Restriction and ample semigroups: constructions and Mária Szendrei's work

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What is this talk about?

The classes of semigroups under consideration:

Inverse, (left) ample and (left) restriction.

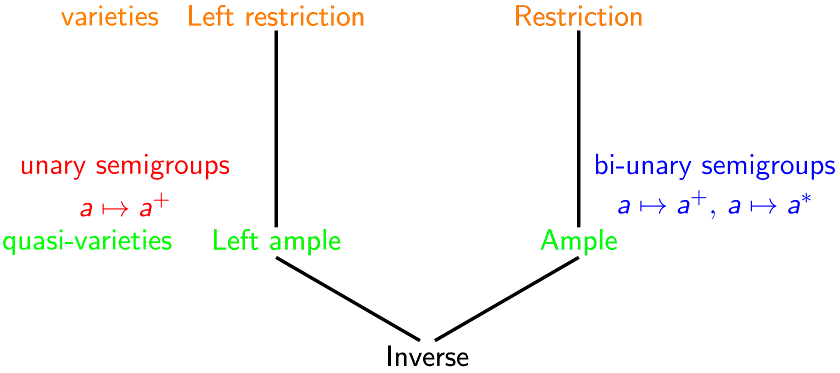
Three cameos illustrating Mária's insights

1. The Szendrei expansion
2. Embedding into W-products
3. Term functions for restriction semigroups:

Constructions

The notion of semidirect product is key.

The semigroups we will consider
Inverse, (left) restriction and (left) ample semigroups



S will always denote a semigroup/(bi-)unary semigroup

$E(S)$ is the set of **idempotents** of S; and $E \subseteq E(S)$

Semidirect products

Let T be a monoid acting on the **left** of a semilattice Y by **morphisms**

That is, there is a map

$$T \times Y \to Y, \ (s, a) \mapsto s \cdot a$$

such that for all $s, t \in T, a, b \in Y$:

$$1 \cdot a = a, \ s \cdot (t \cdot a) = (st) \cdot a \text{ and } s \cdot (a \wedge b) = (s \cdot a) \wedge (s \cdot b).$$

The semidirect product $Y \rtimes T$ on $Y \times T$

$$(a, s)(b, t) = (a \wedge s \cdot b, st).$$

A **reverse** semidirect product $T \ltimes Y$ is obtained by T acting on the **right** of Y by morphisms.

Guiding case: Inverse semigroups

S inverse

1. $E(S)$ is a **semilattice**
2. An inverse semigroup **all** of whose elements are idempotent is a **semilattice**
3. An inverse semigroup with exactly **one** idempotent is a **group**
4. S is naturally **partially ordered**

Many approaches to inverse semigroups

Aim to describe them in terms of **groups** and **semilattices**.

Let G be a group, T a monoid

The semidirect product $Y \rtimes G$ is inverse; $Y \rtimes T$ is left restriction;

$$Y \cong Y \rtimes \{1\} \leq Y \rtimes T.$$

Inverse semigroups: F-inverse and proper

Let S be inverse: some well known facts

1. $\sigma = \langle E(S) \times E(S) \rangle$ is the **least group congruence** on S
2. S is F-**inverse** if every σ-class has a **maximum element**
3. If S is F-inverse then it is **proper**, that is,

$$aa^{-1} = bb^{-1} \text{ and } a\,\sigma\,b \Rightarrow a = b.$$

   Equivalently, $S \to E(S) \times S/\sigma$ *given by*

$$s \mapsto (ss^{-1}, s\sigma)$$

   is a **SET** *embedding*

4. **O'Carroll (1978)** S is proper if and only if embeds into some $Y \rtimes G$
5. **McAlister (1974)** Every inverse semigroup has a proper cover.

Cameo 1: The Szendrei expansion

Birget-Rhodes (1984)

An **expansion** is a functor $\mathcal{E}$ from the category of semigroups to a special subcategory, such that there is a natural tranformation η from $\mathcal{E}$ to the identity functor with $\eta(S)$ surjective for every S.

$$\begin{array}{ccc}
\mathcal{E}(S) & \xrightarrow{\mathcal{E}(\theta)} & \mathcal{E}(T) \\
{\scriptstyle \eta(S)}\downarrow & & \downarrow{\scriptstyle \eta(T)} \\
S & \xrightarrow[\theta]{} & T
\end{array}$$

Cameo 1: The Szendrei expansion

Birget-Rhodes (1984): prefix expansion

$\mathcal{E}(S)$ is given by

$$\widetilde{S}^{\mathcal{R}} := \left\{ \left(\{1, s_1, s_1 s_2, \ldots, s_1 s_2 \ldots s_n\}, s_1 \ldots s_n\right) : s_i \in S, n \geq 1 \right\}$$

with semidirect product multiplication; $\eta(S)$ is π_2.

Szendrei (1989)

For a group G, this expansion is given by

$$\widetilde{G}^{\mathcal{R}} = \{(A, g) : A \in \mathcal{P}_1(G), g \in A\}$$

where $\mathcal{P}_1(S)$ is the set of finite subsets of S containing 1, for any monoid S.

Cameo 1: The Szendrei expansion

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For a group G, this expansion is given by

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where $\mathcal{P}_1(S)$ is the set of finite subsets of S containing 1, for any monoid S.

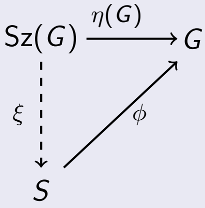
Cameo 1
The Szendrei expansion

1. **Birget-Rhodes (1984)** The free inverse semigroup on X is a subsemigroup of $\mathrm{Sz}(\mathrm{FG}(X))$
2. **Szendrei (1989)** $\mathrm{Sz}(G)$ is F-inverse and $\mathrm{Ker}\,\eta(S) = \sigma$. Further, $\mathrm{Sz}(G)$ has universal properties with respect to being an F-inverse expansion.
3. **Exel (1998), Kellendonk and Lawson (2004)** $G \to \mathrm{Sz}(G)$, $g \mapsto (\{1, g\}, g)$ is a premorphism. Further, $\mathrm{Sz}(G)$ is universal with respect to premorphisms and hence with respect to lifting partial actions to actions.

Cameo 1
The Szendrei expansion

Szendrei (1989)

For every group G, the pair $(\mathrm{Sz}(G), \eta)$ has the property that, whenever S is an F-inverse semigroup and $\phi : S \to G$ is an onto morphism with $\mathrm{Ker}\,\phi = \sigma$, then there is a unique m[a]-morphism $\xi : \mathrm{Sz}(G) \to S$ such that



commutes. This property uniquely determines $\mathrm{Sz}(G)$.

[a] preserving the max elements of σ-classes

Cameo 1
The Szendrei expansion

$$\text{Sz}(S) = \{(A, a) : A \in \mathcal{P}f_1(S), a \in A\}$$

Fountain, Gomes, G., Hollings (1990)-(2007)

All of the above can be extended to **left ample** and **left restriction** semigroups, replacing G with a right cancellative monoid or a monoid.

Kudryavtseva (2018), 11.00 13/07/2018

...and they can be extended to **ample** and **restriction** semigroups, replacing G with a cancellative monoid or a monoid.

(Left) restriction and (left) ample semigroups

A unary semigroup is left restriction if

S satisfies the identities:

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

S is **left ample** if in addition S satisfies the quasi-identity

$$xy = zy \to xy^+ = zy^+.$$

If S is left restriction, then $E = \{a^+ : a \in S\}$ is the **semilattice of projections**. If S is left ample, then $E = E(S)$.

Restriction and ample semigroups

A bi-unary semigroup is restriction (ample) if it is left and right restriction (ample) and the semilattices of projections coincide.

Restriction and ample semigroups: observations, examples

1. A unary semigroup S left restriction iff it embeds into $\mathcal{PT}_S$ where α^+ is the identity map in the domain of α.
2. A unary semigroup is S is left ample iff it embeds into $\mathcal{I}_S$.
3. Inverse semigroups are ample under $a \mapsto a^+ = aa^{-1}$, $a \mapsto a^* = a^{-1}a$.
4. Any bi-unary subsemigroup of an inverse semigroup is ample.
5. Free (left) restriction/ample semigroups embed into free inverse semigroups.
6. (Left) restriction semigroups are (left) Ehresmann; (Left) ample semigroups are(left) adequate.
7. Monoids are **reduced** restriction under $a^+ = a^* = 1$, (right) cancellative monoids are (left) ample.
8. A semidirect product $Y \rtimes M$ where Y is a semilattice and M is a (right cancellative) monoid is left restriction (ample).

Cameo 2
Embedding into W-products

W-product $W(T, Y)$: subsemigroup of **reverse** semidirect product $T \ltimes Y$ where Y is a semilattice and T acts in a special way.

The notion was introduced by **Fountain and Gomes (1992)**; later developed and used by **Gomes, G. and Szendrei**

A left restriction (restriction) semigroup is **proper** if

$$[a^+ = b^+ \text{ and } a\,\sigma\, b] \Rightarrow a = b$$
$$(\text{and also } [a^* = b^* \text{ and } a\,\sigma\, b] \Rightarrow a = b).$$

Here σ is the least congruence identifying the projections.

Fountain and Gomes (1992)

A left ample semigroup is proper if and only if it embeds into a W-product.

Cameo 2
Embedding into W-products

$W(T, Y)$ is **proper restriction (Gomes, Szendrei (2007))**
Further, T is left/right cancellative iff $W(T, Y)$ is right/left ample.

G. and Szendrei (2013)

A left restriction semigroup S embeds into W-product if and only if τ_S is a congruence, where

$$\tau_S = \{(a, b) \in S \times S : a^+ = b^+ \text{ and } a\, \omega_S\, b\}$$

and where ω_S is the least right cancellative congruence on S.

Szendrei (2014)

Gave necessary and sufficient conditions such that any restriction semigroup embeds into a W-product.

Cameo 3
Term functions for restriction semigroups

1. An inverse monoid S is **factorisable** if $S = E(S)H_1$.
2. The corresponding notion for inverse semigroups is called **almost factorisable**.

Lawson (1992)

Every inverse semigroup embeds into an almost factorisable inverse semigroup.

Cameo 3
Term functions for restriction semigroups

For restriction (and ample) semigroups and monoids these notions split into one- and two-sided notions of factorisability.

Gomes, Szendrei (2007)

Let S be a restriction semigroup. Then:

1. S is almost left factorisable if and only if it is a morphic image of a W-**product** of a semilattice by a monoid.
2. S is almost factorisable if and only if it is a morphic image of a semidirect product $Y \rtimes T$ where T **acts on** Y **by automorphisms**.

Cameo 3
Term functions for restriction semigroups

Let S be a bi-unary semigroup. An *n-ary term* function $t(x_1, \dots, x_n) : S^n \to S$ is built from the binary operation, and both unary operations.

e.g. $t(x, y, z, u) = (uz^*)^+ uu^*((xz)^+ y^* xz)^*$

S bi-unary, $H \subseteq S \times S$ and $\rho = \langle H \rangle$.

Standard universal algebra gives $a \,\rho\, b$ iff $a = b$ or there exists a sequence

$$a = t_1(a_1), t_1(b_1) = t_2(a_2), \dots, t_n(b_n) = b$$

where $t_i = t_i(\underline{u_i}, x)$ for some tuple $\underline{u_i} \in S^{n_i}$ where $(a_i, b_i) \in H \cup H^{-1}, 1 \leq i \leq n$.

Cameo 3
Term functions for restriction semigroups

Szendrei (2013)

Gave a simplification for the bi-unary term functions on a restriction semigroup, making use of the identities for restriction semigroups.

Example

$t(x, y, z, u) = (uz^*)^+ uu^*((xz)^+ y^* xz)^* = u(yxz)^*$

Cameo 3
Term functions for restriction semigroups: embedding into almost factorisable

Szendrei (2013)

Every restriction semigroup is embeddable into an **almost left factorisable** restriction semigroup.

Hartmann, G., Szendrei (2017)

Every restriction semigroup is embeddable into an **almost factorisable** restriction semigroup.

Cameo 3
Term functions for restriction semigroups: embedding into almost factorisable

Take $S = F/\rho$ where F is free restriction on X;
embed $F \hookrightarrow Y \rtimes X^*$ for some Y;
consider $\bar{\rho} = \langle \rho \rangle$, the congruence on $Y \rtimes X^*$ generated by ρ;
show $\rho = \bar{\rho} \cap (F \times F)$.

What's behind all of this

- We are trying to understand classes of semigroups extending that of inverse semigroups
- These semigroups arise naturally
- The devil is in the detail
- The move between one- and two-sided constructions is not easy; behind some of this are various notions of partial action, used in C^*-algebras
- Don't give up!!