INDEPENDENCE ALGEBRAS

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1 Introduction

The motivation for this paper arises from an observation concerning the similarity in the ideal structure of End $\mathbf{V}$ and $\mathcal{T}(X)$, where End $\mathbf{V}$ is the endomorphism monoid of a vector space over a division ring and $\mathcal{T}(X)$ is the full transformation monoid on a non-empty set X. In both cases there is a natural definition of *rank* : for $\alpha \in$End $\mathbf{V}$, *rank* α is dim(Im α) and for $\alpha \in \mathcal{T}(X)$, *rank* α is $|\text{Im }\alpha|$. Denoting by $T_n (n \in \mathbf{N})$ those α with rank no greater than n, T_n is an ideal (we allow $0 \in \mathbf{N}$ and $\emptyset$ to be an ideal). Moreover for all $n \in \mathbf{N}$ with $n \geq 1$, the Rees quotient semigroup T_n/T_{n-1} is completely 0-simple.

What then do vector spaces and sets have in common which forces End $\mathbf{V}$ and $\mathcal{T}(X)$ to support a similar pleasing structure? Obviously the notion of rank is a key. Regarding vector spaces and sets as examples of *universal algebras*, in either case any subuniverse B has a well defined rank, namely, the cardinality of any minimal set of generators of B. The rank of $\alpha \in$ End $\mathbf{V}$ or $\alpha \in \mathcal{T}(X)$ is simply the rank of Im α.

In order to see what properties vector spaces and sets share (considered as universal algebras), we investigate $\text{Sg}^{\mathbf{A}}$, the *closure operator* on any universal algebra $\mathbf{A}$ which takes a subset to the subuniverse of A which it generates. In the case of vector spaces and sets, $\text{Sg}^{\mathbf{A}}$ satisfies the *exchange property* (EP) : using the techniques of matroid theory this enables one to give a well defined notion of rank of a subuniverse B, where the rank of B, written $\rho(B)$, is again the cardinality of a minimal set of generators of B. Further, for vector spaces and sets, subsets which are *independent* with regard to the closure operator $\text{Sg}^{\mathbf{A}}$ have the *free subbasis property*, that is, for any independent subset X of A, if $\alpha : X \to A$ is a function, then α can be extended to a homomorphism from $\text{Sg}^{\mathbf{A}}(X)$ to A. If $\mathbf{A}$ is a universal algebra such that $\text{Sg}^{\mathbf{A}}$ has (EP), then a minimal set of generators of A is called a *basis*; bases of A are precisely the maximal independent subsets of A. We therefore define an *independence algebra* to be a universal algebra $\mathbf{A}$ such that $\text{Sg}^{\mathbf{A}}$ has (EP) and for any basis X of A and function $\alpha : X \to A$, α can be extended to an element of End $\mathbf{A}$. This latter condition is called the *free basis property* and is denoted by (F). It is an easy consequence of the properties of independence algebras that independent subsets of such algebras have the free subbasis property.

In Section 2 we enlarge upon the above discussion of closure operators, the exchange property and the free basis property. Weak independence algebras are also defined and we give examples of such algebras, and show which of these are actually independence algebras. Section 3 contains a summary of the required properties of closure operators with (EP). This includes the behaviour of the related rank function ρ. We then introduce the concept of a *preimage basis* for $\alpha \in$ End $\mathbf{A}$, where $\mathbf{A}$ is a weak independence algebra, and develop this idea in the case where $\mathbf{A}$ is an independence algebra.

The main results of this paper are given in Section 4. Defining the *rank* of $\alpha \in$ End $\mathbf{A}$ where $\mathbf{A}$ is a weak independence algebra, to be $\rho(\text{Im } \alpha)$, we show that if $T_n^{\mathbf{A}} = T_n (n \in \mathbf{N})$ is defined as for vector spaces and sets, then $T_n^{\mathbf{A}}$ is an ideal. Further, if $\mathbf{A}$ is an independence algebra then $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ is completely 0-simple, and we give a careful account of the corresponding Rees matrix semigroups. These results are applied in the final section to various examples of independence algebras, yielding a number of corollaries.

Fountain and Lewin [4] have recently investigated the submonoid of the endomorphism monoid of an independence algebra generated by idempotents. The results of [4] are analogous to those proven for sets by Howie [7] and for vector spaces by Erdös [3] and Reynolds and Sullivan [13].

We assume the reader to have a basic knowledge of both universal algebra and algebraic semigroup theory. We recommend [8] and [10] as references. Finally in this introduction we remark that it can be shown that independence algebras are precisely the v^*-algebras introduced by W. Narkiewicz in [5].

2 Independence algebras

As far as possible we follow standard notation and terminology for universal algebra and semigroup theory, as may be found in [1] and [8]. In particular, we use bold face letters to denote algebras, and the corresponding standard face letters to denote the underlying universes. If $\mathbf{A}$ is an algebra and $a_1, ..., a_n \in A$, then a term built from these elements may be written as $t(a_1, ..., a_n)$. Endomorphisms of $\mathbf{A}$ are written on the *right* of their arguments. We differ from convention in that we allow $\emptyset$ to be an ideal of a monoid.

The first step in defining an independence algebra is to consider the notion of a *closure operator*. In this paper we deal with only one specific example of a closure operator, but many of the results we quote are valid for arbitrary closure operators and we therefore state them for such.

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a function, where $\mathcal{P}(A)$ is the set of all subsets of A. Then C is a *closure operator* on A if C satisfies the following conditions for all $X, Y \in \mathcal{P}(A)$:

(i) $X \subseteq C(X)$;
(ii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$;
(iii) C(X)=C(C(X)).

Condition (i) says that C is *extensive*, condition (ii) that C is *order preserving* and condition (ii) that C is *idempotent.*

If $\mathbf{A}$ is any algebra, then $\mathrm{Sg}^{\mathbf{A}}$ is a closure operator on the set A, where for all $X \subseteq A$, $\mathrm{Sg}^{\mathbf{A}}(X)$ is the subuniverse of A generated by X. Note that every subuniverse is the universe of a subalgebra, except in the case where $\mathbf{A}$ has no constants, when the empty set is a subuniverse but is not usually regarded as being the universe of a subalgebra. Of course, $\mathrm{Sg}^{\mathbf{A}}(\emptyset) = \emptyset$ if and only if $\mathbf{A}$ has no constants. If $\mathbf{A}$ has constants, then for any subset X of the constants of $\mathbf{A}$ (where $X = \emptyset$ is allowed), $\mathrm{Sg}^{\mathbf{A}}(X)$ is the subuniverse of A generated by the constants.

It is well known that for any subset X of A, where A is the universe of an algebra $\mathbf{A}$, $\mathrm{Sg}^{\mathbf{A}}(X)$ is the set of terms that can be built from the elements of X. In view of this it is easy to see that $\mathrm{Sg}^{\mathbf{A}}$ is always an *algebraic closure operator*, where a closure operator C on a set B is *algebraic* if for all $X \subseteq B$

$$C(X) = \bigcup\{C(Y) : Y \subseteq X, |Y| < \aleph_0\}.$$

The *exchange property* (EP) for a closure operator C on a set A is defined as follows.

(EP) For all $X \subseteq A$ and $x, y \in A$, if $x \notin C(X)$ but $x \in C(X \cup \{y\})$, then $y \in C(X \cup \{x\})$. If A is a finite set then (EP) says that C is the closure operator associated with a matroid . The concept of an infinite matroid has been well developed and encompasses the theory of algebraic closure operators with (EP). A detailed account of many relevant properties is given in [9], from which we quote extensively in Section 3.

Let $\mathbf{V}$ be a (left) vector space over a division ring $\mathbf{D}$. A straightforward argument gives that $\mathrm{Sg}^{\mathbf{V}}$ has (EP). Now let $\mathbf{A}$ be any algebra such that for every subset B of A, $B = \mathrm{Sg}^{\mathbf{A}}(B)$, that is, every subset is a subuniverse. It is then clear that $\mathrm{Sg}^{\mathbf{A}}$ has (EP). In particular, if $\mathbf{A}$ is a set, that is, an algebra with no operations, then $\mathrm{Sg}^{\mathbf{A}}$ has (EP). Chains (regarded as semigroups) provide further examples of algebras with the property that every subset is a subuniverse. If $\mathbf{G}$ is a group and $\mathbf{S}$ is a (left) $\mathbf{G}$-set, then the orbits of elements of S partition S, from which it is easy to see that $\mathbf{S}$ has (EP).

If an algebra $\mathbf{A}$ satisfies (EP) we say that $\mathbf{A}$ is a *weak independence algebra.* Thus vector spaces, sets, chains and $\mathbf{G}$-sets are weak independence algebras.

The next ingredient of our approach to the definition of an independence algebra is that of an independent subset.

Let C be a closure operator on a set A, and let $X \subseteq A$. Then X is *C-independent* if for all $x \in X$, $x \notin C(X \setminus \{x\})$. If $\mathbf{A}$ is an algebra then we refer to $\mathrm{Sg}^{\mathbf{A}}$-independent sets more simply as *independent* sets. The independent subsets of a vector space are the linearly independent subsets, in the usual sense of linear algebra. If $\mathbf{A}$ is any algebra such that every subset of A is a subuniverse, then clearly every subset of A is independent. In the case of a $\mathbf{G}$-set $\mathbf{S}$, a subset X of S is independent if and only if X contains at most one element from each orbit, that is, $|X \cap Gx| \leq 1$ for each $x \in S$, where Gx is the orbit of x.

Let $\mathbf{A}$ be a weak independence algebra. Recall from the introduction that a *basis* of A is a minimal generating set. The algebra $\mathbf{A}$ is then an *independence algebra* if the *free basis* property (F) holds :

(F) For any basis X of A and function $\alpha : X \to A$, α can be extended to an element of End $\mathbf{A}$.

We show in the next section that if **A** is a weak independence algebra then any basis of A is independent. Moreover, if **A** is an independence algebra and X is an independent subset of A then any function α from X to A can be extended to a homomorphism from $\mathrm{Sg}^{\mathbf{A}}(X)$ to A. Such an extension is unique and may also be denoted by α.

It is well known that any vector space has (F). If **A** is a set, then regarded as an algebra, A is a basis and an endomorphism of **A** is just a function; hence **A** has (F). Thus both vector spaces and sets are examples of independence algebras.

For an example of a weak independence algebra which is *not* an independence algebra, consider a chain **C** where $|C| \geq 2$. Let $x, y \in C$ where $x < y$, that is, $x \neq y$ and $x = xy = yx$. Define $\alpha : \{x, y\} \to C$ by $x\alpha = y$ and $y\alpha = x$. If α could be extended to a homomorphism β from $\{x, y\}$ $=\mathrm{Sg}^{\mathbf{C}}\{x, y\}$ to C, then β must actually be α. But $(xy)\alpha = x\alpha = y$ and $x\alpha y\alpha = yx = x$. In view of the comments following the definition of (F), **C** cannot be an independence algebra.

If **G** is a group and X is a non-empty set, then $F_{\mathbf{G}}(X)$ is the **G**-set defined by

$$F_{\mathbf{G}}(X) = \dot{\bigcup}\{Gx : x \in X\}$$

where $\dot{\bigcup}$ denotes disjoint union and $gx = hy$ for $g, h \in G$ and $x, y \in X$ if and only if $x = y$ and $g = h$. The action of **G** on $F_{\mathbf{G}}(X)$ is the obvious one; $F_{\mathbf{G}}(X)$ is of course the free **G**-set on the set X. If **S** is a **G**-set then **S** is isomorphic to some $F_{\mathbf{G}}(X)$ if and only if the trivial subgroup of **G** is the stabiliser of each $s \in S$.

Consider the case of a **G**-set **S** where there exists $g \in G$ and $s, t \in S$ such that $gs = s$ but $gt \neq t$. It is easy to see that singleton subsets of S are independent. Thus if **S** is an independence algebra, then $\alpha : \{s\} \to S$ defined by $s\alpha = t$ must be extendable to a homomorphism, which we denote also by α, from Gs to S. But then $gt = g(s\alpha) = (gs)\alpha = s\alpha = t$, a contradiction.

However, free **G**-sets provide us with new examples of independence algebras. To see this, suppose that **S** is a **G**-set with the property that all stabilisers of elements of S coincide. Let **H** be the common stabiliser of the elements of S. In the case where **S** is isomorphic to some $F_{\mathbf{G}}(X)$, all stabilisers are trivial; so that in this case **H** exists and is the trivial subgroup of **G**. Suppose that U is a basis of S. It is easy to see that U contains precisely one element from each orbit. Let $\alpha : U \to S$ be a function. Extend the domain of α to S by defining $(gu)\alpha = g(u\alpha)$ for all $g \in G$ and $u \in U$. To see that α is a function, note that if $gu = hv$ where $g, h \in G$ and $u, v \in U$, then since u and v are in the same orbit, $u = v$ so that $h^{-1}g \in H$. Then $(h^{-1}g)(u\alpha) = v\alpha$, giving that $g(u\alpha) = h(v\alpha)$ as required. An easy argument yields that $\alpha \in \mathrm{End}\ \mathbf{S}$. Thus **S** is an independence algebra.

3 Bases and preimage bases

We first give a summary of some properties of C-independent sets and the related notion of a C-basis, where C is an algebraic closure operator with (EP) on a set A. In particular, we consider the *C-rank* of subsets of A. The results we quote are standard; a detailed account is given in [9]. Our study of the endomorphism monoid of a weak independence algebra **A** then begins when we define the *rank* of an endomorphism to be the $\mathrm{Sg}^{\mathbf{A}}$-rank of its image. The notion of preimage basis is introduced and investigated for (weak) independence algebras.

For the remainder of this section, let C be an algebraic closure operator on a set A. Recall that a subset X of A is *C-independent* if for all $x \in X$, $x \notin C(X \setminus \{x\})$; if $Y \subseteq A$ and Y is not C-independent, then Y is said to be *C-dependent*. Note that the empty set is always C-independent. Since C is algebraic, it is easy to see that $X \subseteq A$ is C-independent if and only if every finite subset of X is C-independent; clearly this is also equivalent to every subset of X being C-independent.

A subset B of A is *C-closed* if $C(B) = B$; equivalently, B is C-closed if $B = C(X)$ for some $X \subseteq A$. Given any C-closed subset B and subset X of A, we say that X *C-spans* B if $C(X) = B$. Of course, if X C-spans B then $X \subseteq B$. Our interest lies in C-independent sets which C-span closed sets; a standard argument using Zorn's Lemma shows that if $X_0 \subseteq X \subseteq A$, where X_0 is C-independent, then there is a maximal C-independent subset Y of X containing X_0; clearly Y is a maximal C-independent subset of X. Taking $X_0 = \emptyset$, this shows that given any subset X of A, there is a C-independent set maximal with respect to being contained in X. If in addition C satisfies (EP) then the notions of C-spanning and maximal C-independent are closely connected, as the next result makes clear.

Theorem 3.1 (9) *Let C be an algebraic closure operator on a set A. Then the following conditions are equivalent:*

(i) C has the exchange property;

(ii) for every subset X of A, if Y is a maximal C-independent subset of X, then $C(X) = C(Y)$;

(iii) for every Y and X with $Y \subseteq X \subseteq A$, if Y is C-independent, then there is a C-independent set Z with $Y \subseteq Z \subseteq X$ and $C(X) = C(Z)$.

The next corollary is implicit in [9].

Corollary 3.2 *Let C be an algebraic closure operator with (EP) on a set A, and let $Y \subseteq X \subseteq A$. Then the following conditions are equivalent:*

(i) Y is a maximal C-independent subset of X;

(ii) Y is C-independent and $C(Y) = C(X)$;

(iii) Y is minimal with respect to $C(Y) = C(X)$.

Proof. *(i)* $\Rightarrow$ *(ii)* This is immediate from Theorem 3.1.

(ii) $\Rightarrow$ *(iii)* Suppose that Y is C-independent and $C(Y) = C(X)$. Let Z be a set strictly contained in Y. Then if $y \in Y \setminus Z$, $y \notin C(Y \setminus \{y\}) \supseteq C(Z)$. Thus $C(Z)$ is strictly contained in $C(Y) = C(X)$.

(iii) $\Rightarrow$ *(i)* To see that Y is C-independent, let $y \in Y$. If $y \in C(Y \setminus \{y\})$, then $Y \subseteq C(Y \setminus \{y\})$, from which $C(X) = C(Y) \subseteq C(Y \setminus \{y\}) \subseteq C(X)$ follows. But this contradicts the minimality of Y with repsect to $C(Y) = C(X)$. Thus Y is C-independent. If $Y \subset Z \subseteq X$ and $z \in Z \setminus Y$, then $z \in C(X) = C(Y) \subseteq C(Z \setminus \{z\})$. Thus Y is a maximal C-independent subset of X.

For the remainder of this section, let C be an algebraic closure operator with (EP) on a set A. Let X be a subset of A. We define a *C-basis* of X to be a subset Y of X which is minimal with respect to $C(Y) = C(X)$. Thus a C-basis of X is precisely a maximal C-independent subset of X. In particular, if B is a C-closed subset of A, a subset Y of B is a C-basis for B if and only if one of the following three (equivalent) conditions hold: Y is a maximal C-independent subset of B; Y is C-independent and $C(Y) = B$; Y is minimal with respect to $C(Y) = B$. In the case where $C = \mathrm{Sg}^{\mathbf{A}}$ for a weak independence algebra $\mathbf{A}$, the above definitions are consistent with the previous notion of basis of A.

Let X be a C-independent subset of a C-closed subset B of A. Then $X \subseteq Y \subseteq B$ for some C-basis Y of B; hence $X \cup (Y \setminus X)$ is a C-basis of B and $X \cap (Y \setminus X) = \emptyset$. If Z is any subset of B such that $X \cap Z = \emptyset$ and $X \cup Z$ is a C-basis for B, then we say that Z *extends* X to a C-basis of B.

We now introduce the central concept of C-rank. Let X be a subset of A. Then the *C-rank* of X, written $\rho_C(X)$, is $|Y|$, where Y is a C-basis for X. In view of Corollary 2.3.7 of [9], ρ_C is well-defined.

Proposition 3.3 *[9] Let C be an algebraic closure operator with (EP) on a set A. Then ρ_C is a function. Moreover, for any $X \subseteq A$,*

$$\rho_C(X) = \rho_C(C(X)).$$

The next result is again implicit in [9].

Corollary 3.4 *Let C be an algebraic closure operator with (EP) on a set A, and let $B \subseteq A$ be a C-closed subset with finite C-rank. Then if X is an independent subset of B, $|X| \leq \rho_C(B)$ and $|X| = \rho_C(B)$ if and only if $C(X) = B$.*

Proof. Since any C-independent subset of B can be extended to a C-basis of B, we know that $|X| \leq \rho_C(B)$. Moreover, since $\rho_C(B)$ is finite, $|X| = \rho_C(B)$ if and only if X is a maximal C-independent subset of B. By Corollary 3.2, this is equivalent to $C(X) = B$.

We now concentrate on the case where $\mathbf{A}$ is an algebra and $C = \mathrm{Sg}^{\mathbf{A}}$. Since we are assuming that C has (EP), this is saying that $\mathbf{A}$ is a weak independence algebra. We simplify our terminology by dropping $\mathrm{Sg}^{\mathbf{A}}$ from the notation $\mathrm{Sg}^{\mathbf{A}}$-independent, $\mathrm{Sg}^{\mathbf{A}}$-dependent, $\mathrm{Sg}^{\mathbf{A}}$-closed, $\mathrm{Sg}^{\mathbf{A}}$-spans, $\mathrm{Sg}^{\mathbf{A}}$-rank, $\rho_{\mathrm{Sg}^{\mathbf{A}}}$ and $\mathrm{Sg}^{\mathbf{A}}$-basis. In particular, a closed subset of A is a subuniverse.

Let $\alpha \in \mathrm{End}\ \mathbf{A}$. We define the *rank* of α, written $\rho(\alpha)$, to be $\rho(\mathrm{Im}\ \alpha)$. If $\alpha, \beta \in \mathrm{End}\ \mathbf{A}$ and $\mathrm{Im}\ \alpha = \mathrm{Im}\ \beta$, then clearly $\rho(\alpha) = \rho(\beta)$. On the other hand, if $\mathrm{Ker}\ \alpha = \mathrm{Ker}\ \beta$ then $\mathrm{Im}\ \alpha \cong A/\mathrm{Ker}\ \alpha = A/\mathrm{Ker}\ \beta \cong \mathrm{Im}\ \beta$; it is then clear that $\rho(\alpha) = \rho(\beta)$. These comments become particularly significant when we consider Green's relations on End $\mathbf{A}$ in the next section.

Of course, $\mathrm{Im}\ \alpha$ is a subalgebra of $\mathbf{A}$ for any $\alpha \in \mathrm{End}\ \mathbf{A}$. Choosing a basis Y of $\mathrm{Im}\ \alpha$, we know by Proposition 3.3 that $\rho(\alpha) = \rho(\mathrm{Im}\ \alpha) = \rho(\mathrm{Sg}^{\mathbf{A}}(Y)) = \rho(Y)$. For each $y \in Y$ pick an element $x_y \in A$ with $x_y\alpha = y$, and put $X = \{x_y : y \in Y\}$. Then $\alpha : X \to Y = X\alpha$ is a one-one function. Consequent upon the next result, X is moreover independent.

Lemma 3.5 *Let $\mathbf{A}$ be a weak independence algebra and let $\alpha \in$ End $\mathbf{A}$. Suppose that $X \subseteq A$, $\alpha : X \to X\alpha$ is one-one and $X\alpha$ is independent. Then X is independent.*

Proof. If X were dependent, then there would exist distinct elements $x, x_1, ..., x_n \in X$ with $x = t(x_1, ..., x_n)$. Since α is an endomorphism, $x\alpha = t(x_1, ..., x_n)\alpha = t(x_1\alpha, ..., x_n\alpha)$. But α is one-one on X, so that the independence of $X\alpha$ would be contradicted.

The above discussion leads us to the definition of a preimage basis. Let $\alpha \in \text{End } \mathbf{A}$, where $\mathbf{A}$ is a weak independence algebra. A subset X of A is a *preimage basis* for α if α is one-one on X and $X\alpha$ is a basis for Im α. From Lemma 3.5, X must also be independent. Further, $|X| = |X\alpha| = \rho(X\alpha) = \rho(\text{Sg}^{\mathbf{A}}(X\alpha)) = \rho(\text{Im } \alpha) = \rho(\alpha)$.

Recall that a weak independence algebra $\mathbf{A}$ is an *independence algebra* if for each basis X of A and $\alpha : X \to A$, α can be extended to an endomorphism of $\mathbf{A}$. Since X generates A it is clear that the extension of α to an endomorphism is unique.

Proposition 3.6 *Let* $\mathbf{A}$ *be an independence algebra.*

(I) If $X \subseteq A$ *is independent and* $\alpha : X \to A$*, then* α *can be extended to a homomorphism* $\overline{\alpha} : Sg^{\mathbf{A}}(X) \to A$*.*

(II) If $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$ *and* $\beta : B \to A$ *is a homomorphism, then* β *can be extended to* $\gamma \in End\ \mathbf{A}$*, where Im* $\gamma =$ *Im* β*.*

Proof. (I) Extend X to a basis $X \cup Y$ of A and define $\gamma : X \cup Y \to A$ by

$$\begin{aligned} x\gamma &= x\alpha \quad x \in X \\ y\gamma &= x_0\alpha \quad y \in Y \end{aligned}$$

where x_0 is a fixed element of X. Then γ can be extended to an endomorphism of $\mathbf{A}$, which we also denote by γ. Now put $\overline{\alpha} = \gamma|_{\text{Sg}^{\mathbf{A}}(X)}$: clearly $\overline{\alpha}$ is a homomorphism extending α.

(II) Let X be a basis of B, so that $B = \text{Sg}^{\mathbf{A}}(X)$, and extend X to a basis $X \cup Y$ of A. Let $\alpha = \beta|_X$, and let γ be defined as in the proof of (I). Clearly $\beta = \overline{\alpha} = \gamma|_B$ and

Im $\beta = B\beta = \text{Sg}^{\mathbf{A}}(X)\beta = \text{Sg}^{\mathbf{A}}(X\beta) = \text{Sg}^{\mathbf{A}}(X\alpha) = \text{Sg}^{\mathbf{A}}((X \cup Y)\gamma) = \text{Sg}^{\mathbf{A}}(X \cup Y)\gamma = A\gamma = \text{Im } \gamma$.

The above proof uses the fact that if δ is a homomorphism from an algebra $\mathbf{C}$ to an algebra $\mathbf{D}$ and $U \subseteq C$, then $\text{Sg}^{\mathbf{C}}(U)\delta = \text{Sg}^{\mathbf{D}}(U\delta)$.

An immediate consequence of Proposition 3.6 is that if X is an independent set in an independence algebra $\mathbf{A}$ and $\alpha : X \to A$ is a function, then α can be extended to a homomorphism $\overline{\alpha} : \text{Sg}^{\mathbf{A}}(X) \to A$ and then to $\overline{\overline{\alpha}} \in \text{End } \mathbf{A}$, where Im $\overline{\overline{\alpha}} =$ Im $\overline{\alpha} = \text{Sg}^{\mathbf{A}}(X)\overline{\alpha} = \text{Sg}^{\mathbf{A}}(X\alpha)$. Where there is no danger of ambiguity, both $\overline{\alpha}$ and $\overline{\overline{\alpha}}$ may also be denoted by α. In particular we may follow this convention if X is a basis for A.

At this point we make an observation concerning terms built from independent elements. Let $\mathbf{A}$ be an independence algebra and let $X = \{x_1, ..., x_n\}$ be an independent set of cardinality n. If t, s are terms where $t(x_1, ..., x_n) = s(x_1, ..., x_n)$, then $t(a_1, ..., a_n) = s(a_1, ..., a_n)$ for *any* $a_1, ..., a_n \in A$. To see this, define $\alpha : \{x_1, ..., x_n\} \to A$ by $x_i\alpha = a_i$, and extend α to an endomorphism of $\mathbf{A}$, yielding

$$t(a_1, ..., a_n) = t(x_1\alpha, ..., x_n\alpha) = t(x_1, ..., x_n)\alpha$$

$$= s(x_1, ..., x_n)\alpha = s(x_1\alpha, ..., x_n\alpha) = s(a_1, ..., a_n)$$

as required.

The proof of the next lemma is straightforward.

Lemma 3.7 *Let X be an independent set in an independence algebra $\mathbf{A}$, and let $\alpha : X \to A$ be one-one. If $X\alpha$ is independent then the extension of α to a homomorphism from $Sg^{\mathbf{A}}(X)$ to A is one-one.*

By way of converse to Lemma 3.7 we have

Lemma 3.8 *Let $\mathbf{A}$ be an independence algebra and let α be a one-one homomorphism from a subuniverse B of A to A. Then $X \subseteq B$ is independent if and only if $X\alpha$ is independent.*

Proof. If $X\alpha$ is independent, then Lemma 3.5 and Proposition 3.6 together give that X is independent. Conversely, if X is independent, then letting $\alpha^{-1} : \text{Im } \alpha \to B$ be the inverse of α, the same argument gives that $X\alpha$ is independent.

The next three results concentrate on preimage bases.

Proposition 3.9 *Let $\alpha \in End\ \mathbf{A}$ where $\mathbf{A}$ is an independence algebra, and let X be a preimage basis for α. Then $\alpha|_{Sg^{\mathbf{A}}(X)} : Sg^{\mathbf{A}}(X) \to Im\ \alpha$ is an isomorphism.*

Proof That α is onto is clear. We know that X and $X\alpha$ are independent and $\alpha|_X$ is one-one. Lemma 3.7 gives that α is one-one on $\text{Sg}^{\mathbf{A}}(X)$.

If $\alpha, \beta \in \text{End } \mathbf{A}$, where $\mathbf{A}$ is a weak independence algebra and $\text{Im } \alpha \subseteq \text{Im } \beta$, then by previous comments, any basis of $\text{Im } \alpha$ can be extended to a basis of $\text{Im } \beta$. To prove the corresponding result for kernels and preimage bases we assume that $\mathbf{A}$ is an independence algebra.

Proposition 3.10 *Let $\mathbf{A}$ be an independence algebra and let $\alpha, \beta \in End\ \mathbf{A}$ with $Ker\ \alpha \subseteq Ker\ \beta$. Let X be a preimage basis for β. Then X can be extended to a preimage basis for α. Moreover, if $Ker\ \alpha = Ker\ \beta$, then X is a preimage basis for α.*

Proof. By Proposition 3.9, β is one-one on $\text{Sg}^{\mathbf{A}}(X)$. Since $\text{Ker } \alpha \subseteq \text{Ker } \beta$, α is also one-one on $\text{Sg}^{\mathbf{A}}(X)$. By Lemma 3.8, $X\alpha$ is independent. Extending $X\alpha$ to a basis $X\alpha \cup Y$ of $\text{Im } \alpha$ and choosing for each $y \in Y$ an element x_y of A such that $x_y\alpha = y$, then $X \cup \{x_y : y \in Y\}$ is a preimage basis of α.

Suppose now that $\text{Ker } \alpha = \text{Ker } \beta$. To see that X is a preimage basis of α, we must show that $X\alpha$ spans $\text{Im } \alpha$. Let $a \in \text{Im } \alpha$ and write $a = b\alpha$ where $b \in A$. Now $b\beta = t(x_1, ..., x_n)\beta$ for some $x_1, ..., x_n \in X$ and so as the kernels of α and β coincide we have $a = b\alpha = t(x_1, ..., x_n)\alpha = t(x_1\alpha, ..., x_n\alpha)$ as required.

Corollary 3.11 *Let $\mathbf{A}$ be an independence algebra and let $\alpha, \beta \in End\ \mathbf{A}$ with $Ker\ \alpha = Ker\ \beta$ and $\alpha|_X = \beta|_X$ where X is a preimage basis for α. Then $\alpha = \beta$.*

Proof. Proposition 3.10 says that X is also a preimage basis for β. Since $\alpha|_X = \beta|_X$ we have $\alpha|_{\text{Sg}^{\mathbf{A}}(X)} = \beta|_{\text{Sg}^{\mathbf{A}}(X)}$. Now consider $a \in A$; $a\alpha = t(x_1, ..., x_n)\alpha$ for some $x_1, ..., x_n \in X$. Thus $a\beta = t(x_1, ..., x_n)\beta$ and $t(x_1, ..., x_n)\alpha = t(x_1, ..., x_n)\beta$, giving that $a\alpha = a\beta$; hence $\alpha = \beta$.

Finally in this section we give a result which is central to the proof of Theorem 4.12.

Proposition 3.12 *Let* $\mathbf{A}$ *be an independence algebra and let* $\mathbf{B}$, $\mathbf{C}$ *be subalgebras of* $\mathbf{A}$ *with* $\rho(B) = \rho(C) = n < \aleph_0$. *Let* $\alpha : B \to C$ *be a homomorphism. Then the following conditions are equivalent:*

(i) $\rho(Im\ \alpha) = n$;

(ii) α *is onto;*

(iii) α *is one-one;*

(iv) α *is an isomorphism.*

Proof. Clearly it is sufficient to prove the equivalence of conditions (i), (ii) and (iii).

By Proposition 3.6, α may be extended to $\delta \in \text{End } \mathbf{A}$ with $\text{Im } \alpha = \text{Im } \delta$. Let X be a preimage basis for δ with $X \subseteq B$. By Proposition 3.8, $\delta|_{\text{Sg}^{\mathbf{A}}(X)} = \alpha|_{\text{Sg}^{\mathbf{A}}(X)} : \text{Sg}^{\mathbf{A}}(X) \to \text{Im } \alpha$ is an isomorphism.

If $\rho(\text{Im } \alpha) = n$, then $\rho(\delta) = n$ and $n = \rho(C) = |X\alpha|$ so that as $X\alpha$ is independent, Corollary 3.4 gives

$$\text{Im } \alpha = \text{Sg}^{\mathbf{A}}(X)\alpha = \text{Sg}^{\mathbf{A}}(X\alpha) = C$$

so that α is onto. Clearly if α is onto then $\rho(\text{Im } \alpha) = n$.

Suppose now that α is onto. Thus $|X| = |X\alpha| = n$ and so again by Corollary 3.4, $\text{Sg}^{\mathbf{A}}(X) = B$. Thus α is one-one.

Conversely, if α is one-one then it is immediate from Lemma 3.7 that if Y is a basis of B, then $Y\alpha$ is independent. Since $|Y\alpha| = n$ it follows that $\rho(\text{Im } \alpha) = n$.

4 The endomorphism monoid

Let $\mathbf{A}$ be a weak independence algebra with $\rho(A) = \kappa$. For each cardinal μ with $\mu \leq \kappa$ let

$$T_\mu^{\mathbf{A}} = \{\alpha \in \text{ End } \mathbf{A} : \rho(\alpha) \leq \mu\}.$$

Our first aim is to show that $T_\mu^{\mathbf{A}}$ is an ideal of End $\mathbf{A}$ for all $\mu \leq \kappa$.

Lemma 4.1 *Let* $\alpha, \beta \in$ *End* $\mathbf{A}$ *, where* $\mathbf{A}$ *is a weak independence algebra. Then*

$$\rho(\alpha\beta) \leq min\{\rho(\alpha), \rho(\beta)\}.$$

Proof. Since Im $\alpha\beta$ is a subuniverse of Im β, it is clear from the definition of rank that $\rho(\alpha\beta) \leq \rho(\beta)$.

To see that $\rho(\alpha\beta) \leq \rho(\alpha)$, let X be a basis of Im α. Then

$$\text{Im } \alpha\beta = (\text{ Im } \alpha)\beta = \text{Sg}^{\mathbf{A}}(X)\beta = \text{Sg}^{\mathbf{A}}(X\beta)$$

so that by Proposition 3.3

$$\rho(\alpha\beta) = \rho(\text{Sg}^{\mathbf{A}}(X\beta)) = \rho(X\beta) \leq |X\beta| \leq |X| = \rho(\alpha).$$

Bearing in mind that $\emptyset$ is admitted as an ideal, the next corollary is immediate.

Corollary 4.2 *Let* $\mathbf{A}$ *be a weak independence algebra with* $\rho(A) = \kappa$*, and let* μ *be a cardinal with* $0 \leq \mu \leq \kappa$*. Then* $T_\mu^{\mathbf{A}}$ *is an ideal of* $\mathbf{A}$*.*

To see that End $\mathbf{A}$ contains endomorphisms of every positive rank no greater than $\rho(A)$, we assume that $\mathbf{A}$ is an independence algebra.

Lemma 4.3 *Let* $\mathbf{A}$ *be an independence algebra with* $\rho(A) = \kappa$ *and let* μ *be a cardinal with* $0 < \mu \leq \kappa$*. Then End* $\mathbf{A}$ *contains an endomorphism of rank* μ*.*

Proof. Let X be a basis of A and write X as $X = Y \cup Z$ where $|Y| = \mu$ and $Y \cap Z = \emptyset$. Define $\alpha \in$ End $\mathbf{A}$ by its action on X where

$$\begin{array}{ll} y\alpha = y & y \in Y \\ z\alpha = y_0 & z \in Z, \end{array}$$

and $y_0 \in Y$ is fixed. Then $\rho(\alpha) = \rho(\mathrm{Sg}^{\mathbf{A}}(Y)) = \rho(Y) = \mu$ as required.

We now address the question of whether $T_0^{\mathbf{A}} = \emptyset$.

Lemma 4.4 *Let* $\mathbf{A}$ *be an independence algebra. Then the following conditions are equivalent:*
(i) $\mathbf{A}$ *has constants;*
(ii) $Sg^{\mathbf{A}}(\emptyset) \neq \emptyset$*;*
(iii) $T_0 \neq \emptyset$*.*
Moreover if any (all) of these conditions hold, $T_0^{\mathbf{A}}$ *is a left zero semigroup.*

Proof. The equivalence of (i) and (ii) has already been mentioned. If $T_0^{\mathbf{A}} \neq \emptyset$ then there is an $\alpha \in$ End $\mathbf{A}$ with $\rho(\alpha) = 0$. Hence $\mathbf{A}$ has a non-empty subuniverse of rank 0. But it is easy to see that the only possible candidate for a subuniverse of rank 0 is $\mathrm{Sg}^{\mathbf{A}}(\emptyset)$. Thus if $T_0^{\mathbf{A}} \neq \emptyset$ then $\mathrm{Sg}^{\mathbf{A}}(\emptyset) \neq \emptyset$.

Conversely, suppose that $\mathbf{A}$ has constants. Let X be a basis for A and c a constant. Define $\alpha : X \to A$ by $x\alpha = c$ for all $x \in X$. Then Im $\alpha = \mathrm{Sg}^{\mathbf{A}}(X)\alpha = \mathrm{Sg}^{\mathbf{A}}(X\alpha) = \mathrm{Sg}^{\mathbf{A}}(\{c\}) = \mathrm{Sg}^{\mathbf{A}}(\emptyset)$ so that $\rho(\alpha) = \rho(\mathrm{Sg}^{\mathbf{A}}(\emptyset)) = \rho(\emptyset) = 0$ and $T_0^{\mathbf{A}} \neq \emptyset$.

Suppose now that $T_0^{\mathbf{A}} \neq \emptyset$ and $\alpha, \beta \in T_0^{\mathbf{A}}$. Then Im $\alpha = \mathrm{Sg}^{\mathbf{A}}(\emptyset)$ and clearly β is the constant function when restricted to $\mathrm{Sg}^{\mathbf{A}}(\emptyset)$, since β must map any constant to itself. Thus $\alpha\beta = \alpha$ and $T_0^{\mathbf{A}}$ is a left zero semigroup.

The next two results characterise Green's relations on End $\mathbf{A}$, where $\mathbf{A}$ is an independence algebra.

Proposition 4.5 *Let* $\mathbf{A}$ *be an independence algebra. For* $\alpha, \beta \in$ *End* $\mathbf{A}$*:*
(i) Im $\alpha \subseteq$ *Im* β *if and only if* $\alpha \leq_{\mathcal{L}} \beta$*;*
(ii) Ker $\alpha \subseteq$ *Ker* β *if and only if* $\beta \leq_{\mathcal{R}} \alpha$*;*
(iii) $\rho(\alpha) = \rho(\beta)$ *if and only if* $\alpha \mathcal{D} \beta$*;*
(iv) $\rho(\alpha) \leq \rho(\beta)$ *if and only if* $\alpha \leq_{\mathcal{J}} \beta$*;*
(v) $\mathcal{D} = \mathcal{J}$*.*

Proof. (i) If $\alpha \leq_{\mathcal{L}} \beta$ then $\alpha = \gamma\beta$ for some $\gamma \in \text{End } \mathbf{A}$; hence Im $\alpha =$ Im $\gamma\beta \subseteq$ Im β. Conversely, suppose that Im $\alpha \subseteq$ Im β, and let X be a basis for A. For each $x \in X$ there exists $x' \in A$ with $x\alpha = x'\beta$. Define $\gamma \in \text{End } \mathbf{A}$ by its action on X, setting $x\gamma = x'$ for all $x \in X$. Then $x\gamma\beta = x'\beta = x\alpha$ for all $x \in X$; but $\text{Sg}^{\mathbf{A}}(X) = A$, so that $\gamma\beta = \alpha$ and $\alpha \leq_{\mathcal{L}} \beta$.

(ii) If $\beta \leq_{\mathcal{R}} \alpha$ then $\beta = \alpha\delta$ for some $\delta \in \text{End } \mathbf{A}$. Then if $a\alpha = b\alpha$ for $a, b \in A$ we have $a\alpha\delta = b\alpha\delta$ and so $a\beta = b\beta$. Thus Ker $\alpha \subseteq$ Ker β.

Conversely, suppose that Ker $\alpha \subseteq$ Ker β, and let X be a preimage basis for α. Then $X\alpha$ is independent and so may be extended to a basis $X\alpha \cup Y$ of A. Define $\delta \in \text{End } \mathbf{A}$ by

$$\begin{array}{ll} (x\alpha)\delta = x\beta & x \in X \\ y\delta = w & y \in Y, \end{array}$$

where w is fixed. Note that since α is one-one on X, δ is well defined. Let $a \in A$. Then $a\alpha \in \text{Im } \alpha = \text{Sg}^{\mathbf{A}}(X\alpha) = \text{Sg}^{\mathbf{A}}(X)\alpha$ so that $a\alpha = t(x_1, ..., x_n)\alpha$ for some $x_1, ..., x_n \in X$. Now Ker $\alpha \subseteq$ Ker β so that

$$a\beta = t(x_1, ..., x_n)\beta = t(x_1\beta, ..., x_n\beta) = t(x_1\alpha\delta, ..., x_n\alpha\delta) = t(x_1, ..., x_n)\alpha\delta = a\alpha\delta.$$

Hence $\beta = \alpha\delta \leq_{\mathcal{R}} \alpha$ as required.

(iii) Suppose that $\alpha\mathcal{D}\beta$; then $\alpha\mathcal{R}\gamma\mathcal{L}\beta$ for some $\gamma \in \text{End } \mathbf{A}$. From parts (i) and (ii), Ker $\alpha =$ Ker γ and Im $\gamma =$ Im β. According to the remarks following Corollary 3.4, $\rho(\alpha) = \rho(\gamma) = \rho(\beta)$.

Conversely, assume that $\rho(\alpha) = \rho(\beta)$. Then there are bases X and Y for Im α and Im β repectively, where $|X| = |Y|$. Let $\mu : X \to Y$ be a bijection, and extend μ to $\gamma \in \text{End } \mathbf{A}$ with Im $\gamma = \text{Sg}^{\mathbf{A}}(Y)$; further, extend μ^{-1} to $\delta \in \text{End } \mathbf{A}$ with Im $\delta = \text{Sg}^{\mathbf{A}}(X)$. Proposition 3.6 guarantees the existence of γ and δ. Now Im $(\alpha\gamma) = (\text{Im } \alpha)\gamma = \text{Sg}^{\mathbf{A}}(X)\gamma = \text{Sg}^{\mathbf{A}}(X\gamma) = \text{Sg}^{\mathbf{A}}(X\mu) = \text{Sg}^{\mathbf{A}}(Y) = \text{Im } \beta$, so that $\beta\mathcal{L}\alpha\gamma$ by part (i). Since $\alpha\gamma\delta = \alpha$ we have $\alpha\mathcal{R}\alpha\gamma$, whence $\alpha\mathcal{D}\beta$.

(iv) If $\alpha \leq_{\mathcal{J}} \beta$ then Lemma 4.1 gives that $\rho(\alpha) \leq \rho(\beta)$. Suppose now that $\rho(\alpha) \leq \rho(\beta)$. Let X be a basis of Im β and choose $Y \subseteq X$ with $|Y| = \rho(\alpha)$. Let γ be the identity automorphism of $\text{Sg}^{\mathbf{A}}(Y)$. By Proposition 3.6, we may extend γ to $\overline{\gamma} \in \text{End } \mathbf{A}$ with Im $\overline{\gamma} =$ Im γ. Note that Im $\beta\overline{\gamma} = (\text{Im } \beta)\overline{\gamma} = \text{Sg}^{\mathbf{A}}(Y)$, so that $\rho(\beta\overline{\gamma}) = |Y| = \rho(\alpha)$. Then $\alpha\mathcal{D}\beta\overline{\gamma}$ so that $\alpha\mathcal{J}\beta\overline{\gamma} \leq_{\mathcal{J}} \beta$.

(v) This is an immediate consequence of (iii) and (iv).

Corollary 4.6 *Let* $\mathbf{A}$ *be an independence algebra and let* $\alpha, \beta \in$ *End* $\mathbf{A}$. *Then:*

(i) Im $\alpha =$ *Im* β *if and only if* $\alpha\mathcal{L}\beta$;

(ii) Ker $\alpha =$ *Ker* β *if and only if* $\alpha\mathcal{R}\beta$.

Lemma 4.1 yields immediately that if $\mathbf{A}$ is a weak independence algebra, then

$$I_{\mu}^{\mathbf{A}} = \{\alpha \in \text{ End } \mathbf{A} : \rho(\alpha) < \mu\}$$

is an ideal for all cardinals μ. In particular, $T^{\mathbf{A}}$ is an ideal, where $T^{\mathbf{A}} = I_{\aleph_0}^{\mathbf{A}}$. In other words, $T^{\mathbf{A}}$ consists of the endomorphisms of $\mathbf{A}$ of finite rank.

We now consider the Rees quotients of End $\mathbf{A}$ of the form $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$, where $\mathbf{A}$ is an independence algebra with $\rho(A) \geq n$ and $n \in \mathbf{N}$. Note that these semigroups always have a zero; the only awkward case is when $n = 1$ and $\mathrm{Sg}^{\mathbf{A}}(\emptyset) = \emptyset$ so that $T_0^{\mathbf{A}} = \emptyset$. But by convention, $\emptyset$ is an ideal of End $\mathbf{A}$ so that $\{\emptyset\}$ is the zero of $T_1^{\mathbf{A}}/T_0^{\mathbf{A}}$. First we show that they are regular, by proving that End $\mathbf{A}$ is regular and then employing a standard technique of semigroup theory. If $\alpha \in$ End $\mathbf{A}$ has rank n, then we make no notational distinction between the endomorphism α and its image in the Rees quotient $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$.

Proposition 4.7 *Let* $\mathbf{A}$ *be an independence algebra. Then End* $\mathbf{A}$ *is regular.*

Proof. Let $\alpha \in$ End $\mathbf{A}$ and let X be a preimage basis for α. Extend $X\alpha$ to a basis $X\alpha \cup Y$ of A and define $\beta \in$ End $\mathbf{A}$ by its action on $X\alpha \cup Y$ where

$$\begin{array}{ll} x\alpha\beta = x & x \in X \\ y\beta = y & y \in Y, \end{array}$$

since α is one-one on X, β is well defined. For $x \in X$, $x\alpha\beta\alpha = x\alpha$, so that $\beta\alpha$ is the identity when restricted to Im $\alpha = \mathrm{Sg}^{\mathbf{A}}(X\alpha)$. Hence $\alpha\beta\alpha = \alpha$.

Corollary 4.8 *Let* $\mathbf{A}$ *be an independence algebra. Then* $T_\mu^{\mathbf{A}}$ *is regular for all cardinals* $\mu > 0$. *Further, if* $n \in \mathbf{N}$ *and* $n \leq \rho(A)$, *then* $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ *is regular.*

Proof. If a, b are elements of a semigroup and $a = aba$, then $a = a(bab)a$ also. It is then clear that every ideal of a regular semigroup is regular. In particular, if $\alpha \in$ End $\mathbf{A}$ and $\rho(\alpha) = n$, then choosing β with $\alpha = \alpha\beta\alpha$ we have also that $\alpha = \alpha\gamma\alpha$ where $\gamma = \beta\alpha\beta$. But then Lemma 4.1 gives that $\rho(\alpha) \leq \rho(\gamma) \leq \rho(\alpha)$ so that $\rho(\gamma) = n$. Thus α is a regular element of $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$.

We are now in a position to prove the first of our two theorems concerning the endomorphism monoid of an independence algebra.

Theorem 4.9 *Let* $\mathbf{A}$ *be an independence algebra with* $\rho(A) \geq n$ *for some* $n \in \mathbf{N}$. *Then* $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ *is completely 0-simple.*

Proof. We have already shown that $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ is regular.

Let $n \in \mathbf{N}$ and let ϵ, η be idempotents of rank n and suppose that $\epsilon\eta = \epsilon = \eta\epsilon$. Then $\rho(\epsilon\eta) = \rho(\eta) = n$ and Im $\epsilon\eta \subseteq$ Im η, so that Im $\epsilon\eta =$ Im η and by Corollary 4.6, $\eta\mathcal{L}\epsilon\eta$. Thus $\eta\mathcal{L}\epsilon$ and since ϵ is idempotent, $\epsilon = \eta\epsilon = \eta$. This shows that the non-zero idempotents of $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ are primitive.

From Theorem III 3.5 of [8] it remains to show that 0 is a prime ideal of $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$. This is equivalent to showing that if $\alpha, \beta \in$ End $\mathbf{A}$ and $\rho(\alpha) = \rho(\beta) = n$, then there exists $\gamma \in$ End $\mathbf{A}$ with $\rho(\gamma) = \rho(\alpha\gamma\beta) = n$.

With α, β as in the previous paragraph, choose a basis U of Im α and a preimage basis V for β. Then $|U| = |V| = n$ so there exists a bijection $\theta : U \to V$. By Proposition 3.6 we may extend θ to an element $\overline{\theta}$ of End $\mathbf{A}$ such that Im $\overline{\theta} = \mathrm{Sg}^{\mathbf{A}}(V)$, so that $\rho(\overline{\theta}) = n$. Now $u\overline{\theta}\beta = u\theta\beta$ for all $u \in U$. Since β is a bijection from $\mathrm{Sg}^{\mathbf{A}}(V)$ to Im β it follows by

Lemma 3.7 that the restriction of $\overline{\theta}\beta$ to Im $\alpha = \mathrm{Sg}^{\mathbf{A}}(U)$ is a bijection from Im α to Im β. Thus Im $\alpha\overline{\theta}\beta =$ Im β so that $\rho(\alpha\overline{\theta}\beta) = \rho(\beta) = n$ as required.

The celebrated theorem of Rees [12] states that every completely 0-simple semigroup is isomorphic to a regular Rees matrix semigroup. The question now arises of describing the Rees matrix semigroups to which the completely 0-simple semigroups $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ and $T_0^{\mathbf{A}}$ (where $\mathrm{Sg}^{\mathbf{A}}(\emptyset) \neq \emptyset$) are isomorphic.

For the remainder of this section let $\mathbf{A}$ be an independence algebra and let $n \in \mathbf{N}$ be less than the rank of A. Put $\mathbf{Q} = T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$; we also allow $\mathbf{Q} = T_0^{\mathbf{A}}$ in the case where $\mathrm{Sg}^{\mathbf{A}}(\emptyset) \neq \emptyset$. In the former case $\mathbf{Q}$ is completely 0-simple, and in the latter Lemma 4.4 gives that $\mathbf{Q}$ is completely simple. Our aim is to give an explicit description of a regular Rees matrix semigroup isomorphic to $\mathbf{Q}$.

Let

$$I = \{i : i \text{ is the kernel of } \alpha \in \text{ End } \mathbf{A} \text{ where } \rho(\alpha) = n\}$$

and let

$$\Lambda = \{\lambda : \lambda \text{ is a subuniverse of } A \text{ with } \rho(\lambda) = n\}.$$

Note that if $n = 0$ then by assumption $\mathrm{Sg}^{\mathbf{A}}(\emptyset) \neq \emptyset$ and $\Lambda = \{\mathrm{Sg}^{\mathbf{A}}(\emptyset)\}$.

For each $i \in I$ choose and fix a preimage basis $\{u_1^i, ..., u_n^i\}$ of α where Ker $\alpha = i$ (and $\rho(\alpha) = n$). By Proposition 3.10, $\{u_1^i, ..., u_n^i\}$ is a preimage basis of any $\beta \in$ End $\mathbf{A}$ with Ker $\alpha =$ Ker β; recall also that if Ker $\alpha =$ Ker β then $\rho(\alpha) = \rho(\beta)$. For each $i \in I$ denote by B_i the subuniverse $\mathrm{Sg}^{\mathbf{A}}(\{u_1^i, ..., u_n^i\})$.

For each $\lambda \in \Lambda$ choose and fix a basis $\{v_1^\lambda, ..., v_n^\lambda\}$ of λ. Now fix $\lambda_0 \in \Lambda$ and label $v_j^{\lambda_0}$ as w_j for $j \in \{1, ..., n\}$. Put $B = \mathrm{Sg}^{\mathbf{A}}(\{w_1, ..., w_n\}) = \lambda_0$ and let $\mathbf{G}$ be the group of automorphisms of $\mathbf{B}$. We shall show that $\mathbf{Q}$ is isomorphic to $\boldsymbol{\mathcal{M}}^{\mathbf{0}}(\mathbf{G}; I, \Lambda; P)$ for some regular sandwich matrix P (or to $\boldsymbol{\mathcal{M}}(\mathbf{G}; I, \Lambda; P)$ in the completely simple case).

In order to construct P we begin with the following observation.

Lemma 4.10 *Let $\alpha, \beta, \gamma, \delta \in$ End $\mathbf{A}$ all have rank n and suppose that Im $\alpha =$ Im γ, Ker $\beta =$ Ker δ and $\rho(\alpha\beta) = n$. Then $\rho(\gamma\delta) = n$.*

Proof. By Proposition 4.5, $\alpha\mathcal{L}\gamma$ and $\beta\mathcal{R}\delta$. Hence $\alpha\beta\mathcal{L}\gamma\beta\mathcal{R}\gamma\delta$ so that $\rho(\alpha\beta) = \rho(\gamma\delta)$, again by Proposition 4.5.

Consider now $\alpha, \beta \in$ End $\mathbf{A}$, both of rank n, with Im $\alpha = \lambda$ and Ker $\beta = i$. If $\rho(\alpha\beta) < n$ put $p_{\lambda i} = 0$, where 0 is a symbol which has not previously occured. If $\rho(\alpha\beta) = n$ then define $p_{\lambda i} \in$ End $\mathbf{B}$ by

$$w_j p_{\lambda i} = t_j(w_1, ..., w_n), 1 \leq j \leq n,$$

where $v_j^\lambda \beta = t_j(u_1^i, ..., u_n^i)\beta$. We remark that $\{v_1^\lambda\beta, ..., v_n^\lambda\beta\}$ is an independent set of cardinality n, for

$$n = \rho(\text{ Im } \alpha\beta) = \rho(\text{ Im } \beta|_{\text{ Im } \alpha}) = \rho(\mathrm{Sg}^{\mathbf{A}}(\{v_1^\lambda\beta, ..., v_n^\lambda\beta\}) = \rho(\{v_1^\lambda\beta, ..., v_n^\lambda\beta\}).$$

Lemma 4.11 *For any $i \in I$ and $\lambda \in \Lambda$, $p_{\lambda i}$ is well-defined and if $p_{\lambda i} \neq 0$, then $p_{\lambda i} \in G$.*

Proof. Let $\alpha, \beta, \gamma, \delta \in \text{End } \mathbf{A}$ and suppose that $\text{Im } \alpha = \text{Im } \gamma = \lambda$ and $\text{Ker } \beta = \text{Ker } \delta = i$. In view of the previous lemma, $\rho(\alpha\beta) = n$ if and only if $\rho(\gamma\delta) = n$.

Suppose that $\rho(\alpha\beta) = \rho(\gamma\delta) = n$, and $v_j^\lambda \beta = t_j(u_1^i, ..., u_n^i)\beta$ and $v_j^\lambda \delta = s_j(u_1^i, ..., u_n^i)\delta, 1 \leq j \leq n$. We wish to show that $t_j(w_1, ..., w_n) = s_j(w_1, ..., w_n)$ for all j with $1 \leq j \leq n$. Since $\text{Ker } \beta = \text{Ker } \delta$ we have $t_j(u_1^i, ..., u_n^i)\beta = v_j^\lambda \beta = s_j(u_1^i, ..., u_n^i)\beta$, which yields $t_j(u_1^i, ..., u_n^i) = s_j(u_1^i, ..., u_n^i)$, since β is one-one on the subuniverse generated by a preimage basis. Since $\{u_1^i, ..., u_n^i\}$ is a set of n independent elements, it follows that $t_j(w_1, ..., w_n) = s_j(w_1, ..., w_n)$. Hence $p_{\lambda i}$ is well defined. It remains to show that $p_{\lambda i} \in G$.

In view of Proposition 3.12, it is sufficient to show that $\{t_1(w_1, ..., w_n), ..., t_n(w_1, ..., w_n)\}$ is an independent set of cardinality n. Suppose that

$$t_j(\overline{w}) \in \text{Sg}^{\mathbf{A}}(\{t_1(\overline{w}), ..., t_{j-1}(\overline{w}), t_{j+1}(\overline{w}), ..., t_n(\overline{w})\})$$

where $\overline{w} = (w_1, ..., w_n)$; without loss of generality we may assume that $j = 1$. Hence

$$t_1(\overline{w}) = s(t_2(\overline{w}), ..., t_n(\overline{w})) \tag{1}$$

for some term s. But $\{w_1, ..., w_n\}$ is an independent set of n elements, so we may replace each w_j in (1) by $u_j^i\beta, 1 \leq j \leq n$ obtaining

$$v_1^\lambda \beta = s(v_2^\lambda \beta, ..., v_n^\lambda \beta),$$

contradicting the fact that $\{v_1^\lambda \beta, ..., v_n^\lambda \beta\}$ is an independent set of cardinality n. It follows that $\{t_1(\overline{w}), ..., t_n(\overline{w})\}$ is an independent set of cardinality n.

We are now in a position to define a $\Lambda \times I$ matrix P over $G \cup \{0\}$ by putting $P = (p_{\lambda i})$. Then $\boldsymbol{\mathcal{M}}^{\mathbf{0}} = \boldsymbol{\mathcal{M}}^{\mathbf{0}}(\mathbf{G}; I, \Lambda; P)$ is a Rees matrix semigroup. In the case where $\mathbf{Q} = T_0^{\mathbf{A}}, B = \text{Sg}^{\mathbf{A}}(\emptyset)$ and $\mathbf{G}$ is the trivial group: let $p_{\lambda i} = 1$ for all $i \in I$, where $G = \{1\}$ and put $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}(\mathbf{G}; I, \Lambda; P)$. We show that $\boldsymbol{\mathcal{M}}^{\mathbf{0}}$ (or $\boldsymbol{\mathcal{M}}$) is isomorphic to $\mathbf{Q}$. Then where $\mathbf{Q}$ has a zero, zero must be a prime ideal of $\boldsymbol{\mathcal{M}}^{\mathbf{0}}$ and we deduce that P is regular in the sense that every row and column of P contains at least one non-zero entry.

In order to construct an isomorphism $\psi : Q \to \mathcal{M}^0(\mathcal{M})$ we begin by defining for each $i \in I$ and $\lambda \in \Lambda$ isomorphisms $\mu_i : B \to B_i$ and $\tau_\lambda : \lambda \to B$ by putting

$$\begin{aligned} w_j \mu_i &= u_j^i \quad 1 \leq j \leq n \\ v_j^\lambda \tau_\lambda &= w_j \quad 1 \leq j \leq n. \end{aligned}$$

Where $\mathbf{Q} = T_0^{\mathbf{A}}, B = B_i = \lambda$ and we put $\mu_i = \tau_\lambda = 1$.

Consider now $\alpha \in \text{End } \mathbf{A}$ where $\rho(\alpha) = n$. Then if $i = \text{Ker } \alpha$ and $\lambda = \text{Im } \alpha$, we know that $\alpha' = \alpha|_{B_i} : B_i \to \lambda$ is an isomorphism. Hence $\mu_i \alpha' \tau_\lambda : B \to B$ is an automorphism of $\mathbf{B}$, that is, $\mu_i \alpha' \tau_\lambda \in G$. We therefore define $\psi : Q \to \mathcal{M}^0(\mathcal{M})$ by $0\psi = 0$ and $\alpha\psi = (i, \mu_i \alpha' \tau_\lambda, \lambda)$, where $i = \text{Ker } \alpha$ and $\lambda = \text{Im } \alpha$.

Theorem 4.12 *With* $\mathbf{Q}$, $\boldsymbol{\mathcal{M}}^{\mathbf{0}}$ *(*$\boldsymbol{\mathcal{M}}$*) and* ψ *defined as above,* ψ *is an isomorphism from* $\mathbf{Q}$ *to* $\boldsymbol{\mathcal{M}}^{\mathbf{0}}$ *(*$\boldsymbol{\mathcal{M}}$*).*

Proof. We begin by showing that ψ is one-one. Let $\alpha, \beta \in$ End $\mathbf{A}$ where $\rho(\alpha) = \rho(\beta) = n$ and suppose that $\alpha\psi = \beta\psi = (i, \theta, \lambda)$. By definition of ψ, Ker $\alpha =$ Ker $\beta = i$ and Im $\alpha =$ Im $\beta = \lambda$. The set $\{u_1^i, ..., u_n^i\}$ is a preimage basis for α and β and since $\mu_i\alpha'\tau_\lambda = \mu_i\beta'\tau_\lambda$ and μ_i, τ_λ are isomorphisms, we have $\alpha' = \beta'$. Thus by Corollary 3.11, $\alpha = \beta$.

To see that ψ is onto, consider an arbitrary non-zero element (i, θ, λ) of $\mathcal{M}^0$ or $\mathcal{M}$ and choose $\alpha \in$ End $\mathbf{A}$ with Ker $\alpha = i$. Extend $\{u_1^i, ..., u_n^i\}$ to a basis $\{u_1^i, ..., u_n^i\} \cup X$ of A and define $\beta \in$ End $\mathbf{A}$ by

$$\begin{array}{ll} u_j^i\beta = u_j^i\mu_i^{-1}\theta\tau_\lambda^{-1} & 1 \leq j \leq n \\ x\beta = t(u_1^i\beta, ..., u_n^i\beta) & x \in X \end{array}$$

where $x\alpha = t(u_1^i\alpha, ..., u_n^i\alpha)$. Note that $\mu_i^{-1}\theta\tau_\lambda^{-1} : B_i \to \lambda$ is an isomorphism so that Im $\beta = \lambda$. In particular, $\{u_1^i\beta, ..., u_n^i\beta\}$ is an independent set of cardinality n so that for any $x \in X, x\beta = t(u_1^i\beta, ..., u_n^i\beta)$ if and only if $x\alpha = t(u_1^i\alpha, ..., u_n^i\alpha)$. We claim that Ker $\beta = i$, so that

$$\begin{aligned} \beta\psi &= (i, \mu_i\beta'\tau_\lambda, \lambda) \\ &= (i, \mu_i\mu_i^{-1}\theta\tau_\lambda^{-1}\tau_\lambda, \lambda) = (i, \theta, \lambda) \end{aligned}$$

as required.

Consider $a \in A$; then $a = s(\overline{u^i}, \overline{x})$ where $\overline{u^i} = (u_1^i, ..., u_n^i)$ and $\overline{x} = (x_1, ..., x_m)$ for some $x_1, ..., x_m \in X$. Now $a\alpha = t(\overline{u^i})\alpha$ if and only if

$$s(\overline{u^i\alpha}, \overline{x\alpha}) = t(\overline{u^i\alpha})$$

where $\overline{u^i\alpha} = (u_1^i\alpha, ..., u_n^i\alpha)$ and $\overline{x\alpha} = (x_1\alpha, ..., x_m\alpha)$. Rewriting $x_k\alpha$ as $t_k(\overline{u^i\alpha})$ for $1 \leq k \leq m$, we have $a\alpha = t(\overline{u^i})\alpha$ if and only if

$$s(\overline{u^i\alpha}, t_1(\overline{u^i\alpha}), ..., t_m(\overline{u^i\alpha})) = t(\overline{u^i\alpha}). \tag{2}$$

But $\{u_1^i\alpha, ..., u_n^i\alpha\}$ and $\{u_1^i\beta, ..., u_n^i\beta\}$ are both independent sets of cardinality n, so that (2) is equivalent to

$$s(\overline{u^i\beta}, t_1(\overline{u^i\beta}), ..., t_m(\overline{u^i\beta})) = t(\overline{u^i\beta})$$

where $\overline{u^i\beta} = (u_1^i\beta, ..., u_n^i\beta)$. But this is equivalent to $s(\overline{u^i\beta}, \overline{x\beta}) = t(\overline{u^i\beta})$ and then to $s(\overline{u^i}, \overline{x})\beta = t(\overline{u^i})\beta$, where $\overline{x\beta} = (x_1\beta, ..., x_m\beta)$. Hence $a\alpha = t(\overline{u^i})\alpha$ if and only if $a\beta = t(\overline{u^i})\beta$.

Suppose now that $a, b \in A$ and $a\alpha = b\alpha$. We know that $a\alpha = b\alpha$ can be written as $a\alpha = b\alpha = t(\overline{u^i})\alpha$ and the above argument gives that $a\beta = t(\overline{u^i})\beta = b\beta$. Conversely, if $a\beta = b\beta$ then using the fact that $\{u_1^i, ..., u_n^i\}$ is clearly a preimage basis for β, we obtain that $a\alpha = b\alpha$. Thus Ker $\beta = i$ as required.

Finally we must show that ψ is a homomorphism. This is clear in the case where $\mathbf{Q} = T_0^{\mathbf{A}}$. Suppose that $\mathbf{Q} = T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$. Let $\alpha, \beta \in$ End $\mathbf{A}$ have rank n, where $\alpha\psi = (i, \mu_i\alpha'\tau_\lambda, \lambda)$ and $\beta\psi = (k, \mu_k\beta'\tau_\xi, \xi)$. Then $\alpha\beta = 0$ if and only if $\rho(\alpha\beta) < n$ if and only if $p_{\lambda k} = 0$; it follows that if $\alpha\beta = 0$ then $(\alpha\beta)\psi = \alpha\psi\beta\psi$.

We assume now that in $\mathbf{Q}$, $\alpha\beta \neq 0$, that is, $\rho(\alpha\beta) = n$. Since $\mathbf{Q}$ is completely 0-simple, $\alpha\mathcal{R}\alpha\beta\mathcal{L}\beta$ so that by Proposition 4.5, Ker $\alpha =$ Ker $\alpha\beta$ and Im $\alpha\beta =$ Im β. hence

$$(\alpha\beta)\psi = (i, \mu_i(\alpha\beta)'\tau_\xi, \xi)$$

and

$$\alpha\psi\beta\psi = (i, \mu_i\alpha'\tau_\lambda p_{\lambda k}\mu_k\beta'\tau_\xi, \xi).$$

The task is to show that $(\alpha\beta)' = \alpha'\tau_\lambda p_{\lambda k}\mu_k\beta'$. Now Im $\alpha = \lambda$ and for $1 \leq j \leq n$,

$$v_j^\lambda \tau_\lambda p_{\lambda k} = w_j p_{\lambda k} = t_j(w_1, ..., w_n)$$

where $v_j^\lambda \beta = t_j(u_1^k, ..., u_n^k)\beta$, since Ker $\beta = k$. Then

$$v_j^\lambda \tau_\lambda p_{\lambda k}\mu_k\beta = t_j(w_1, ..., w_n)\mu_k\beta = t_j(u_1^k, ..., u_n^k)\beta = v_j^\lambda\beta.$$

Hence $\tau_\lambda p_{\lambda k}\mu_k\beta$ agrees with β on Im α. That is, $\alpha\tau_\lambda p_{\lambda k}\mu_k\beta = \alpha\beta$ so that certainly $\alpha'\tau_\lambda p_{\lambda k}\mu_k\beta' = (\alpha\beta)'$ and ψ is an isomorphism.

5 Applications

Our first application of Theorem 4.12 is to the case of the full transformation monoid $\mathcal{T}(X)$ on a non-empty set X. As previously commented, if we regard $\mathcal{T}(X)$ as the algebra on X with empty set of operations, then $\mathcal{T}(X)$ is an independence algebra. Proposition 4.5 and Corollary 4.6 hold; these results appear in Section 2.2 of [1] and are due to Miller and Doss [2] and Suschkewitsch [14]. The structure of the principal factors in the case where X is finite appears in Section 3.2 of [1] and is credited to Hewitt and Zuckermann [6]. The construction in [1] of the Rees matrix semigroups is in fact virtually the same as ours, the only difference in approach is that in [1] the authors consider partitions on X rather than preimage bases. In our notation if $\alpha \in \mathcal{T}(X)$ and $\rho(\alpha) = n$ then a preimage basis $\{x_1, ..., x_n\}$ of α is a cross section of ξ, where ξ is the partition of X induced by Ker α. Of course, any partition of X into n subsets induces an equivalence relation which is the kernel of some function with rank n. The translation of Theorem 4.12 to the case of $\mathcal{T}(X)$ is given below, where P and the required isomorphism are constructed as in that theorem.

Corollary 5.1 *Let X be a non-empty set with $|X| \geq n$ and $n \in \mathbf{N}$. Putting $\mathbf{A} = \mathcal{T}(X)$, the principal factor $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ is isomorphic to $\mathcal{M}^0(\mathcal{S}_n; I, \Lambda; P)$ where $\mathcal{S}_n$ is the symmetric group on n elements, I indexes the equivalence relations on X with n classes and Λ indexes the subsets of X of cardinality n.*

We now turn our attention to the structure of End $\mathbf{V}$, where $\mathbf{V}$ is a left vector space over a division ring $\mathbf{D}$. In *Rings and Semigroups* [11] Petrich describes in detail not only the Rees factors $T_n^{\mathbf{V}}/T_{n-1}^{\mathbf{V}}$, but also Rees factors in certain subrings of End $\mathbf{V}$. The subrings in question are dense rings of linear maps of $\mathbf{V}$ having finite rank; they have an alternative description as rings of linear maps of $\mathbf{V}$ of finite rank having an adjoint in a dual vector space $\mathbf{U}$. If we take $\mathbf{V}^*$ to be the set of all linear forms on $\mathbf{V}$, (that is, linear maps from V to D, where $\mathbf{D}$ has the natural structure as a left vector space), then $\mathbf{V}^*$ is a right vector space under pointwise addition and scalar multiplication defined by

$$x(f\lambda) = (xf)\lambda$$

for all $x \in V, f \in V^*$ and $\lambda \in D$. Moreover $(\mathbf{V}^*, \mathbf{V})$ is a dual pair with bilinear form $(v, f) = vf$ where $v \in V, f \in V^*$. Lemma I.1.4 of [11] says that each $\alpha \in$ End $\mathbf{V}$ has an adjoint α^* in End $\mathbf{V}^*$, where the linear maps of $\mathbf{V}^*$ are written on the *left* of their arguments. If $\alpha \in$ End $\mathbf{V}$ then α^* is obtained as follows. For every $f \in V^*$, let $\alpha^* f \in V^*$ be the function from V to D determined by setting $x(\alpha^* f) = (x\alpha)f$ for all $x \in V$. Putting $\mathbf{U} = \mathbf{V}^*$, Theorem I.4.6 of [11] describes the structure of the Rees quotients $T_n^{\mathbf{V}}/T_{n-1}^{\mathbf{V}}$, for $n \geq 1$. The following result shows that this description can be obtained from the Rees matrix semigroups $\mathcal{M}^0(\mathbf{G}; I, \Lambda; P)$ of Theorem 4.12. We remark that Exercise 6 on page 57 of [1] leads a discussion of the structure of $T_n^{\mathbf{V}}/T_{n-1}^{\mathbf{V}}$.

Corollary 5.2 *[11] Let* $\mathbf{V}$ *be a left vector space over a division ring* $\mathbf{D}$ *and* $n \in \mathbf{N}$ *where* $n \leq$ *dim* $\mathbf{V}$. *Put*

$$J = \{j : j \text{ is an } n\text{-dimensionsal subspace of } \mathbf{V}^*\}$$

and

$$\Lambda = \{\lambda : \lambda \text{ is an } n\text{-dimensional subspace of } \mathbf{V}\}.$$

For every $\lambda \in \Lambda$ *fix a basis* $\{w_1^\lambda, ..., w_n^\lambda\}$ *of* λ *and for every* $j \in J$ *fix a basis* $\{f_1^j, ..., f_n^j\}$ *of* j. *Let*

$$\mathcal{M}^0 = \mathcal{M}^0(\mathbf{GL}(n, \mathbf{D}); J, \Lambda; Q)$$

where $Q = (q_{\lambda j})$ *with*

$$q_{\lambda j} = \begin{pmatrix} w_1^\lambda \\ . \\ . \\ w_n^\lambda \end{pmatrix} (f_1^j, ..., f_n^j) = \begin{pmatrix} w_1^\lambda f_1^j & . & . & w_1^\lambda f_n^j \\ . & . & . & . \\ . & . & . & . \\ w_n^\lambda f_1^j & . & . & w_n^\lambda f_n^j \end{pmatrix}$$

if this matrix is in $GL(n, D)$, *and* $q_{\lambda j} = 0$ *otherwise. Then* $T_n^{\mathbf{A}}/T_{n-1}^{\mathbf{A}}$ *is isomorphic to* $\mathcal{M}^0$.

Proof. Recall from Section 4 that $\mathbf{G}$ is the group of automorphisms of $\mathrm{Sg}^{\mathbf{A}}(\{w_1, ..., w_n\})$. Let $\theta : G \to GL(n, D)$ be the isomorphism defined by

$$\nu\theta = H = (h_{ij})$$

where

$$w_i\nu = h_{i1}w_1 + ... + h_{in}w_n, 1 \leq i \leq n.$$

The next step is to define an isomorphism $\psi : I \to J$. Let $i \in I$ and choose $\alpha \in$ End $\mathbf{V}$ with Ker $\alpha = i$. Then $\{u_1^i, ..., u_n^i\}$ is a preimage basis for α; we use this preimage basis to determine $g_1^i, ..., g_n^i \in V^*$ by writing, for each $v \in V$,

$$v\alpha = (vg_1^i)(u_1^i\alpha) + ... + (vg_n^i)(u_n^i\alpha).$$

Since $\{u_1^i\alpha, ..., u_n^i\alpha\}$ is a basis for Im α, certainly $g_j^i, 1 \leq j \leq n$ are functions from V to D; that they are linear is a consequence of the linearity of α. Moreover they are independent of the choice of α, for if $\beta \in$ End $\mathbf{V}$ and Ker $\beta = i$, then as

$$v - \sum_j (vg_j^i)u_j^i \in \text{ Ker } \alpha = \text{ Ker } \beta,$$

we have

$$v\beta = \sum_j (vg_j^i)(u_j^i\beta).$$

If $\lambda_1, ..., \lambda_n \in D$ and $g_1^i\lambda_1 + ... + g_n^i\lambda_n = 0$, then for each $j \in \{1, ..., n\}$,

$$0 = u_j^i 0 = u_j^i(\sum_k g_k^i\lambda_k) = \sum_k (u_j^i g_k^i)\lambda_k.$$

Since $\{u_1^i\alpha, ..., u_n^i\alpha\}$ is a basis of Im α,

$$u_j^i g_k^i = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

Hence $\lambda_j = 0$ for $1 \leq j \leq n$ and so $\{g_1^i, ..., g_n^i\}$ is a linearly independent subset in $\mathbf{V}^*$. This enables us to define a function $\psi : I \to J$ by $i\psi = \mathrm{Sg}^{\mathbf{V}^*}(\{g_1^i, ..., g_n^i\})$. To see that ψ is one-one, suppose that $i, h \in I$ where $i =$ Ker α and $h =$ Ker β and $i\psi = h\psi$. Then

$$\mathrm{Sg}^{\mathbf{V}^*}(\{g_1^i, ..., g_n^i\}) = \mathrm{Sg}^{\mathbf{V}^*}(\{g_1^h, ..., g_n^h\}).$$

If $v \in$ Ker α then

$$0 = v\alpha = \sum_k (vg_k^i)(u_k^i\alpha)$$

so that $vg_k^i = 0, 1 \leq k \leq n$. Since each g_k^h is in the span of $\{g_1^i, ..., g_n^i\}$, $vg_k^h = 0$ for $1 \leq k \leq n$. Consequently, $v\beta = 0$ and $v \in$ Ker β. Together with the dual argument this yields that Ker α =Ker β, that is, $i = h$ and ψ is one-one.

Consider now $j \in J$. By Lemma I.2.3 of [11] there exist linearly independent vectors $v_1, ..., v_n \in V$ such that

$$v_i f_k^j = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

The linearity of $f_1^j, ..., f_n^j$ implies that $\alpha \in$ End $\mathbf{V}$, where α is defined by

$$v\alpha = (vf_1^j)v_1 + ... + (vf_n^j)v_n, v \in V.$$

Since $v_j\alpha = v_j$ for $1 \leq j \leq n, \rho(\alpha) = n$. Let $i =$ Ker α. We know that $\{u_1^i\alpha, ..., u_n^i\alpha\}$ and $\{v_1, ..., v_n\}$ are bases of Im α. Let $H \in GL(n, D)$ be the matrix such that

$$H \begin{pmatrix} u_1^i\alpha \\ \cdot \\ \cdot \\ u_n^i\alpha \end{pmatrix} = \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

Then if $\begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \lambda_n \end{pmatrix}$ is the co-ordinate vector of $w \in$ Im α with respect to the basis $\{v_1, ..., v_n\}$, $H^t \begin{pmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \lambda_n \end{pmatrix}$ is the co-ordinate vector of w with respect to the basis $\{u_1^i\alpha, ..., u_n^i\alpha\}$. For each $v \in V$,

$$v\alpha = \sum_k (vg_k^i)(u_k^i\alpha) = \sum_k (vf_k^j)v_k$$

so that

$$v\alpha = \sum_k \sum_\ell h_{\ell k}(vf_\ell^j)(u_k^i\alpha),$$

where $H = (h_{k\ell})$. Hence for each $k \in \{1, ..., n\}$.

$$vg_k^i = \sum_\ell h_{\ell k}(vf_\ell^j) = \sum_\ell (vf_\ell^j)h_{\ell k} = \sum_\ell v(f_\ell^j h_{\ell k}) = v(\sum_\ell f_\ell^j h_{\ell k}).$$

Therefore $g_k^i = \sum_\ell f_\ell^j h_{\ell k} \in j$. A similar argument shows that each f_k^j is in $i\psi$; consequently, $i\psi = j$ and ψ is onto.

For each $i \in I$, let H_i be the matrix in $GL(n, D)$ such that

$$(g_1^i, ..., g_n^i) = (f_1^{i\psi}, ..., f_n^{i\psi})H_i,$$

and for each $\lambda \in \Lambda$ let K_λ be the matrix in $GL(n, D)$ such that

$$\begin{pmatrix} v_1^\lambda \\ \cdot \\ \cdot \\ v_n^\lambda \end{pmatrix} = K_\lambda \begin{pmatrix} w_1^\lambda \\ \cdot \\ \cdot \\ w_n^\lambda \end{pmatrix}.$$

Finally, let $\chi : \Lambda \to \Lambda$ be the identity map. By Theorem III 2.8 of [8], to show that $\boldsymbol{\mathcal{M}}^0(\mathbf{G}; I, \Lambda; P)$ and $\boldsymbol{\mathcal{M}}^0(\mathbf{GL}(n, \mathbf{D}); J, \Lambda; Q)$ are isomorphic, it is enough to show that for all $\lambda \in \Lambda$ and $i \in I$

$$p_{\lambda i} = K_\lambda q_{\lambda\chi i\psi} H_i.$$

Let $\lambda \in \Lambda$ and $i \in I$ and choose $\alpha, \beta \in$ End $\mathbf{V}$ with Im $\alpha = \lambda$ and Ker $\beta = i$. Then $\{v_1^\lambda\beta, ..., v_n^\lambda\beta\}$ spans Im $\alpha\beta$. Write

$$v_\ell^\lambda\beta = t_{\ell 1}(u_1^i\beta) + ... + t_{\ell n}(u_n^i\beta), 1 \leq j \leq n.$$

Then with $T = (t_{\ell k})$,

$$\begin{pmatrix} v_1^\lambda\beta \\ \cdot \\ \cdot \\ v_n^\lambda\beta \end{pmatrix} = T \begin{pmatrix} u_1^i\beta \\ \cdot \\ \cdot \\ u_n^i\beta \end{pmatrix}.$$

Note that $p_{\lambda i} = 0$ if and only if T is *not* invertible and if T *is* invertible, then $p_{\lambda i}\theta = T$. But

$$T = (v_\ell^\lambda g_k^i) = \begin{pmatrix} v_1^\lambda \\ \cdot \\ \cdot \\ v_n^\lambda \end{pmatrix} (g_1^i, ..., g_n^i) = K_\lambda \begin{pmatrix} w_1^\lambda \\ \cdot \\ \cdot \\ w_n^\lambda \end{pmatrix} (f_1^{i\psi}, ..., f_n^{i\psi})H_i$$

so that T is invertible if and only if

$$\begin{pmatrix} w_1^\lambda \\ \cdot \\ \cdot \\ w_n^\lambda \end{pmatrix} (f_1^{i\psi}, ..., f_n^{i\psi})$$

is invertible. Hence $p_{\lambda i} = 0$ if and only if $q_{\lambda\chi i\psi} = 0$. Further, if $p_{\lambda i}$ is not zero, then

$$p_{\lambda i}\theta = T = K_\lambda q_{\lambda\chi i\psi} H_i$$

as required.

Our final example concerns End $\mathbf{S}$, where $\mathbf{G}$ is a group and $\mathbf{S} = F_{\mathbf{G}}(X)$ is the free $\mathbf{G}$ -set over a non-empty set X. It follows from Theorem 4.12 that if $n \leq |X|$ then $T_n^{\mathbf{S}}/T_{n-1}^{\mathbf{S}}$ is isomorphic to $\boldsymbol{\mathcal{M}}^{\mathbf{0}}(\mathbf{H}; I, \Lambda; P)$ where $\mathbf{H}$ is the group of automorphisms of $Gx_1\dot{\cup}...\dot{\cup}Gx_n$ for some fixed $x_1, ..., x_n \in X$. Let $\boldsymbol{\mathcal{S}_n}$ be the symmetric group on n variables. The *wreath product* $\mathbf{G}\wr_N\boldsymbol{\mathcal{S}_n}$, where $N = \{1, ..., n\}$ is defined by

$$\mathbf{G} \wr_N \boldsymbol{\mathcal{S}_n} = G^n \times \mathcal{S}_n$$

with multiplication given by

$$(g_1, ..., g_n, \alpha)(h_1, ..., h_n, \beta) = (gh_{1\alpha}, ..., gh_{n\alpha}, \alpha\beta).$$

Then $\theta : H \to G \wr_N \mathcal{S}_n$ defined by

$$\alpha\theta = (g_1, ..., g_n, \overline{\alpha})$$

where for $i \in \{1, ..., n\}$, $x_i\alpha = g_i x_{i\overline{\alpha}}$ is an isomorphism. Our last Corollary again calls upon Theorem III 2.8 of [8]. The author is grateful to Dr. J.D.P. Meldrum for the above description of $\mathbf{H}$ as a wreath product.

Corollary 5.3 *Let* $\mathbf{S} = F_{\mathbf{G}}(X)$ *and* θ *be as above. Then if* $1 \leq n \leq |X|$, $T_n^{\mathbf{S}}/T_{n-1}^{\mathbf{S}}$ *is isomorphic to* $\boldsymbol{\mathcal{M}}^{\mathbf{0}}(\mathbf{G}\wr_N\boldsymbol{\mathcal{S}_n}; I, \Lambda; P\theta')$ *where* $P\theta' = (p_{\lambda i}\theta)$, *defining* $0\theta = 0$.

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