A Short History of Inverse Semigroups

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Overview

1. A little theory
2. Pseudogroups
3. V. V. Wagner and generalised groups
4. G. B. Preston and inverse semigroups
5. Concluding remarks

What is an inverse semigroup?

Let S be a semigroup; $s' \in S$ is a generalised inverse for $s \in S$ if

$$ss's = s \qquad \text{and} \qquad s'ss' = s'.$$

Call S an inverse semigroup if every element has precisely one generalised inverse.

Equivalently, an inverse semigroup is a semigroup in which

1. every element has at least one generalised inverse;
2. idempotents commute with each other.

Why?

A partial bijection on a set X is a bijection $A \to B$, where $A, B \subseteq X$. We compose partial transformations α, β (left $\to$ right) on the domain

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1}$$

and put $x(\alpha\beta) = (x\alpha)\beta$, for $x \in \operatorname{dom} \alpha\beta$.

Let $\mathcal{I}_X$ denote the collection of all partial bijections on X, under this composition. Every $\alpha : A \to B$ in $\mathcal{I}_X$ has an inverse $\alpha^{-1} : B \to A$ in $\mathcal{I}_X$. In fact, $\mathcal{I}_X$ is an inverse semigroup — the symmetric inverse semigroup on X.

Idempotents in $\mathcal{I}_X$: partial identity transformations I_A, $A \subseteq X$. These commute: $I_A I_B = I_{A \cap B} = I_{B \cap A} = I_B I_A$.

The Wagner–Preston representation

Every inverse semigroup can be embedded in a symmetric inverse semigroup.

Map $s \in S$ to the partial transformation $\rho_s \in \mathcal{I}_S$ with $\operatorname{dom} \rho_s = Ss^{-1}$ and $x\rho_s = xs$, for $x \in \operatorname{dom} \rho_s$.

The Erlanger Programm

Felix Klein: every geometry may be regarded as the theory of invariants of a particular group of transformations.

$$\text{Groups} \longleftrightarrow \text{Geometries}$$

Veblen and Whitehead:

> *"[the Erlanger Programm] ... supplied a principle of classification by which it is possible to get a bird's-eye view of the relations between a large number of important geometries."*

However...

Veblen and Whitehead:

> *"... long before the Erlanger Programm had been formulated there were geometries in existence which did not properly fall within its categories, namely the Riemannian geometries."*
>
> *"There is, therefore, a strong tendency among contemporary geometers to seek a generalization of the Erlanger Programm which can replace it as a definition of geometry by means of the group concept."*

Seek algebraic structure which would serve to describe the symmetries of any geometry.

Veblen and Whitehead

The foundations of differential geometry (1932).

‘Axiomatising’ differential geometry

“Any mathematical science is a body of theorems deduced from a set of axioms. A geometry is a mathematical science.”

‘Axiomatisation’: abstract description of the structure of partial homeomorphisms.

Pseudogroups

A pseudogroup Γ is a collection of partial homeomorphisms between open subsets of a topological space such that Γ is closed under composition and inverses, where we compose $\alpha, \beta \in \Gamma$ only if im $\alpha =$ dom β.

Use pseudogroups of 'regular' (i.e., one-one) partial homeomorphisms to classify 'geometric objects' ('invariants').

> *"[classification] in the spirit of the Erlanger Programm."*

‘Abstraction’ of pseudogroups

Look for abstract structure corresponding to pseudogroup, just as abstract group corresponds to group of permutations.

But partially-defined operation difficult to work with, so first seek to ‘complete’ the operation in a pseudogroup.

Schouten and Haantjes

On the theory of the geometric object (1937).

A new composition

Compose partial transformations α, β only if $\operatorname{im} \alpha \subseteq \operatorname{dom} \beta$.

But still not fully defined.

Moreover, don't have such nice properties as $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$.

Stanisław Gołąb

Über den Begriff der "Pseudogruppe von Transformationen" (1939).

Clarifying pseudogroups

Gołąb's goal: to 'tighten up' the concept of a pseudogroup.

Now compose α, β if $\operatorname{im} \alpha \cap \operatorname{dom} \beta \neq \emptyset$.

But still partial!

Two types of pseudogroups

'Pseudogruppen im weiteren Sinne' $\mathfrak{G}$:

1. For any $\alpha \in \mathfrak{G}$, $\operatorname{dom}\alpha$ is an open set.
2. For any $\alpha \in \mathfrak{G}$ and an open set S, $\alpha|_S \in \mathfrak{G}$.
3. Can compose $\alpha, \beta \in \mathfrak{G}$ only if $\operatorname{im}\alpha \cap \operatorname{dom}\beta \neq \emptyset$.
4. For any $\alpha \in \mathfrak{G}$, $a \in \operatorname{dom}\alpha$, there is an open set $D \subseteq \operatorname{dom}\alpha$ and a transformation $\beta \in \mathfrak{G}$ such that $a \in D$, $\operatorname{dom}\beta = D\alpha$ and $\alpha\beta = I_D$.

'Pseudogruppen im engeren Sinne' $\mathfrak{G}$:

5. Let $\alpha, \beta \in \mathfrak{G}$ and let $S \subseteq \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ be and open set such that $\alpha|_S = \beta|_S$. Then $\alpha|_{\operatorname{dom}\alpha \cap \operatorname{dom}\beta} = \beta|_{\operatorname{dom}\alpha \cap \operatorname{dom}\beta}$.
6. Let $\alpha \in \mathfrak{G}$ and let T be an open set. Then there exists $\beta \in \mathfrak{G}$ such that $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$, $\operatorname{dom}\beta \cap T \neq \emptyset$ and $\beta|_{\operatorname{dom}\alpha} = \alpha$.

Two types of pseudogroups

'Wide' pseudogroup much the same as previous notions of pseudogroup.

'Narrow' pseudogroups introduced because it is possible to construct a group from any such:

Define equivalence relation R on $\mathfrak{G}$: $\alpha \, R \, \beta$ iff for any extensions α^* of α and β^* of β in $\mathfrak{G}$ for which there exists a nonempty $S \subseteq \operatorname{dom} \alpha^* \cap \operatorname{dom} \beta^*$, $\alpha^*|_S = \beta^*|_S$.

In fact, R is a congruence and $\mathfrak{G}/R$ is a group.

(R — minimum group congruence; 'narrow' pseudogroup $\sim$ E-unitary inverse semigroup.)

Incomplete composition

Have arrived at a composition which is almost fully defined: $\alpha\beta$ exists only if $\operatorname{im}\alpha \cap \operatorname{dom}\beta \neq \emptyset$.

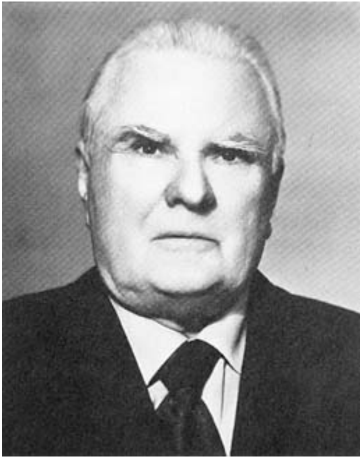
It only remains to take account of the possibility of that $\operatorname{im}\alpha \cap \operatorname{dom}\beta = \emptyset$.

To modern eyes, this is easy: in this case, put $\alpha\beta = \varepsilon$, the empty transformation on X — this acts as a zero in $\mathcal{I}_X$.

Schein:

> *"It is interesting to compare the history of the integers. Positive integers came first. Negative integers followed. Zero came last (the number of objects in a set without objects was the most difficult psychological step)."*

Viktor Vladimirovich Wagner



Виктор Владимирович Вагнер (1908–1981).

Wagner and differential geometry

Added appendix ‘The theory of differential objects and the foundations of differential geometry’ to Russian translation of Veblen and Whitehead’s ‘The foundations of differential geometry’.

Develops rigorous theory of geometric objects, building on the work of Gołąb, et. al.

Pseudogroups of partial one-one transformations emerge as being important.

Composition of partial transformations

'On the theory of partial transformations' (1952):

A partial transformation α on a set X may be expressed as a binary relation:

$$\{(x, y) \in \operatorname{dom}\alpha \times \operatorname{im}\alpha : x\alpha = y\} \subseteq X \times X.$$

Then composition of partial transformations is a special case of that of binary relations:

$$x\,(\rho \circ \sigma)\,y \iff \exists z \in X \text{ such that } x\,\rho\,z \text{ and } z\,\sigma\,y.$$

Since $\emptyset \subseteq X \times X$, the empty transformation now appears naturally in the theory.

Semigroups of binary relations

Let $\mathfrak{B}(A \times A)$ be the semigroup of all binary relations on a set A.

$\mathfrak{B}(A \times A)$ is ordered by $\subset$, which is compatible with composition.

Canonical symmetric transformation $^{-1}$: $x\,\rho^{-1}\,y \iff y\,\rho\,x$.

$\mathfrak{M}(A \times A)$: collection of all partial one-one transformations on A.

$^{-1}$ and $\subset$ may be expressed in terms of composition in $\mathfrak{M}(A \times A)$:

$$\rho_2 = \rho_1^{-1} \iff \rho_1\rho_2\rho_1 = \rho_1 \text{ and } \rho_2\rho_1\rho_2 = \rho_2;$$

$$\rho_1 \subset \rho_2 \iff \exists\rho \text{ such that } \rho_1\rho\rho_1 = \rho_1,\ \rho_2\rho\rho_2 = \rho_2 \text{ and } \rho\rho_2\rho = \rho.$$

Generalised groups (1952)

First page: modern definition of an inverse semigroup, here called a generalised group.

Theorem. Every symmetric semigroup of partial one-one transformations of a set forms a generalised group with respect to composition of partial transformations.

Theorem. Every generalised group may be represented as a generalised group of partial one-one transformations.

Back to differential geometry

Let M be an n-dimensional differentiable manifold. Such a manifold has a coordinate atlas A: a set of partial one-one transformations from M into $\mathbb{R}^n$. Each $\kappa \in A$ represents a local system of coordinates; $\kappa(m) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are the coordinates of $m \in M$.

Apply a ternary operation to $\kappa, \lambda, \mu \in A$:

$$[\kappa\,\lambda\,\mu] = \kappa \circ \lambda^{-1} \circ \mu.$$

More generally, we can apply such an operation to $\mathfrak{B}(A \times B)$ — the collection of all binary relations between sets A and B, i.e., subsets of $A \times B$.

Abstract ternary operations

Let K be a set. We define an abstract ternary operation $[\cdot\ \cdot\ \cdot]$ on K.

Call the operation pseudo-associative if

$$[[k_1\ k_2\ k_3]\ k_4\ k_5] = [k_1\ [k_4\ k_3\ k_2]\ k_5] = [k_1\ k_2\ [k_3\ k_4\ k_5]]\,.$$

In this case, call K a semiheap.

A semiheap forms a heap if

$$[k_1\ k_2\ k_2] = [k_2\ k_2\ k_1] = k_1.$$

$\mathfrak{B}(A \times B)$ forms a semiheap.

Generalised heaps

A semiheap K is a generalised heap if:

$$[[k\ k_1\ k_1]\ k_2\ k_2] = [[k\ k_2\ k_2]\ k_1\ k_1],$$
$$[k_1\ k_1\ [k_2\ k_2\ k]] = [k_2\ k_2\ [k_1\ k_1\ k]],$$
$$[k\ k\ k] = k.$$

Let $\mathfrak{K}(A \times B) \subseteq \mathfrak{B}(A \times B)$ be the collection of all partial one-one transformations from a set A to a set B.

$\mathfrak{K}(A \times B)$ forms a generalised heap.

Every abstract generalised heap may be embedded in some $\mathfrak{K}(A \times B)$.

Semigroups and semiheaps

Let S be a semigroup with involution: $(s')' = s, \quad (s_1 s_2)' = s_2' s_1'$.

Define a ternary relation $[s_1\ s_2\ s_3] = s_1 s_2' s_3$. S forms a semiheap under this operation.

Let K be a semiheap. $b \in K$ is a biunitary element if $\forall k \in K$

$$[k\ b\ b] = [b\ b\ k] = k.$$

For any biunitary element $b \in K$, define a binary operation and involution by

$$s_1 s_2 = [s_1\ b\ s_2] \qquad \text{and} \qquad s' = [b\ s\ b].$$

Under these operations, K is a semigroup with involution.

Other types of semiheaps

semiheap	$\longleftrightarrow$	semigroup
heap	$\longrightarrow$	group

Other types of semiheaps

semiheap	$\longleftrightarrow$	semigroup
heap	$\longleftrightarrow$	group
generalised heap	$\longleftrightarrow$	generalised group

In terms of binary relations

$$\text{Semiheap } \mathfrak{B}(A \times B) \quad \longleftrightarrow \quad \text{Semigroup } \mathfrak{B}(A \times A)$$

$$\text{Generalised heap } \mathfrak{K}(A \times B) \quad \longleftrightarrow \quad \text{Generalised group } \mathfrak{K}(A \times A)$$

Gordon Preston

b. 1925.

Preston’s DPhil thesis

‘Some problems in the theory of ideals’ (Oxford, 1953).

Defines suitable notion of ‘ideal’ for universal algebras and obtains analogues of results for ideals in rings.

Inverse semigroups appear in final chapter of thesis.

Inspired by a paper by David Rees.

David Rees and partial transformations

Studies partial one-one transformations of set; composes α, β only if $\operatorname{im} \alpha \cap \operatorname{dom} \beta \neq \emptyset$.

Regular set of partial transformations Σ:

1. if $\alpha, \beta \in \Sigma$, then $\alpha\beta$ exists and belongs to Σ;
2. if $\alpha \in \Sigma$, then $\alpha^{-1} \in \Sigma$.

Defines $\sim$ on Σ by $\alpha \sim \beta$ iff α, β have a common subtransformation. Shows that $G = \Sigma / \sim$ is a group.

In fact, Rees' $\sim$ is Gołąb's R (i.e., σ, the minimum group congruence).

Uses all this to provide (new?) proof that a commutative, cancellative semigroup can be embedded in a group.

Preston's mapping semigroups

Preston sought to axiomatise Rees' regular sets of partial transformations.

Let S be a semigroup with 0. For $a \in S$, denote by E_a the set of idempotents e for which $ea \neq 0$. Call S a mapping semigroup if

M1. if $s \in S$, then there is at least one element $e \in S$ for which $es = s$ and such that the equation $sx = e$ has a solution $x \in S$;

M2. if e, f are idempotents in S, then $ef = fe$;

M3. if $a \in S$ and $E_a \neq \emptyset$, then there is at least one $e \in E_a$ such that $ef = f$ and $f^2 = f$ together imply that either $f = 0$ or $f = e$.

A mapping semigroup S is a complete mapping semigroup if:

M4. if $a, b \in S$ have the property that, for all $x, y \in S$, $xay = 0$ if and only if $xby = 0$, then $a = b$.

Preston's mapping semigroups

So mapping semigroups are certainly inverse semigroups — but apparently with two extra conditions.

In fact, there is only one extra condition, since M3 is a consequence of M1 and M2.

But M4 is independent of M1 and M2.

Don't quite have a complete axiomatisation: an arbitrary inverse semigroup with 0 need not satisfy M4.

Inverse semigroups

Inspired by Whitehead's incorrect comments, Preston developed a refined version of his work on mapping semigroups.

Defines inverse semigroups by M1 and M2, in series of 3 papers in 1954.

Quite 'group-like' in approach: introduces notion of 'normal' inverse subsemigroup which is the 'kernel' of a congruence.

Inverse semigroups with minimal right ideals

Second paper of 1954.

Studies ideals in inverse semigroups.

For example, shows that $S = \bigcup_{e \in P} SeS$, where $P \subseteq E(S)$ is the collection of primitive idempotents of S.

Representations of inverse semigroups

Only paper of the three to deal with partial transformations.

Subtle difference: compose α, β in the usual way if $\operatorname{im}\alpha \cap \operatorname{dom}\beta \neq \emptyset$; if $\operatorname{im}\alpha \cap \operatorname{dom}\beta = \emptyset$, define $\alpha\beta = 0$, for some abstract symbol 0. Demand further that $0^2 = 0$ and $\alpha 0 = 0 = 0\alpha$.

Another subtle point to note: studies bijections $\alpha : A \to A'$ and $\beta : B \to B'$, and considers whether or not $A' \cap B = C$ is empty. Does not consider A, B, C, etc. to be subsets of some all-containing set.

Wagner–Preston representation

‘Semigroup of (1-1) mappings’ is an inverse semigroup; every inverse semigroup can be embedded in a semigroup of (1-1) mappings.

Proves that M1 and M2 are independent.

(Completely simple semigroup satisfies M1 but not M2; but

	a	e	f
a	e	e	f
e	e	e	f
f	f	f	f

satisfies M2 but not M1, since *a* has no left identity.)

The connection is made...

At end of third paper: note thanking B. H. Neumann for drawing Preston's attention to a paper by E. S. Lyapin, citing Wagner...

Subsequent approaches to inverse semigroups

1. proper/E-unitary inverse semigroups, etc.;
2. fundamental inverse semigroups and Munn representations;
3. inductive groupoids.

The End