x_{k+1}, \cdots, x_n$ do not, where $0 \leq k \leq n$. Then $[x]\ \widetilde{\mathcal{R}}\ [x_1 \circ \cdots \circ x_k]$.*

Proof. Let $e = e_1 \circ \cdots \circ e_m$ be a reduced word such that $[e]$ is an idempotent in $\mathscr{GP}$. Suppose that $[e][x] = [x]$. Then

$$[e_1 \circ \cdots \circ e_m][x_1 \circ \cdots \circ x_n] = [x_1 \circ \cdots \circ x_n].$$

Then $s(e) \subseteq s(x)$, and since both e and x are reduced we have

$$[e][x] = [z_1 \circ \cdots \circ z_n]$$

where $z_i = x_i$ for $i \in I$ and $z_j = e_{i_j} x_j$ for $j \in J$, with $I \cap J = \emptyset$, $I \cup J = \{1, \cdots, n\}$ and $i \mapsto i_j$ a bijection $\{1, \cdots, m\} \to J$. From Remark 3.13(iii) we have that $e_{i_j} x_j = x_j$ for $j \in J$, so that $J \subseteq \{1, \cdots, k\}$ and so $[e][x_1 \circ \cdots \cdots \circ x_k] = [x_1 \circ \cdots \cdots \circ x_k]$. The result follows. $\square$

Lemma 5.7. *Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be complete blocks. Suppose that $x_1, \cdots, x_k, y_1, \cdots, y_l$ have idempotent left identities in the corresponding vertex semigroups but $x_{k+1}, \cdots, x_n, y_{l+1}, \cdots, y_m$ do not, for some $0 \leq k \leq n, 0 \leq l \leq m$. If $[x] \; \widetilde{\mathcal{R}} \; [y]$ in $\mathscr{GP}$ then $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$ and so $k = l$.*

Proof. By Lemma 5.6,

$$[x_1 \circ \cdots \circ x_k] \; \widetilde{\mathcal{R}} \; [y_1 \circ \cdots \circ y_l].$$

Assume that $s(x_1 \circ \cdots \circ x_k) \neq s(y_1 \circ \cdots \circ y_l)$. If $k = l = 0$ we are done. Otherwise, without loss of generality, let $\gamma = s(x_j) \in s(x_1 \circ \cdots \circ x_k)$. Since x_j has an idempotent left identity, there must exist an idempotent $u \in S_\gamma$ such that $u x_j = x_j$, so that

$$[u][x_1 \circ \cdots \circ x_k] = [x_1 \circ \cdots \circ x_{j-1} \circ u x_j \circ x_{j+1} \circ \cdots \circ x_k] = [x_1 \circ \cdots \circ x_k].$$

Since $[x_1 \circ \cdots \circ x_k] \; \widetilde{\mathcal{R}} \; [y_1 \circ \cdots \circ y_l]$, we have $[u][y_1 \circ \cdots \circ y_l] = [y_1 \circ \cdots \circ y_l]$ and so $\gamma = s(u) \in s(y_1 \circ \cdots \circ y_l)$. The result follows. $\square$

In what follows, when we have two complete blocks $x_1 \circ \cdots \circ x_n, y_1 \circ \cdots \circ y_m \in X^+$ with $s(x) = s(y)$(and so $n = m$), without loss of generality, we always assume $s(x_i) = s(y_i)$ for all $1 \leq i \leq n$.

Lemma 5.8. *Let $x = x_1 \circ \cdots \circ x_n, y = y_1 \circ \cdots \circ y_m \in X^+$ be complete blocks such that all letters contained in x and y have idempotent left identities in the corresponding vertex semigroups. Then $[x] \; \widetilde{\mathcal{R}} \; [y]$ if and only if $s(x) = s(y), n = m$ and $x_i \; \widetilde{\mathcal{R}} \; y_i$ for all $1 \leq i \leq n$.*

Proof. Suppose that $[x] \; \widetilde{\mathcal{R}} \; [y]$. Then $s(x) = s(y)$ and $n = m$ by Lemma 5.7. Let $1 \leq i \leq n$ and $u \in S_{s(x_i)}$ be an idempotent in $S_{s(x_i)}$ such that $u x_i = x_i$. Then

$$[u][x] = [x_1 \circ \cdots \circ x_{i-1} \circ u x_i \circ x_{i+1} \circ \cdots \circ x_n] = [x_1 \circ \cdots \circ x_{i-1} \circ x_i \circ x_{i+1} \circ \cdots \circ x_n] = [x]$$

implying $[u][y] = [y]$, so that

$$[y_1 \circ \cdots \circ y_{i-1} \circ u y_i \circ y_{i+1} \circ \cdots \circ y_n] = [y_1 \circ \cdots \circ y_{i-1} \circ y_i \circ y_{i+1} \circ \cdots \circ y_n].$$

By Remark 3.13 (iii), $u y_i = y_i$. Together with the dual arguments, we have $x_i \; \widetilde{\mathcal{R}} \; y_i$.

The converse is a direct application of Lemma 5.2. $\square$

We now come to our characterisation of $\widetilde{\mathcal{R}}$ on $\mathscr{GP}$.

Theorem 5.9. *Let $[u], [v] \in \mathscr{GP}$ have left Foata normal forms with left-most blocks $x = x_1 \circ \cdots \circ x_n$ and $y = y_1 \circ \cdots \circ y_m \in X^+$, respectively. Suppose that $x_1, \cdots, x_k$ and $y_1, \cdots, y_l$ are the elements of x, y, respectively that have idempotent left identities, in the* *corresponding vertex semigroups, where* $0 \le k \le n$ *and* $0 \le l \le m$*. Then* $[u] \;\widetilde{\mathcal{R}}\; [v]$ *if and only if* $s(x_1 \circ \cdots \circ x_k) = s(y_1 \circ \cdots \circ y_l)$*,* $k = l$ *and* $x_i \;\widetilde{\mathcal{R}}\; y_i$ *for all* $1 \le i \le k$*.*

Proof. This follows immediately from Corollary 5.1, Lemma 5.6 and Lemma 5.8. □

The relation $\mathcal{R}^*$ is always a left congruence on any semigroup S, however, the relation $\widetilde{\mathcal{R}}$ is not. We might expect that if $\widetilde{\mathcal{R}}$ is a left congruence on every vertex semigroup of $\mathscr{GP}$ then it would be a left congruence on $\mathscr{GP}$. However, this is not true in general.

Example 5.10. Suppose that $u \in S_\alpha, v \in S_\beta$, where $\alpha \neq \beta$, such that neither have idempotent left identities. Then, by Lemma 5.9, $[u] \;\widetilde{\mathcal{R}}\; [v]$ in $\mathscr{GP}$. Choose $a \in S_\alpha, e^2 = e \in S_\alpha$ with $eau = au$ but $ea \neq a$. Such a configuration exists; for example, we could take S_α to be a null semigroup, $e = 0$ and $a, u \neq 0$. Then $[e][a][u] = [a][u]$. However, $[e][a][v] \neq [a][v]$, as otherwise, we would have $[ea] = [a]$ by Lemma 3.10, giving $ea = a$, a contradiction.

In fact, the behaviour explicated in Example 5.10 is the only block to $\widetilde{\mathcal{R}}$ being a left congruence. Recall in what follows that we assume $|V| \ge 2$.

Theorem 5.11. *The relation* $\widetilde{\mathcal{R}}$ *is a left congruence on* $\mathscr{GP}$ *if and only if:*

(1) *for each* $\alpha \in V$ *the relation* $\widetilde{\mathcal{R}}$ *is a left congruence on* S_α*;*
(2) *if* $\alpha, \beta \in V$ *with* $\alpha \neq \beta$ *such that there exists* $u \in S_\alpha, v \in S_\beta$ *such that* u, v *have no idempotent left identities, then if* $e = e^2, a \in S_\alpha$*, if* $eau = au$ *then* $ea = a$*.*

Proof. The necessity is clear from Example 5.10, and the easily verifiable fact that for any $\alpha \in V$ and $a, b \in S_\alpha$ we have $a \,\widetilde{\mathcal{R}}\, b$ in S_α if and only if $[a] \,\widetilde{\mathcal{R}}\, [b]$ in $\mathscr{GP}$.

Suppose now that (1) and (2) hold. Let $[w], [u], [v] \in \mathscr{GP}$ and suppose that $[u] \,\widetilde{\mathcal{R}}\, [v]$. Let $[u_1], [v_1]$ be the first blocks of $[u], [v]$ in left Foata normal form, with $u_1 = p_1 \circ \cdots \circ p_h \in X^+$ and $v_1 = q_1 \circ \cdots \circ q_k \in X^+$. Let $w = w_1 \circ \cdots \circ w_n \in X^+$. We have $[u] \,\mathcal{R}^*\, [u_1]$ and $[v] \,\mathcal{R}^*\, [v_1]$ so $[u_1] \,\widetilde{\mathcal{R}}\, [v_1]$. Suppose that $[e]$ is idempotent and $[e][w][u] = [w][u]$. Since $[u] \,\mathcal{R}^*\, [u_1]$ we immediately have $[e][w][u_1] = [w][u_1]$. Recalling that e is a complete block, write $e = e_1 \circ \cdots \circ e_m \in X^+$, where $e_i^2 = e_i$ for $1 \le i \le m$.

Without loss of generality we may assume that we can reduce $w \circ u_1$ to

$$x = v_1 \circ \cdots \circ v_n \circ p_{h'} \circ \cdots \circ p_h = x_1 \circ \cdots \circ x_t$$

for some $1 \le h' \le h$, where for $1 \le i \le n$ we have $v_i = w_i$ or $v_i = w_i p_{i_j}$, and

$$\{p_{i_j} : 1 \le i \le n\} = \{p_1, \cdots, p_{h'-1}\}.$$

Further, $t = n + h - h' + 1$. Notice that since w is reduced, each letter of u_1 glues to at most one letter of w.

Since both e and x are reduced, and the length of the reduced form of $e \circ x$ equals that of x, it follows that we can reduce $e \circ x$ to

$$y = x'_1 \circ \cdots \circ x'_t$$

where x'_i is x_i or $e_{k_i} x_i$ such that $\{e_{k_i} : 1 \le i \le t\} = \{e_1, \cdots, e_m\}$. Notice again that each letter of e glues to exactly one letter of x.

Recall that we have $[y] = [x]$. Let $s(x_i) = s(x_i') = \alpha$. If i is the unique index such that $s(x_i) = \alpha$, then from Lemma 4.1 we immediately have $x_i = x_i'$. If $s(x_j) = \alpha$ for some $j \neq i$, then we must have $i < j$, and as x is reduced, there is a j' with $i < j' < j$ such that $s(x_j) = \beta$ and $(\alpha, \beta) \notin E$. The dual of Lemma 4.3 now gives that $x_i = x_i'$.

Consider i with $1 \leq i \leq t$ and suppose that $x_i' = e_{k_i} x_i$ and $s(x_i') = s(x_i) = \alpha$. We have $x_i = w_i$, or $x_i = w_i p_{i_j}$, or $x_i = p_l$ for some $h' \leq l \leq h$. If $x_i = w_i p_{i_j}$ then we have $e_{k_i} w_i p_{i_j} = w_i p_{i_j}$. It follows by (2) that either p_{i_j} has an idempotent left identity, $e_{k_i} w_i = w_i$ or there are no elements of any other $S_\beta, \beta \neq \alpha$ having idempotent left identities.

Using Theorem 5.9 we consider the above cases. We first deal with the essentially degenerate case where there is an i with $x_i = w p_{i_j}$ where p_{i_j} has no idempotent left identity and $e_{k_i} w_i \neq w_i$. Using (2) it follows that for all $\beta \neq \alpha$ there are no elements of S_β having idempotent left identities. In particular, this applies to all the letters of u_1. Consequently, for $1 \leq j \leq t$ if $x_j' = e_{k_j} x_j = x_j$ we have $x_j = w_j$, and so $[e][w] = [w]$, giving $[e][w][v] = [v]$. We can therefore now assume that the above situation does not arise, for any $1 \leq i \leq t$.

If p_{i_j} does not have a idempotent left identity, and $e_{k_i} w_i = w_i$, then v_1 contains no q_l with $s(q_l) = \alpha$ such that q_l has an idempotent left identity.

If p_{i_j} does have an idempotent left identity then it follows without loss of generality that there exists q_{i_j} such that $s(p_{i_j}) = s(q_{i_j}) = \alpha$ and $p_{i_j} \, \widetilde{\mathcal{R}} \, q_{i_j}$ in S_α; since $\widetilde{\mathcal{R}}$ is a left congruence in S_α we deduce that $e_{k_i} w_i q_{i_j} = w_i q_{i_j}$.

The other case to consider is where $x_i' = e_{k_i} x_i$ and $x_i = p_l$. Again without loss of generality we have $s(p_l) = s(q_l) = \alpha$ and $p_l \, \widetilde{\mathcal{R}} \, q_l$ in S_α, so that $e_{k_i} q_l = q_l$.

Again without loss of generality we may assume that $q_1, \cdots, q_{k'}$ are the elements of v_1 that occur as some q_{i_j} or q_l above. It follows that $[e][w][v_1'] = [w][v_1']$, where $v_1' = q_1 \circ \cdots \circ q_{k'}$ and then finally $[e][w][v_1] = [v_1]$. The result follows. □

The semigroups appearing in our final result are sometimes called weakly left abundant with (CL).

Corollary 5.12. *The graph product $\mathscr{GP} = \mathscr{GP}(\Gamma, \mathcal{S})$ with respect to Γ of left Fountain semigroups $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ where for each $\alpha \in V$ we have $\widetilde{\mathcal{R}}$ is a left congruence on S_α, is left Fountain and has $\widetilde{\mathcal{R}}$ as a left congruence.*

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