Coherence and Uniform Interpolation

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The Craig Interpolation Theorem

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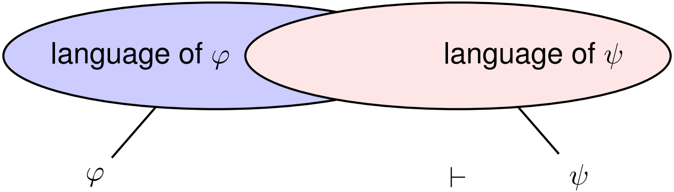

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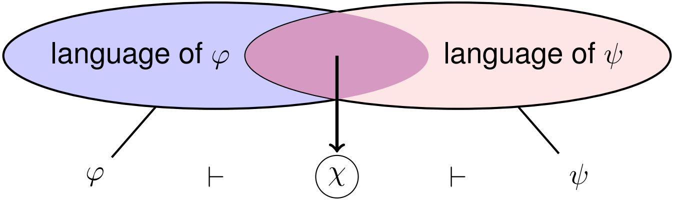
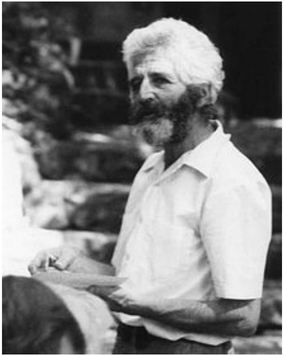
If φ *and* ψ *are sentences of first-order logic such that* $\varphi \vdash \psi$, *then there exists a sentence* χ *with* $\mathrm{Rel}(\chi) \subseteq \mathrm{Rel}(\varphi) \cap \mathrm{Rel}(\psi)$ *such that*

$$\varphi \vdash \chi \quad \textit{and} \quad \chi \vdash \psi.$$



Origins

"Although I was aware of the mathematical interest of questions related to elimination problems in logic, my main aim, initially unfocused, was to try to use methods and results from logic to clarify or illuminate a topic that seems central to empiricist programs: In epistemology, the relationship between the external world and sense data; in philosophy of science, that between theoretical constructs and observed data."



William Craig (2008).

Interpolation in Classical Logic

Theorem (Craig 1957)

If $\varphi(\overline{x}, \overline{y})$ and $\psi(\overline{y}, \overline{z})$ are propositional formulas such that $\varphi \vdash_{\mathbf{CL}} \psi$, then there exists a formula $\chi(\overline{y})$ such that $\varphi \vdash_{\mathbf{CL}} \chi$ and $\chi \vdash_{\mathbf{CL}} \psi$.

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For example...

$\varphi = \neg(x \to y)$

$\psi = y \to \neg z$

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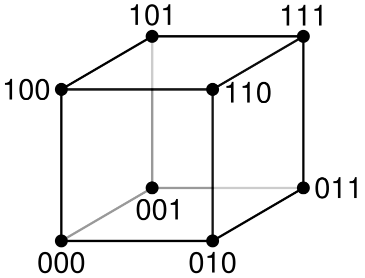

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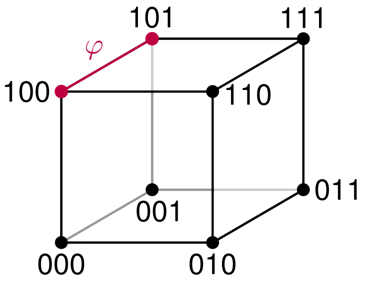

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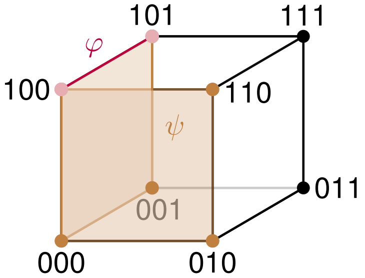

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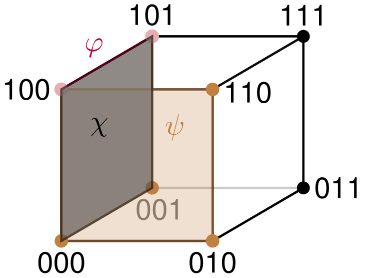

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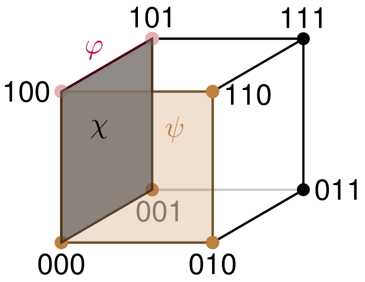

In fact, for *any* formula $\psi'(y, \overline{z})$,

$$\varphi \vdash_{\mathbf{CL}} \psi' \quad \Longrightarrow \quad \chi \vdash_{\mathbf{CL}} \psi'.$$

Uniform Interpolation in Intuitonistic Logic

Theorem (Pitts 1992)

For any formula $\varphi(\overline{x}, \overline{y})$ *of intuitionistic propositional logic* IL, *there exist* **left** *and* **right uniform interpolants**, *i.e., formulas*

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Theorem (Ghilardi and Zawadowski 1997)

The first-order theory of Heyting algebras admits a model completion.

Interpolation and Coherence

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there exists $\chi(\overline{y})$ such that

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Coherence: for any $\varphi(\overline{x}, \overline{y})$, there exists $\varphi^R(\overline{y})$ such that

$$\varphi(\overline{x}, \overline{y}) \vdash_{\mathrm{IL}} \psi(\overline{y}) \quad \Longleftrightarrow \quad \varphi^R(\overline{y}) \vdash_{\mathrm{IL}} \psi(\overline{y}).$$

This Talk

What does uniform interpolation mean **algebraically**?

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if for any $\mathbf{A} \in \mathcal{V}$ and homomorphism $e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}$,

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We also write $\Sigma \models_{\mathcal{V}} \Delta$ if $\Sigma \models_{\mathcal{V}} s \approx t$ for all $s \approx t \in \Delta$.

Deductive Interpolation

$\mathcal{V}$ admits **deductive interpolation** if whenever $\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z})$, there exists a set of equations $\Delta(\overline{y})$ such that

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Equivalently, $\mathcal{V}$ admits deductive interpolation if for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

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Congruences

A **congruence** Θ on an algebra $\mathbf{A}$ is an equivalence relation satisfying

$$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \Theta \quad \Longrightarrow \quad \langle \star(a_1, \ldots, a_n), \star(b_1, \ldots, b_n)\rangle \in \Theta$$

for every n-ary operation $\star$ of $\mathbf{A}$.

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Note. The congruences of **A** always form a complete lattice $\mathrm{Con}\,\mathbf{A}$.

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ is

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}} \quad \text{where } s\,\Theta_{\mathcal{V}}\,t \iff \mathcal{V} \models s \approx t.$$

We write t to denote both a term t in $\mathbf{Tm}(\overline{x})$ and $[t]$ in $\mathbf{F}(\overline{x})$.

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{s \approx t\}$ *with variables in* $\overline{x}$*,*

$$\Sigma \models_{\mathcal{V}} s \approx t \quad \Longleftrightarrow \quad \langle s, t \rangle \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma),$$

where $\mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$ *is the congruence on* $\mathbf{F}(\overline{x})$ *generated by* Σ*.*

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Note that the pair $\langle i^*, i^{-1} \rangle$ is an **adjunction**, i.e.,

$$i^*(\Theta) \subseteq \Psi \iff \Theta \subseteq i^{-1}(\Psi).$$

Deductive Interpolation Again

The following are equivalent:

(1) $\mathcal{V}$ admits **deductive interpolation**, i.e., for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \quad \Longleftrightarrow \quad \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).$$

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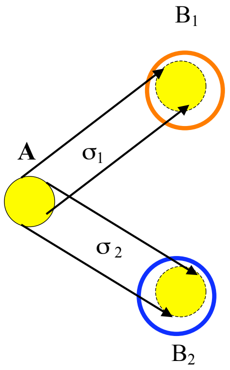
(2) For any finite sets $\overline{x}$, $\overline{y}$, $\overline{z}$, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y}) & \xrightarrow{\;i^{-1}\;} & \mathrm{Con}\,\mathbf{F}(\overline{y}) \\
\Big\downarrow j^{*} & & \Big\downarrow l^{*} \\
\mathrm{Con}\,\mathbf{F}(\overline{x}, \overline{y}, \overline{z}) & \xrightarrow[\;k^{-1}\;]{} & \mathrm{Con}\,\mathbf{F}(\overline{y}, \overline{z})
\end{array}$$

where i, j, k, and l denote inclusion maps between free algebras.

The Amalgamation Property



$\mathcal{V}$ admits the **amalgamation property** if for any $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in \mathcal{V}$, and embeddings $\sigma_1 : \mathbf{A} \to \mathbf{B}_1$, $\sigma_2 : \mathbf{A} \to \mathbf{B}_2$,

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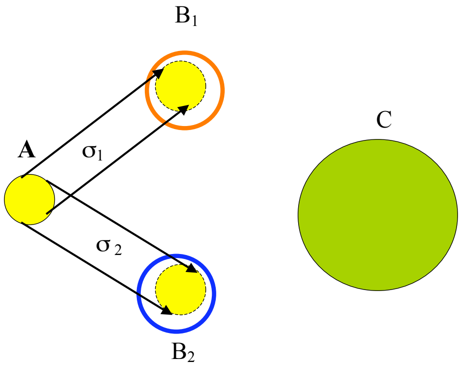

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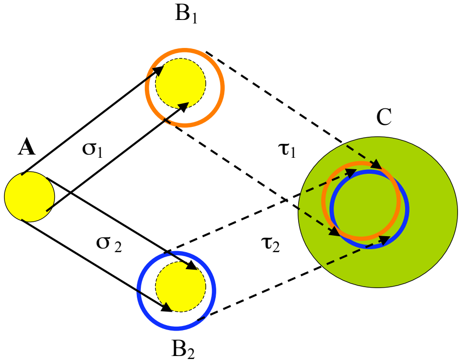

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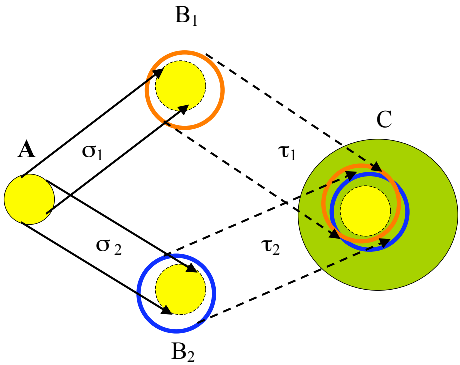

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A Bridge Theorem

Theorem (Pigozzi, Bacsich, Maksimova, Czelakowski,. . .)

A variety with the congruence extension property admits the deductive interpolation property if and only if it admits the amalgamation property.

But Now. . .

Can we describe **uniform interpolation** algebraically?

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Right Uniform Deductive Interpolation

$\mathcal{V}$ has **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

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Equivalently, $\mathcal{V}$ has deductive interpolation and for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

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$$i^*\colon \mathrm{Con}\,\mathbf{F}(\overline{y}) \to \mathrm{Con}\,\mathbf{F}(\overline{x},\overline{y}); \qquad \Theta \mapsto \mathrm{Cg}_{\mathbf{F}(\overline{x},\overline{y})}(i[\Theta])$$

$$i^{-1}\colon \mathrm{Con}\,\mathbf{F}(\overline{x},\overline{y}) \to \mathrm{Con}\,\mathbf{F}(\overline{y}); \qquad \Psi \mapsto i^{-1}[\Psi] = \Psi \cap \mathrm{F}(\overline{y})^2.$$

The **compact lifting of** i restricts i^* to $\mathrm{KCon}\,\mathbf{F}(\overline{y}) \to \mathrm{KCon}\,\mathbf{F}(\overline{x},\overline{y})$; it has a right adjoint if i^{-1} restricts to $\mathrm{KCon}\,\mathbf{F}(\overline{x},\overline{y}) \to \mathrm{KCon}\,\mathbf{F}(\overline{y})$.

The Missing Ingredient

Theorem (Kowalski and Metcalfe 2017)

The following are equivalent:

(1) *For any finite set of equations* $\Sigma(\overline{x}, \overline{y})$*, there is a finite set of equations* $\Delta(\overline{y})$ *such that*

$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}) \quad \Longleftrightarrow \quad \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}).$$

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(2) *For finite* $\overline{x}$, $\overline{y}$, *the compact lifting of* $\mathbf{F}_{\mathcal{V}}(\overline{y}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ *has a right adjoint; that is,* $\Theta \in \mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in \mathrm{KCon}\,\mathbf{F}(\overline{y})$.

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Examples include any locally finite variety, abelian groups, abelian ℓ-groups, MV-algebras, Heyting algebras, diagonalizable algebras...

The variety of groups is *not* coherent, however, since every finitely generated recursively presented group embeds into some finitely presented group (Higman 1961).

A Useful Lemma

Lemma

If $\mathbf{A} \in \mathcal{V}$ *is finitely presented and isomorphic to* $\mathbf{F}(\overline{x})/\Theta$ *for some finite set* $\overline{x}$ *and* $\Theta \in \mathrm{Con}\,\mathbf{F}(\overline{x})$, *then* Θ *is compact.*

Proof Sketch

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The following are equivalent:

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(3) $\Rightarrow$ (2) Consider finite $\overline{x}$, $\overline{y}$ and $\Theta \in \mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y})$. Then $\mathbf{F}(\overline{x}, \overline{y})/\Theta$ is finitely presented and, by coherence, so is $\mathbf{F}(\overline{y})/(\Theta \cap F(\overline{y})^2)$.

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(2) $\Rightarrow$ (3) Consider a finitely presented $\mathbf{A} \in \mathcal{V}$ with a finitely generated subalgebra $\mathbf{B}$ that is not finitely presented. Let $\overline{x}, \overline{y}$ be finite sets of generators of $\mathbf{A}, \mathbf{B}$, and construct a new presentation $\mathbf{F}(\overline{x}, \overline{y})/\Theta$ of $\mathbf{A}$ with $\Theta \in \mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y})$.

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Another Bridge Theorem

Theorem (Kowalski and Metcalfe 2017)

A variety with the congruence extension property admits the right uniform deductive interpolation property if and only if it is coherent and admits the amalgamation property.

A Failure of Coherence

A **modal algebra** consists of a Boolean algebra equipped with a unary operation $\Box$ satisfying $\Box(x \wedge y) \approx \Box x \wedge \Box y$ and $\Box\top \approx \top$.

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Note that $\mathcal{K}$ does admit a uniform "implicative" interpolation property (Ghilardi 1995, Visser 1996, Bilkova 2007).

Proof Sketch

Let $\boxdot x = \Box x \wedge x$, and define

$$\Sigma = \{y \leq x, x \leq z, x \approx \boxdot x\} \quad \text{and} \quad \Delta = \{y \leq \boxdot^k z \mid k \in \mathbb{N}\}.$$

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Then also $\Sigma \subseteq \ker(e)$, and hence $\Sigma \not\models_{\mathcal{K}} \varepsilon(y, z)$. $\Box$

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Theorem (Kowalski and Metcalfe 2017)

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Then $\mathcal{V} \models t^n(x) \approx t^{n+1}(x)$ for some $n \in \mathbb{N}$.

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- any variety of residuated lattices that is closed under canonical extensions and does not satisfy $x^{n+1} \approx x^n$ for any $n \in \mathbb{N}$;
- the variety of lattices (first proved by Schmidt 1983).

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We have seen that. . .

- the "logical" deductive interpolation property corresponds to the "algebraic" amalgamation property;
- right uniform interpolation requires also coherence.

What more is required for the existence of a model completion for the first-order theory? Is there a fixpoint characterization of coherence?

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