Comparison Semigroups and Function Algebra

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The theory of *relation algebras* was Tarski's attempt to model the algebra of binary relations on a set X – subsets of $X \times X$.

Many operations are defined:

- relational composition $\cdot$ (generalises function composition)
- all the usual set operations: $\cup$, $\cap$, $'$, plus "nullaries" 0 (the empty relation) and U (the universal relation)
- relational converse $*$ (reverse all pairs)
- the diagonal relation 1 (= identity function)

Tarski came up with an apparently comprehensive set of laws for relation algebra. They are algebras $(B, \cdot, 0, 1, U, \cap, \cup, ', ^*)$ for which:

- $(B, 0, U, \cap, \cup, ')$ is a Boolean algebra
- $(B, 0, 1, \cdot)$ is a monoid with zero
- $x^{**} = x$, $(x \cdot y)^* = y^* \cdot x^*$
- $(x \cup y) \cdot z = (x \cdot z) \cup (y \cdot z)$
- $(x^* \cdot (x \cdot y)') \cup y' = y'$

But this class of “relation algebras” still fails to capture actual algebras of relations!

There are relation algebras that are not concrete algebras of relations on any set X.

The class of concrete relation algebras is an infinitely based variety (needs an infinite set of equational axioms to specify it).

So there is no “finite axiomatization” of concrete relation algebras.

How about the set $P(X)$ of all partial functions $X \to X$?

These are functions whose domain may not be all of X, ie “functional binary relations”.

They are good models for the action of computer programs (undefined means “doesn’t halt”).

Possible operations now include composition (still associative), but also several set-theoretic and other operations.

These operations are:

- *intersection* $f \cap g$
- *set difference* $f \backslash g$
- nullaries such as 0 and the identity map 1
- *domain* $D(f)$: the restriction of 1 to the domain of f
- *tie* $f \bowtie g$: the restriction of 1 to where f, g do not disagree
- *preferential union* $f \sqcup g$: f where it is defined together with g wherever f is undefined.

Each of these operations, when combined with composition, has previously been axiomatized.

Some combinations of them have been axiomatized as well.

For example:

- abstract function $\{\}$-semigroups are exactly… *semigroups*

Axiom:

$$(fg)h = f(gh).$$

Also axiomatizes *transformations* on a set.

Injective transformations are axiomatized by the above plus:

$$fh = gh \ \Rightarrow f = g \text{ and } fh = h \Rightarrow f = 1.$$

But for injective partial functions, there is no finite axiomatization!

It is a proper quasivariety. Axioms have been found (Schein, 1961).

If inversion is added in, we get the finitely based variety of *inverse semigroups*: (Vagner, Preston).

Example: function semigroups with intersection

Axiomatized by Garvac'kii, 1971. A semigroup (of course), plus:

- $f \cap f = f$
- $f \cap g = g \cap f$
- $f \cap (g \cap h) = (f \cap g) \cap h$

There are also interactions with composition:

We have:

- $f(g \cap h) = fg \cap fh$

- $(f_1 \cap f_2 \cap f_3)g \cap (f_1 \cap f_2) = (f_1 \cap f_3) \cap (f_1 \cap f_3)g$.

Injective functions axiomatized as above plus:

$$fg \cap fh \cap kh \cap kg = fg \cap fh \cap kh.$$

Other examples include:

Function semigroups with domain D.

- Schweizer and Sklar (all but – 1967)
- Trokhimenko (1973)
- Jackson and Stokes (2001)
- Manes (2006).

Functions semigroups with set difference $\setminus$:

axiomatized by Schein (1992).

Function semigroups with $\bowtie$ and $\emptyset$ (equivalently, D and $\setminus$):

Jackson and Stokes (2010).

(Can add preferential union there too.)

So, much work has been done on enriched semigroups of functions and binary relations on a set.

Many more combinations of operations (and relations) have been axiomatized over the last several decades.

How about semigroups of *transformations*?

The semigroup law $f(gh) = (fg)h$ axiomatizes algebras of transformations under composition.

Do any other operations make sense on transformations?

For transformations f, g, h, k on X, define

$$(f,g)[h,k](x) := \begin{cases} h(x) & \text{if } f(x) = g(x) \\ k(x) & \text{otherwise.} \end{cases}$$

This is the *comparison* operation on $T(X)$.

It is just a pointwise-defined *switching function* operation.

$(T(X), (\ ,\ )[\ ,\ ], \cdot)$ is an algebra with one quaternary and one binary operation. Hence so is any subalgebra.

Can we axiomatize such subalgebras?

First, let's drop the semigroup part of the signature, and just model functions $X \rightarrow Y$ closed under comparison (formally defined as above).

In fact Kennison has axiomatized these, as *comparison algebras*:

- $(a,a)[b,c] = b$
- $(a,b)[c,c] = c$
- $(a,b)[c,d] = (b,a)[c,d]$
- $(a,b)[a,b] = b$

Plus...

- $(a,b)[(c_1,d_1)[e_1,f_1],(c_2,d_2)[e_2,f_2]]$
  $= ((a,b)[c_1,c_2],(a,b)[d_1,d_2])[(a,b)[e_1,e_2],(a,b)[f_1,f_2]]$

Let us call a set with a switching function a *switching algebra*.

Switching algebras are easily seen to be comparison algebras.

Hence so is the algebra of functions $X \to Y$ under comparison,

as it is a direct product of $|X|$ copies of the switching algebra on Y.

Let's show every comparison algebra is $\cong$ to a functional one.

Let A be a non-trivial comparison algebra with $a, b \in A$ unequal.

Define $(x, y) \in \rho_{a,b} \;\Leftrightarrow\; y = (a, b)[x, y]$.

This is the smallest congruence containing (a, b).

Now note that A is simple if and only if $\rho_{a,b} = \nabla$ for all unequal a, b.

That is, $(a, b)[x, y] = y$ for all $a, b, x, y \in A$ with $a \neq b$.

On the other hand, $(a, a)[x, y] = x$ from our laws!

So A is simple if and only if it is a switching algebra. (Kennison, 1981.)

Now also note that $\theta_{a,b}$ given by

$$(x, y) \in \theta_{a,b} \;\Leftrightarrow\; x = (a, b)[x, y]$$

is a congruence, *not* containing a, b.

Use Zorn to extend $\theta_{a,b}$ to a congruence $\tau_{a,b}$ maximal with respect to not containing (a, b).

But for any $x, y \in A$, $x \; \rho_{a,b} \; (a, b)[x, y] \; \theta_{a,b} \; y$.

So $\rho_{a,b} \circ \theta_{a,b}$ is the full relation.

It follows that $\tau_{a,b}$ is a maximal congruence on A.

So $A/\tau_{a,b}$ is a simple comparison algebra, hence a switching algebra.

Now note that $\bigcap_{a \neq b} \tau_{a,b} = \Delta$.

So A is a subdirect product of switching algebras.

But any direct product of switching algebras can be interpreted as a functional comparison algebra:

view the index set as domain, and the union of the algebras as the co-domain.

Theorem: Every comparison algebra embeds in a functional one.

How to adapt this to transformation cases, with composition present too?

The following laws on $T(X)$ are routinely verified.

- composition $\cdot$ is associative (of course)
- $(a, b)[c, d] \cdot e = (a, b)[ce, de]$
- $e \cdot (a, b)[c, d] = (ea, eb)[ec, ed]$

Call the resulting variety of signature $(4, 2)$ the class of *comparison semigroups*.

A *comparison monoid* is a comparison semigroup in which there is an identity element (modelling the identity transformation).

Are the comparison semigroup/monoid axioms complete for the intended models?

We extend the previous approach for comparison algebras.

Call any congruence on the comparison algebra reduct of a comparison semigroup S a *pre-congruence* on S.

A pre-congruence on S is a right congruence on $(S, \cdot)$:

if $x \ \theta \ y$ then

$$ya = (x, y)[x, y] \cdot a = (x, y)[xa, ya] \ \theta \ (x, x)[xa, ya] = xa.$$

Now suppose S has 1, and θ is a maximal precongruence.

For each $a \in S$, define $\psi_a \in T(S/\theta)$ by setting

$$\psi_a(\overline{x}) = \overline{xa}.$$

(If S is not a monoid, need to add in a 1-like element to S/θ.)

Then $f_\theta : S \to T(S/\theta)$ such that $a \mapsto \psi_a$ is a monoid homomorphism (very easy to see).

But because S/θ is a switching algebra, it also follows easily that f_θ is a comparison monoid homomorphism.

Note that for $a \neq b$, under $\theta = \tau_{a,b}$, a maximal pre-congruence,

$$\psi_a(\bar{1}) = \bar{a} \neq \bar{b} = \psi_b(\bar{1}).$$

So $(a, b) \notin ker(f_{\tau_{a,b}})$.

Hence once again,

$$\bigcap_{\theta \text{ a maximal precongruence on } S} ker(f_\theta) = \Delta,$$

so we get a faithful subdirect product representation.

But the functional examples are closed under subalgebras and direct products (not hard to show). So...

Theorem: Every comparison semigroup embeds into $T(X)$ for some X.

But that’s not all!

Thanks to a neat trick, we can also model partial function semigroups equipped with a natural comparison operation.

How to extend the definition of comparison to $P(X)$?

For $f, g, h, k \in P(X)$, define

$$(f,g)[h,k](x) := \begin{cases} h(x) & \text{if } f(x) = g(x) \textit{or both are undefined} \\ k(x) & \text{otherwise.} \end{cases}$$

In fact, *all the same laws hold as for* $T(X)$.

One way to see this is to introduce a new element 0 to X and for $f \in P(X)$, let all undefined values map to 0 (and 0 map to 0 too).

In this way, $P(X)$ faithfully embeds into $T(X')$ where $X' = X \cup \{0\}$.

So *the same axiomatization holds for partial transformations as for transformations.*

This is what happens with semigroups of transformations and partial transformations: “one size fits all”.

However, we might also like to have a zero element $\emptyset$, modelling the empty function in $P(X)$.

(Doesn't make sense for $T(X)$.)

The following additional laws prove necessary and sufficient.

- $\emptyset$ is a zero of the semigroup

...and that's all we need.

(Not hard to extend the proof to cover this.)

With $\emptyset$ in hand, many established operations on $P(X)$ can be expressed.

- Intersection (Garvackii) is expressible via $f \cap g := (f, g)[f, \emptyset]$.

- Set-theoretic difference (Schein) is $f \backslash g := (f, f \cap g)[\emptyset, f]$.

- Preferential union (Jackson and Stokes):

  $f \sqcup g$ is f together with g where it does not conflict with f

$$f \sqcup g := (f, \emptyset)[g, f]$$

With $\emptyset$ *and* 1 in hand, even more can be expressed.

- The domain of f (Trokhimenko, Schweizer and Sklar, Jackson and Stokes, Manes etc etc.):

$$D(f) := (f, \emptyset)[\emptyset, 1]$$

- Domain complement P (Jackson and Stokes, 2010):

$$P(f) := (f, \emptyset)[1, \emptyset]$$

- Hence also restriction of the identity to the region of non-disagreement of f, g (Jackson and Stokes):

$$(f \bowtie g) := D(f \cap g) \sqcup P(f)P(g)$$

Conversely,

$$(f, g)[h, k] = (f \bowtie g)h \sqcup k.$$

So there is an equivalence of operation sets:

$$\{\emptyset, 1, \cdot, (\ ,\ )[\ ,\ ]\} \leftrightarrow \{\emptyset, \cdot, \bowtie, \sqcup\} \leftrightarrow \{\cdot, \cap, \sqcup, P\}.$$

The first of these won't handle *reducts* such as $\{\emptyset, \cdot, \bowtie\}$.

On the plus side it makes sense for *transformations*.