Congruence Lattices of Partition Monoids

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Aim and credits

- Describe the congruence lattice of the partition monoid $\mathcal{P}_n$ and its various important submonoids.
- By way of introduction: congruence lattices of symmetric groups and full transformation monoids.
- Plus a quick introduction to partition monoids.
- Joint work with: James East, James Mitchell and Michael Torpey.

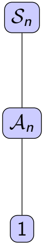
Normal subgroups of the symmetric group

Theorem

The alternating group $\mathcal{A}_n$ is the only proper normal subgroup of $\mathcal{S}_n$ ($n \neq 1, 2, 4$).

Remark

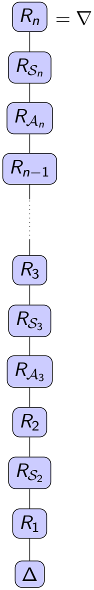
- Exceptions: $\mathcal{S}_1$, $\mathcal{S}_2$ (too small) and $\mathcal{S}_4$ (because of the Klein 4-group K_4).
- The normal subgroups of any group form a (modular) lattice.
- $\mathrm{Norm}(G) \cong \mathrm{Cong}(G)$.



Congruences of the full transformation monoid $\mathcal{T}_n$

Theorem (A.I. Mal'cev 1952)

$\mathrm{Cong}(\mathcal{T}_n)$ *is the chain shown on the right.*



Green's structure of $\mathcal{T}_n$

The following are well known:

- $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{im}\alpha = \operatorname{im}\beta$.
- $\alpha\mathcal{R}\beta \Leftrightarrow \ker\alpha = \ker\beta$.
- $\alpha\mathcal{J}\beta \Leftrightarrow \operatorname{rank}\alpha = \operatorname{rank}\beta$.
- All $\mathcal{J}(=\mathcal{D})$-classes are regular.
- The maximal subgroups corresponding to the idempotents of rank r are all isomorphic to $\mathcal{S}_r$.

Ideals of $\mathcal{T}_n$ and Rees congruences

- Every ideal of $\mathcal{T}_n$ has the form
$$I_r = \{\alpha \in \mathcal{T}_n \,:\, \operatorname{rank} \alpha \leq r\}.$$
- All ideals are principal, and they form a chain.
- To every ideal I_r there corresponds a (Rees) congruence
$$R_r = \Delta \cup (I_r \times I_r).$$

Group-induced congruences

- Consider a typical $\mathcal{J}$-class $J_r = \{\alpha \in \mathcal{T}_n \,:\, \operatorname{rank}\alpha = r\}$.
- Let $\overline{J}_r$ be the corresponding principal factor.
- $\overline{J}_r \cong \mathcal{M}^0[\mathcal{S}_r; K, L; P]$ – a Rees matrix semigroup.
- For every $N \trianglelefteq \mathcal{S}_r$, the semigroup $\mathcal{M}^0[\mathcal{S}_r/N; K, L; P/N]$ is a quotient of $\overline{J}_r$.
- Let ν_N be the corresponding relation on J_r.
- $R_N = \Delta \cup \nu_N \cup (I_{r-1} \times I_{r-1})$ is a congruence on $\mathcal{T}_n$.
- Intuitively R_N: collapses I_{r-1} to a single element (zero); collapses each $\mathcal{S}_r$ in J_r to $\mathcal{S}_r/N$, and correspondingly collapses the non-group $\mathcal{H}$-classes; leaves the rest of $\mathcal{T}_n$ intact.

Proof outline of Mal'cev's Theorem

- Verify that all the congruences R_r and R_N form a chain.
- This relies on the fact that the ideals form a chain, and that congruences on each $\mathcal{S}_r$ form a chain.
- It turns out that all these congruences are principal.
- For every pair $(\alpha, \beta) \in \mathcal{T}_n \times \mathcal{T}_n$, determine the congruence $(\alpha, \beta)^\sharp$ generated by it, and verify it is one of the listed congruences.
- Since every congruence is a join of principal congruences, conclude that there are no further congruences on $\mathcal{T}_n$.

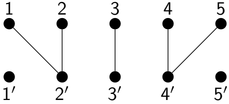
Further remarks on Cong($\mathcal{T}_n$)

- Mal'cev also describes Cong($\mathcal{T}_X$), X infinite.
- Analogous results have been proved for:
  - full matrix semigroups (Mal'cev 1953);
  - symmetric inverse monoids (Liber 1953);
  - and many others.
- In all instances, Cong(S) is a chain.

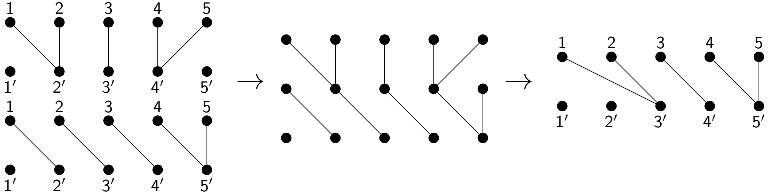
From transformations to partitions

View mappings graphically, e.g:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 4 \end{pmatrix} =$$



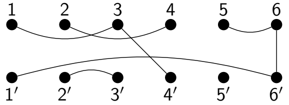
Composition:



Partition monoid $\mathcal{P}_n$

Partition = a set partition of $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$.

For example: $\alpha = \{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\}$

$=$ 

$\in \mathcal{P}_6$.

Some useful parameters:

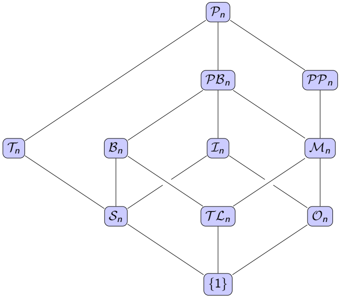
$\operatorname{dom} \alpha = \{1, 3, 5, 6\} \quad \ker \alpha = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$

$\operatorname{codom} \alpha = \{1', 4', 6'\} \quad \operatorname{coker} \alpha = \{\{1', 6'\}, \{2', 3'\}, \{4'\}, \{5'\}\}$

$$\operatorname{rank} \alpha = 2.$$

Partition monoid $\mathcal{P}_n$: some remarks

- $\mathcal{P}_n$ contains $\mathcal{S}_n$, $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{O}_n$ as submonoids.
- It also contains: Brauer monoid, Motzkin monoid, Temperely–Lieb (Jones) monoid.



- They form a basis from which their name-sakes algebras are built – connections with Mathematical Physics, Representation Theory and Topology.
- Elements of $\mathcal{P}_n$ can be viewed as partial bijections between quotients of $\{1, \ldots, n\}$.

Green's relations on $\mathcal{P}_n$

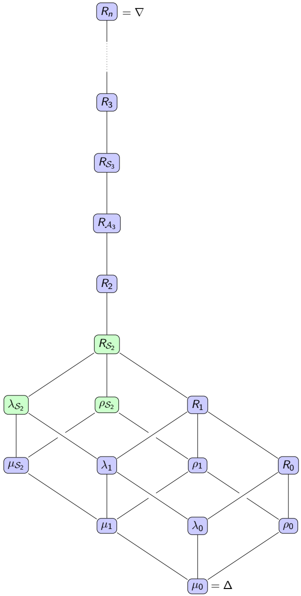
- $\alpha\mathcal{R}\beta \Leftrightarrow \operatorname{ker}\alpha = \operatorname{ker}\beta$ & $\operatorname{dom}\alpha = \operatorname{dom}\beta$.
- $\alpha\mathcal{L}\beta \Leftrightarrow \operatorname{coker}\alpha = \operatorname{coker}\beta$ & $\operatorname{codom}\alpha = \operatorname{codom}\beta$.
- $\alpha\mathcal{J}\beta \Leftrightarrow \operatorname{rank}\alpha = \operatorname{rank}\beta$.
- All $\mathcal{J}(=\mathcal{D})$-classes are regular.
- The maximal subgroups corresponding to the idempotents of rank r are all isomorphic to $\mathcal{S}_r$.

Ideals of $\mathcal{P}_n$, and congruences arising

- Every ideal of $\mathcal{P}_n$ has the form $I_r = \{\alpha \in \mathcal{P}_n \ : \ \text{rank}\,\alpha \leq r\}$.
- All ideals are principal, and they form a chain.
- To every ideal I_r there corresponds a congruence $R_r = \Delta \cup (I_r \times I_r)$.
- Analogous to $\mathcal{T}_n$, we also have congruences R_N for $N \trianglelefteq \mathcal{S}_r$.
- One difference though: The minimal ideal of $\mathcal{P}_n$ (partitions of rank 0) is a proper rectangular band.
- (As opposed to a right zero semigroup of constant mappings in $\mathcal{T}_n$.)

Cong($\mathcal{P}_n$)

Theorem
[J. East, J.D. Mitchell, NR, M. Torpey]
Cong($\mathcal{P}_n$) is the lattice shown on the right.



$\mathcal{R}$ and $\mathcal{L}$ on the minimal ideal

Theorem (Folklore?)

Let S be a finite monoid with the minimal ideal M. The relations $\rho_0 = \Delta \cup \mathcal{R}\restriction_M$ and $\lambda_0 = \Delta \cup \mathcal{L}\restriction_M$ are congruences on S.

Retractions

A (computational) inspection of the congruence μ_1 yields:

$$\mu_1 = \{(\alpha, \beta) \in I_1 \times I_1 \ : \ \ker \alpha = \ker \beta,\ \operatorname{coker} \alpha = \operatorname{coker} \beta\} \cup \Delta$$

It is a congruence, because the following mapping is a retraction:

$$I_1 \to I_0,\ \alpha \mapsto \widehat{\alpha} \in I_0,\ \ker \alpha = \ker \widehat{\alpha},\ \operatorname{coker} \alpha = \operatorname{coker} \widehat{\alpha}.$$

Definition

Let S be a semigroup and $T \leq S$. A homomorphism $f : S \to T$ with $f{\restriction}_T = 1_T$ is called a retraction.

Congruence triples

Definition

Let S be a finite monoid with minimal ideal M. A triple $\mathcal{T} = (I, f, N)$ is a congruence triple if:

- I is an ideal;
- $f : I \to M$ is a retraction;
- N is a normal subgroup of a maximal subgroup in a $\mathcal{J}$-class 'just above' I;
- All elements of N act the same way on M, i.e. $|xN| = |Nx| = 1$ $(x \in M)$.

A family of congruences

Definition

To every congruence triple $\mathcal{T}$ associate three relations:

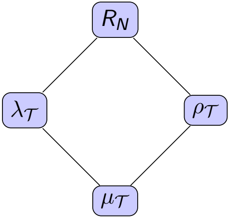
- $\lambda_{\mathcal{T}} = \Delta \cup \nu_N \cup \{(x,y) \in I \times I \,:\, f(x)\mathcal{L}f(y)\}$;
- $\rho_{\mathcal{T}} = \Delta \cup \nu_N \cup \{(x,y) \in I \times I \,:\, f(x)\mathcal{R}f(y)\}$;
- $\mu_{\mathcal{T}} = \Delta \cup \nu_N \cup \{(x,y) \in I \times I \,:\, f(x)\mathcal{H}f(y)\}$.

Theorem

$\lambda_{\mathcal{T}}$, $\rho_{\mathcal{T}}$ and $\mu_{\mathcal{T}}$ are congruences.

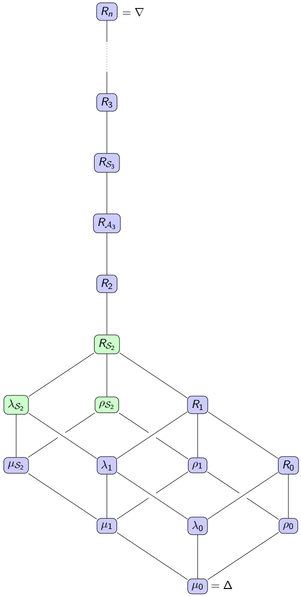
Theorem

The congruences $\lambda_{\mathcal{T}}$, $\rho_{\mathcal{T}}$ and $\mu_{\mathcal{T}}$, together with R_N, form a diamond lattice.



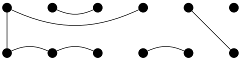
Cong($\mathcal{P}_n$) explained

- Key fact: $(I_1, \alpha \mapsto \widehat{\alpha}, \mathcal{S}_2)$ is a congruence triple on $\mathcal{P}_n$.
- It induces two 'smaller' congruence triples $(I_1, \alpha \mapsto \widehat{\alpha}, \{1\})$ and $(I_0, 1, \{1\})$.
- The rest is the same as for $\mathcal{T}_n$.
- But: not all congruences are principal!



Planar partition monoid

- Planar partition: can be drawn without edges crossing.
- Edges need not be straight, but have to be contained within the rectangle with corners $1, 1', n, n'$.



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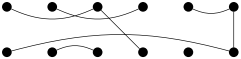
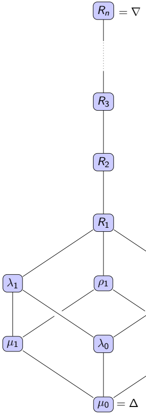
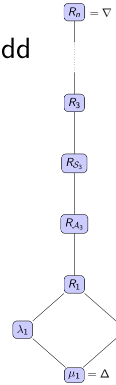




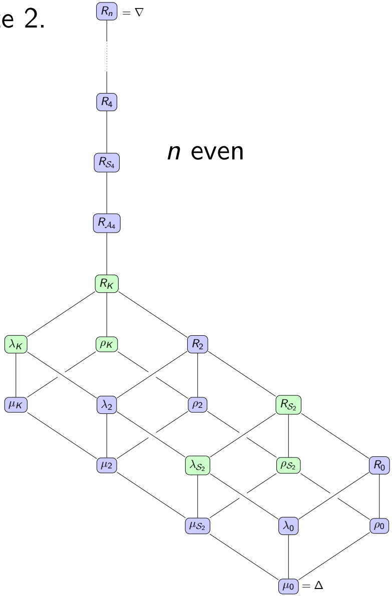
Brauer monoid $\mathcal{B}_n$

$\mathcal{B}_n$ = partitions with blocks of size 2.

n odd

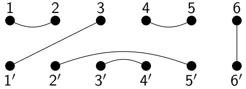


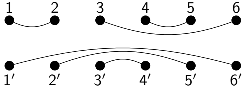
n even



$\mathcal{B}_n$ (n even): key retraction

- An $\alpha \in \mathcal{B}_n$ with $\operatorname{rank} \alpha = 2$ has precisely two transversal blocks $\{i, j'\}$, $\{k, l'\}$.
- Let $\widehat{\alpha} \in I_0$ be obtained from α by replacing those two blocks by $\{i, k\}$, $\{j, l\}$.

$\alpha =$ 

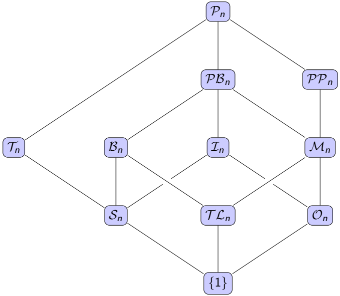
$\mapsto$ 

$= \widehat{\alpha}$

- $(I_2, \alpha \mapsto \widehat{\alpha}, K \trianglelefteq \mathcal{S}_4)$ is a congruence triple.
- Three further derived triples: $(I_2, \alpha \mapsto \widehat{\alpha}, \{1\})$, $(I_0, 1, \mathcal{S}_2 \trianglelefteq \mathcal{S}_2)$, $(I_0, 1, \{1\})$.

Concluding remarks

- Congruence lattices determined for all partition monoids shown in the diagram.
- Work was crucially informed by computational evidence obtained using GAP package Semigroups (J.D. Mitchell et al.)



- All the congruences are instances of the construction(s) described here.
- The work to determine the principal congruences is still case-specific.
- Related work: J. Araújo, W. Bentz, G.M.S. Gomes, Congruences on direct products of transformation and matrix monoids.

Some speculations about future work...

- Develop a general theory of generators for the congruences introduced here.
- For example: Under which genereal conditions are the congruences R_N, $\rho_{\mathcal{T}}$, $\lambda_{\mathcal{T}}$ and μ_T principal?
- The answer is likely to be couched in terms of groups, Rees matrix semigroups, and the actions on $\mathcal{R}$- and $\mathcal{L}$-classes.
- To what extent does this point to a general approach towards computing (and understanding) congruence lattices of arbitrary semigroups?
- What are families of semigroups to which one could turn next, in search of interesting behaviours and patterns?

THANK YOU FOR LISTENING!