# Non-commutative Stone dualities

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This talk is based on joint work carried out in collaboration with Ganna Kudryavtseva.

## The big idea

Generalize classical Stone duality to a non-commutative setting using semigroups and quantales.

Regard this theory as *non-commutative frame theory*.

Explore applications to $C^*$-algebras, group theory, tilings, . . .

## 0. Stone duality

The following is a classical theorem due to Marshall H. Stone.

**Theorem** *The category of unital Boolean algebras is dual to the category of* Boolean spaces *— that is, compact Hausdorff topological spaces with a basis of clopen sets.*

In general terms, this theorem links algebra, in the guise of Boolean algebras, with topology.

The Boolean algebras should be regarded as *commutative* structures.

The aim of this talk is to show you how this theorem may be generalized in such a way that the *commutative* algebraic structures are replaced by *non-commutative* structures.

## 1. Motivations

The following are all related and provided important examples and motivations.

- The work of Ehresmann on ordered categories from the 1950s.

- The theory of frames and locales.

- The work of Renault, Paterson, Kellendonk, Lenz, Exel relating inverse semigroups, étale groupoids and $C^*$-algebras.

- The paper by Pedro Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

## 2. First example

We shall replace Boolean algebras by *monoids* with extra structure. To explain what that extra structure is, we shall examine a concrete example in detail. This will motivate the whole talk.

Let $X$ be a non-empty set. Denote by B($X$) the set of all binary relations on $X$. Equip this set with the usual multiplication of binary relations. This turns B($X$) into a monoid with zero 0, the empty relation.

The monoid $B(X)$ has an obvious involution which plays an important role in Resende's work.

But other examples show that an involution is not necessary.

For example, replace B($X$) by B($A$) where $A$ is any reflexive and transitive relation on $X$.

We might choose $X = \{1, \ldots, n\}$ and the relation to be $\leq$, for instance.

Thus we do not want to construct our theory by assuming that we have an involution.

Denote by $E$ the set of all binary relations which are *partial identities*. That is, binary relations $a$ such that $(y, x) \in a$ implies that $y = x$. These are idempotents in $\mathsf{B}(X)$ that we call *projections*. These form a commutative idempotent submonoid.

Let $a$ be an arbitrary binary relation. Define

$$\lambda(a) = \{(x, x) \in X \times X \colon \exists y, (y, x) \in a\}$$

and

$$\rho(a) = \{(y, y) \in X \times X \colon \exists x, (y, x) \in a\}.$$

Both of these elements are projections.

We have a monoid $S$, a submonoid of idempotents $E$ which is commutative and two maps $\lambda, \rho \colon S \to E$ in which the following axioms are satisfied.

(ES1) If $a \in E$ then $\lambda(a) = a = \rho(a)$.

(ES2) $a\lambda(a) = a = \rho(a)a$.

(ES3) $\lambda(\lambda(a)b) = \lambda(ab)$ and $\rho(a\rho(b)) = \rho(ab)$.

Such a monoid is called an *Ehresmann monoid* (w.r.t the set of *projections* $E$.)

Every Ehresmann monoid is equipped with a *natural partial order* $\preceq$ defined by

$$a \preceq b \Leftrightarrow a = eb = bf$$

where $e$ and $f$ are projections.

Not necessarily compatible with the multiplication.

This will play an important role later.

Why Ehresmann monoids?

They are monoids whose structure is determined by a category.

Let $S$ be an Ehresmann monoid.

Define a partial binary operation $\cdot$ where $a \cdot b = ab$ if $\lambda(a) = \rho(b)$ and undefined otherwise.

**Proposition** *With the above definition,* $(S, \cdot)$ *is a category whose set of identities is the set of projections of the Ehresmann monoid.*

**Question** How are arbitrary Ehresmann monoids constructed from categories?

A *quantale* is a (sup-)lattice-ordered semigroup in which multiplication distributes over joins. Our quantales will be *unital*. The identity denoted by $e$.

A *frame* is a sup-lattice in which finite meets distribute over arbitrary joins.

A *unital quantal frame* is a unital quantale which is also a frame.

An *Ehresmann quantal frame* is a unital quantal frame that is an Ehresmann monoid with respect to the set of projections $e^{\downarrow}$ and in which $\lambda$ and $\rho$ are sup-maps.

**Proposition** *For any reflexive and transitive relation $A$ on a set $X$, the monoid* $\mathsf{B}(A)$ *is an Ehresmann quantal frame.*

Ehresmann quantal frames satisfying an additional property to be described later will be the main algebraic objects we shall study.

## 3. Second example

The monoid $\mathsf{B}(X)$ is the set of all subsets of $X \times X$.

We can regard $X \times X$ as a category, in fact a groupoid with the discrete topology.

More generally, let $C$ be a topological category in which the maps $\mathbf{d}$, $\mathbf{r}$ and $\mathbf{m}$ are all open and where the space of identities $C_o$ is an open subset.

Denote by $\mathsf{O}(C)$ the set of all open subsets of $C$. Denote by $E$ the set of all open subsets of $C_o$. For $A \in \mathsf{O}(C)$ define

$\lambda(A) = \{\mathbf{d}(a) \colon a \in A\}$ and $\rho(A) = \{\mathbf{r}(a) \colon a \in A\}$.

Since both $\mathbf{d}$ and $\mathbf{r}$ are open maps, we have well-defined maps

$$\lambda, \rho \colon \mathsf{O}(C) \to E.$$

If $A, B \in \mathsf{O}(C)$ define $AB$ to be the binary operation $\mathsf{O}(C)$, well-defined because $\mathbf{m}$ is open.

**Proposition** *With the above definitions,* $\mathsf{O}(C)$ *is an Ehresmann quantal frame.*

Our results so far hint at a correspondence linking

- Ehresmann quantal frames.
- Certain topological categories.

It is more convenient to work with another formulation of what we mean by a ‘space’.

## 4. Frames and locales

We follow Resende and work in the first instance not with topological spaces but with locales.

A *frame* is a complete lattice satisfying the infinite distributive law. A *morphism* between frames is a map preserving finite meets and arbitrary joins.

The opposite of the category of frames is called the category of *locales*.

With each topological space $X$, we may associate its frame $\Omega(X)$ of open subsets.

With each locale $A$ we may associate the space pt($A$) whose points are the completely prime filters on $A$. The open sets are those sets of the form $V_a$, where $a \in A$, and $V_a$ consists of all completely prime filters containing $a$.

The following is a standard result.

**Theorem**

1. *There is an adjunction $\Omega \dashv pt$ between the category of spaces and the category of frames.*

2. *This adjunction restricts to an equivalence between* sober *spaces and* spatial *locales.*

We shall accordingly work with *localic categories* rather than topological categories.

This consists of two locales $C_1$ and $C_0$ where $C_1$ should be regarded as the arrows and $C_0$ as the identities. These are connected by locale maps

$$u\colon C_0 \to C_1, \quad d, r\colon C_1 \to C_0, \quad m\colon C_1 \times_{C_0} C_1 \to C_1$$

that satisfy the obvious conditions to give us a category.

A locale map is said to be *semiopen* if its associated frame map preserves arbitrary meets.

A locale map is said to be *open* if it is semiopen and satisfies an algebraic condition called the *Frobenius reciprocity condition*.

## 4. Theorem

A *quantal localic category* is a localic category where the maps $d, r, u$ are open and $m$ is semiopen.

An Ehresmann quantal frame $Q$ is said to be *multiplicative* if the multiplication map

$$Q \otimes_{e\downarrow} Q \to Q$$

has a right adjoint that preserves arbitrary joins.

We omit describing the morphisms that make sense of the following statement.

**The correspondence theorem** *There is a duality between multiplicative Ehresmann quantal frames and quantal localic categories.*

This theorem provides a common framework for

- Resende's work on the relationship between étale groupoids, quantales, and inverse semigroups.

- Work by Renault, Paterson, Kellendonk, Lenz, Exel etc on the role of of inverse semigroups in the theory of $C^*$-algebras.

We shall now show that there is also a connection with the York school of semigroup theory.

## 5. Restriction monoids

Multiplicative Ehresmann quantal frames seem like rather abstract structures. We now show how to construct an interesting class of examples and, in the process, refine the statement of our main theorem.

An Ehresmann semigroup is said to be a *restriction semigroup* if it satisfies the following two axioms:

(RS1) $ea = a\lambda(ea)$ for all projections $e$.

(RS2) $af = \rho(af)a$ for all projections $f$.

On a restriction semigroup the natural partial order is compatible with the multiplication.

Elements $a$ and $b$ are said to be *compatible*, denoted $a \sim b$ if and only if $a\lambda(b) = b\lambda(a)$ and $\rho(b)a = \rho(a)b$.

A *complete restriction monoid* is a restriction monoid whose projections form a frame, all joins of compatible subsets exist and products distribute over the joins that exist.

These generalize *pseudogroups* which are complete inverse monoids.

Let $Q$ be an Ehresmann quantale.

This is equipped with two orders $\leq$ and $\preceq$.

Observe that

$$a \preceq b \Rightarrow a \leq b.$$

We use the relationship between these two orders to define an important class of elements.

An element $a \in Q$ is called a *partial isometry* if for all $b \in Q$ we have that

$$b \leq a \Rightarrow b \preceq a.$$

We denote the set of partial isometries of $Q$ by $\mathsf{PI}(Q)$. This is not always a submonoid.

**Example** Let $X = \{1, 2\}$ and let $A$ be the reflexive and transitive relation $\leq$. The set of partial isometries of $\mathsf{B}(A)$ is $I(X) \setminus \{(2, 1)\}$. In particular, it need not be an inverse monoid.

## Motivating example

Let $C$ be a topological category as before.

Assume in addition that the topology is $T_0$.

Recall that $A \subseteq C$ is a *local bisection* if $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ implies $a = b$, and dually.

**Result** *The open local bisections of $C$ are precisely the open partial isometries of $C$. They form a complete restriction monoid.*

Let $S$ be a complete restriction monoid. Denote by $\mathsf{L}^{\vee}(S)$ the set of all order ideals of $S$ closed under compatible joins.

**Theorem** $\mathsf{L}^{\vee}(S)$ *is an Ehresmann quantal frame. In addition,*

1. *The partial isometries of* $\mathsf{L}^{\vee}(S)$ *form a submonoid isomorphic to* $S$*.*

2. *Each element of* $\mathsf{L}^{\vee}(S)$ *is a join of partial isometries.*

3. $\mathsf{L}^{\vee}(S)$ *is multiplicative.*

A *restriction quantal frame* is an Ehresmann quantal frame $Q$ in which the top element of $Q$ is a join of partial isometries and the partial isometries form a submonoid.

An *étale localic category* is a localic category in which the maps $u$ and $m$ are open and $d$ and $r$ are étale.

**Theorem** *The category of complete restriction monoids is equivalent to the category of restriction quantal frames.*

These are the two theorems from this talk that you should take home with you.

**Main theorem** *The category of complete restriction monoids is dual to the category of étale localic categories.*

By incorporating involutions into the above theory, we may deduce the main theorem of Resende (2007).

**Theorem** *The category of pseudogroups is dual to the category of étale localic groupoids.*

We may replace localic categories by topological but we pay the usual price.

**Theorem** *The category of spatial complete restriction monoids is dually equivalent to the category of sober étale topological groupoids.*