PERFECTION FOR POMONOIDS

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Abstract. A *pomonoid* S is a monoid equipped with a partial order that is compatible with the binary operation. In the same way that M-acts over a *monoid* M correspond to the representation of M by transformations of sets, S-posets correspond to the representation of a pomonoid S by order preserving transformations of posets.

Following standard terminology from the theories of R-modules over a unital ring R, and M-acts over a monoid M, we say that a pomonoid S is *left poperfect* if every left S-poset has a projective cover.

Left perfect rings were introduced in 1960 in a seminal paper of Bass [1] and shown to be precisely those rings satisfying M_R, the descending chain condition on principal right ideals. In 1971, inspired by the results of Bass and Chase [6], Isbell was the first to study left perfect monoids [13]. The results of [13], together with those of Fountain [11], show that a monoid is left perfect if and only if it satisfies a finitary condition dubbed Condition (A), in addition to M_R. Moreover, M_R can be replaced by another finitary condition, namely Condition (D).

A further characterisation of left perfect rings was given in [6], where Chase proved that a ring is left perfect if and only if every flat left module is projective; the corresponding result for M-acts was demonstrated in [11].

In this paper we continue the study of left poperfect pomonoids, recently initiated in [18]. We show, as in [18] that a pomonoid S is left poperfect if and only if it satisfies (M_R) and the 'ordered' version Condition ($\mathrm{A^O}$) of Condition (A) and further, these conditions are equivalent to every strongly flat left S-poset being projective. On the other hand, we argue via an analysis of direct limits that Conditions (A) and ($\mathrm{A^O}$) are equivalent, so that a pomonoid S is left perfect if and only if it is left poperfect. We also give a characterisation of left poperfect monoids involving the ordered version of Condition (D). Our results and many of our techniques certainly correspond to those for monoids, but we must take careful account of the partial ordering on S, and in places introduce alternative strategies to those found in [13], [11] and [18].

Dedicated to the memory of Douglas Munn

1. Introduction

A *pomonoid* is a monoid S partially ordered by $\leq$, such that $\leq$ is compatible with the semigroup operation. That is, for all $a, b, c, d \in S$, if $a \leq b$ and $c \leq d$, then $ac \leq bd$. The standard example of a pomonoid is an inverse monoid under the natural partial order, which is given by the rule that $a \leq b$ if and only if $a = aa^{-1}b$.

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Let M be a monoid and let U be a set. We say that U is a left *M-act* if there is a map $M \times U \to U$, written $(m, u) \mapsto mu$, such that for all $m, n \in M$ and $u \in U$,

$$1\,u = u \text{ and } m(nu) = (mn)u.$$

A map $\phi : U \to V$ from a left S-act U to a left S-act V is called an *S-morphism* if it respects the action of S, that is, $(su)\phi = s(u\phi)$ for all $s \in S$ and $u \in U$. The collection of left M-acts, together with M-morphisms, forms a category which we denote by **M-Act**. The category **Act-M** of *right M-acts* and appropriate M-morphisms is defined dually. We give further brief details of acts as necessary, referring the reader to the comprehensive survey [16].

Now let S be a pomonoid and let A be a partially ordered set. We say that A is a left *S-poset* if A is a left S-act and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \le t$ then $sa \le ta$, and if $a \le b$ then $sa \le sb$. We say that an order preserving S-morphism $\phi : A \to B$ from a left S-poset A to a left S-poset B is an *S-pomorphism.* The collection of left S-posets, together with S-pomorphisms, forms a category which we denote by **S-Pos**. The category **Pos-S** of *right S-posets* and appropriate S-pomorphisms is defined dually. Note that for S-acts and S-posets epimorphisms are onto, and monomorphisms are one-one [16, 4]. In this article we deal with both left and right S-posets (and M-acts); an unspecified S-poset (or M-act) will always be a left S-poset (left M-act).

The study of M-acts over a monoid M has been well established since the late 1960s. On the other hand, the investigation of S-posets was initiated by Fakhruddin in the 1980s [9], [10], but lay fallow until this decade, which has seen a flurry of papers on this topic, mostly concentrating on projectivity, and various notions of flatness for S-posets. Definitions and concepts relating to flatness are given in Section 2; an excellent survey is given in [3]. It is worth pointing out in this Introduction that S-posets (indeed, pomonoids) are not merely algebras, they are relational structures. As such, care is needed to take account of the partial order relation, particularly when considering congruences.

A left S-poset A over a pomonoid S is called a *cover* for a left S-poset B if there exists an S-poset epimorphism (an *S-po-epimorphism*) $\beta : A \to B$, such that any restriction of β to a proper S-subposet of A is not an S-po-epimorphism. Such a map β is called a *coessential S-po-epimorphism.* The pomonoid S is said to be *left poperfect* if every left S-poset has a projective cover: our aim in this article is to investigate left poperfect pomonoids. We introduce the terminology *po*perfect in order to distinguish the two possible definitions of left perfection of a pomonoid S, that is, as a monoid and as a pomonoid. In fact, they transpire to be equivalent.

The analogous notion of a left perfect monoid was introduced in [13]. Characterisations of left perfect monoids were given by Isbell in [13] and subsequently by Fountain [11] and Kilp [15]. Since their results inform ours, we now pause to explain them.

A submonoid T of a monoid M is *right unitary* if $a, ba \in T$ implies that $b \in T$.

Lemma 1.1. [16, Corollary 1.4.9] *A submonoid T of a monoid M is right unitary if and only if T is the ρ-class of the identity, for some left congruence ρ on S.*

Let M be a monoid. A submonoid T of M is *right collapsible* if for any $a, b \in T$ we can find $c \in T$ with $ac = bc$. For convenience we list some finitary conditions that we need below:

Condition (A): every left M-act satisfies the ascending chain condition for cyclic subacts;

Condition (D): every right unitary submonoid of M contains a minimal left ideal generated by an idempotent;

Condition (K): every right collapsible submonoid of M contains a right zero;

Condition ($\mathbf{M_R}$): M satisfies the descending chain condition on principal right ideals.

Theorem 1.2. [13, 11, 15] *The following conditions are equivalent for a monoid M:*
(i) *M is left perfect;*
(ii) *M satisfies (A) and (D);*
(iii) *M satisfies (A) and (M_R);*
(iv) *every strongly flat left M-act is projective;*
(v) *M satisfies (A) and (K).*

In a series of steps we prove the ordered analogue of Theorem 1.2. Some of our techniques are taken from those used in the monoid case, but these need careful adjustment to deal with the orderings involved; for some steps we develop new strategies. After giving the requisite background results in Section 2, we concentrate in Section 3 on characterising those pomonoids S such that every strongly flat S-poset is projective, and show that these are precisely those that satisfy Conditions (M_R) and ($\mathrm{A^O}$), the ordered version of Condition (A), defined as follows:

Condition ($\mathbf{A^O}$): every left S-poset satisfies the ascending chain condition on cyclic S-subposets.

Conditions (A) and ($\mathrm{A^O}$) are intimately related to the behaviour of direct limits of sequences of copies of S. Careful analysis of these direct limits enables us to show that (A) and ($\mathrm{A^O}$) are equivalent for a pomonoid.

In Section 4 we turn our attention explicitly to poperfect pomonoids. We investigate conditions under which a subpomonoid is the ρ-class of the identity, for some left S-poset congruence ρ: we call such subpomonoids *right po-unitary subpomonoids*. We show that a pomonoid S is left poperfect if and only if it satisfies ($\mathrm{D^O}$), the ordered version of (D), defined for a pomonoid S as follows:

Condition ($\mathbf{D^O}$): every right po-unitary subpomonoid of S contains a minimal left ideal generated by an idempotent.

We observe that if ρ is a left S-poset congruence on S such that S/ρ is strongly flat, then S/ρ is strongly flat as a left S-act: it follows from [14] that ρ-class of the identity is right collapsible. In Section 5 we show that all strongly flat cyclic left S-posets are projective if and only if S satisfies (K).

In Section 6 we show that in the presence of Condition ($\mathrm{A^O}$), Conditions (M_R) and ($\mathrm{D^O}$) are equivalent. One way is relatively straightforward, but to show that ($\mathrm{D^O}$) follows from (M_R) we require a mixture of the techniques of [13] and a classic semigroup theoretic argument. This completes the proof of the following theorem.

Theorem 1.3. *The following conditions are equivalent for a pomonoid S:*
(i) *S is left poperfect;*
(ii) *S satisfies (A^O) and (D^O);*
(iii) *S satisfies (A^O) and (M_R);*
(iv) *every strongly flat left S-poset is projective;*
(v) *S satisfies (A^O) and (K).*

Since (A) and ($\mathrm{A^O}$) are interchangeable, the conditions of the above result are also equivalent to those in Theorem 1.2.

Some of the minor results in this paper have recently been announced, without proof, in [26]. We note, however, that the author of [26] does not distinguish between congruence classes of S-poset congruences, and congruence classes of S-act congruences, a distinction we feel to be necessary. More significantly, Pervukhin and Stepanova [18] have recently shown some of the equivalences in Theorem 1.3. For completeness we provide proofs, whilst making reference to [26] and [18].

2. Preliminaries

In this section we outline the concepts related to pomonoids and S-posets needed for the rest of the article; for definitions relating to acts over monoids, we refer the reader to the monograph [16]. Throughout, S will denote a pomonoid. We have already introduced the categories **S-Pos** and **Pos-S** of left and right S-posets. An *S-subposet* of an S-poset A is a subset of A partially ordered by the restriction of the ordering on A, that is closed under the action of S (called *regular* S-subposets in [17]). As for acts, a pomonoid S can be regarded as both a left and a right S-poset over itself; more generally, left and right ideals of S are left and right S-subposets, respectively.

We now consider the notion of congruence for S-posets. For information on the approach to congruences on general ordered structures, we refer the reader to [2] and [8], and for further information pertaining to congruences on S-posets to [25] and [5].

Definition 2.1. Let A be a left S-poset. An *S-poset congruence* on A is an equivalence relation ρ such that for $a, b \in A$ and $s \in S$, if $a \,\rho\, b$, then $sa \,\rho\, sb$ (that is, ρ is an S-act congruence) such that in addition, A/ρ may be partially ordered in such a way that the natural map $\nu_\rho : A \to A/\rho$ is order preserving.

Such a congruence is also referred to as a *(left) order-congruence* or a *(left) po-congruence*, particularly where we are regarding a pomonoid S as a left S-poset.

We observe that S-act and S-poset congruences on pomonoids are different relations. For example, consider the pomonoid $\mathbb{N}$ of natural numbers under multiplication, with the usual ordering. Certainly $\equiv \pmod 2$ is an $\mathbb{N}$-act congruence. But it cannot be an $\mathbb{N}$-poset congruence. For, if it were, we would have in the quotient $\mathbb{N}/\equiv$ that

$$[1] \leq [2] \leq [3] = [1],$$

so that $[1] = [2]$, a contradiction.

Let A be a left S-poset. For the purposes of this paper we give one description of the S-poset congruence generated by $H \subseteq A \times A$. First we say that $a \leq_H b$ if and only if there exists $n \geq 0$, $(c_1, d_1), \ldots, (c_n, d_n) \in H \cup H^{-1}$ and $s_1, \ldots, s_n \in S$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, \ldots, s_nd_n \leq b.$$

Then $\leq_H$ is reflexive (take $n = 0$), transitive, contains the relation $\leq$ and compatible with the action of S. It follows that the relation $\equiv_H$ given by $a \equiv_H b$ if and only if $a \leq_H b \leq_H a$ is is an S-act congruence. Moreover, $A/\equiv_H$ may be partially ordered by

$$[a] \preceq_H [b] \text{ if and only if } a \leq_H b,$$

and the natural map $A \to A/\equiv_H$ is an S-poset morphism. That is, $\equiv_H$ is an S-poset congruence, the *S-poset congruence generated by H*. Notice that for any $(a, b) \in H$, $[a] = [b]$.

We now consider free, projective and flat S-posets. Freeness and projectivity are defined in the standard categorical manner.

An S-poset A is *free* on $X \subseteq A$ if for any S-poset B and map $j : X \to B$ there is a unique S-pomorphism $\theta : A \to B$ such that $i\,\theta = j$, where $i : X \to A$ is inclusion, i.e. the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] & B \\ A \arrow[ur, "\theta"'] & \end{tikzcd}$$

commutes.

We now show how to construct a free S-poset over a pomonoid S. First, for a symbol x we let $Sx = \{sx \mid s \in S\}$ be a set of formal expressions in one-one correspondence with S; Sx becomes a left S-poset (isomorphic to the left ideal S) if we define $s(tx) = (st)x$ for all $s, t \in S$ and $sx \leq tx$ if and only if $s \leq t$ in S. Let X be a non-empty set: the disjoint union of S-posets $\bigcup_{x \in X} Sx$ is then an S-poset with ordering given by $sx \leq ty$ if and only if $x = y$ and $s \leq t$. The next result is easy to verify.

Theorem 2.2. *A left S-poset A is free on a set X if and only if $A \cong \bigcup_{x \in X} Sx$.*

An S-poset P is *projective* if for any onto S-pomorphism $g : A \to B$ and for any S-pomorphism $f : P \to B$ there exists a S-pomorphism $h : P \to A$ such that the following diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

commutes. We will denote the class of projective S-posets by $\mathcal{P}$.

Proposition 2.3. [23] *Let S be a pomonoid. Then*
(i) *Se is projective S-poset for any idempotent $e \in S$;*
(ii) *a disjoint union of S-posets P_i is projective if and only if each P_i is projective for every $i \in I$;*
(iii) *a left S-poset is projective if and only if it is isomorphic to a disjoint union of S-posets of the form Se, where e is idempotent.*

Definition 2.4. A left S-poset A is called *cyclic* if $A = Sa$ for some $a \in A$.

It is clear that A is cyclic if and only if A is isomorphic to S/ρ for some left po-congruence on S. We remark that, from Proposition 2.3, an indecomposable projective S-poset A is cyclic and therefore of the form Sa, where there is an S-po-isomorphism $\phi : Sa \to Se$ for some idempotent $e \in E(S)$, with $a\phi = e$. Consequently, for any $s, t \in S$ we have that $sa \leq ta$ if and only if $se \leq te$; we say that a is *ordered right e-cancellative*. In fact the following is true.

Lemma 2.5. [22] *Let λ be a left po-congruence on S then S/λ is projective if and only if there exists an idempotent $e \in S$ such that $1\,\lambda\,e$ and $[s] \leq [t]$ implies $se \leq te$.*

Notions of flatness for S-posets are all derived from the use of the tensor product $A \otimes B$ of a right S-poset A and a left S-poset B. For this article we do not need to go into the technical details of tensor products, but refer the reader to [5]. As for the case of acts over monoids, but unlike the case for modules over unital rings, there are several differing notions flatness for S-posets. We are interested here in strong flatness.

A left S-poset B is *strongly flat* if the functor $- \otimes B$ from **Pos-S** to the category **Pos** of partially ordered sets, preserves subpullbacks and subequalisers. Before stating our next result, we remark that in **S-Pos**, direct limits of directed systems of S-posets exist, as observed in [5], where they are referred to as *directed colimits*.

Theorem 2.6. [5] *The following conditions are equivalent for a left S-poset B:*

(i) *B is strongly flat;*

(ii) *B is a direct limit of finitely generated free left S-posets;*

(iii) *B satisfies (P) and (E):*

Condition (P): for all $b, b' \in B, s, s' \in S$, if $sb \leq s'b'$, then there exists $b'' \in B$ and $u, u' \in S$ such that $b = ub''$, $b' = u'b''$ and $su \leq s'u'$;

Condition (E): for all $b \in B, s, s' \in S$, if $sb \leq s'b$, then there exists $b' \in B$ and $u \in S$ such that $b = ub'$ and $su \leq s'u$.

We will denote the class of strongly flat S-posets by $\mathcal{SF}$.

The notion of strong flatness simplifies for cyclic left S-posets.

Lemma 2.7. [22] *The following conditions are equivalent for a cyclic left S-poset $A = Sa$:*

(i) *A is strongly flat;*

(ii) *A satisfies Condition (E);*

(ii) *for any $s, t \in S$, if $sa \leq ta$, then there exists $u \in S$ such that $a = ua$ and $su \leq tu$.*

Consequently, we can easily deduce the following.

Corollary 2.8. [21] *Let ρ be a left po-congruence on a pomonoid S. Then S/ρ is strongly flat if and only if for any $s, t \in S$, if $[s] \leq [t]$, then there exists $u \in S$ such that $su \leq tu$ and $1 \,\rho\, u$.*

The next observation is straightforward, and we will employ it from time to time to simplify our approach to strongly flat S-posets. It follows from an analysis of direct limits of free S-acts and S-posets. An argument directly from interpolation conditions is given in [18].

Lemma 2.9. [18] *Let A be a strongly flat left S-poset. Then A is strongly flat as a left S-act.*

3. Pomonoids for which $\mathcal{SF} = \mathcal{P}$

Just as for R-modules over a unital ring R, and M-acts over a monoid M, any projective S-poset is strongly flat [5], that is $\mathcal{SF} \subseteq \mathcal{P}$. A natural question, which we address in this section, asks under what conditions on S do we have that $\mathcal{P} \subseteq \mathcal{SF}$?

We have two strategies to answer this question. Both involve a careful study of direct limits of free left S-acts versus free left S-posets over a pomonoid S. One approach is to then consider under which conditions S-morphisms automatically become S-pomorphisms, and call upon the result of [13, 11]. Details may be found in the thesis of the second author [20]. We prefer here a more direct strategy, on the way making clear a number of arguments sketched in [13].

The construction in the next result is crucial to this article, particularly in understanding the connections between perfection and poperfection for a pomonoid S. It is implicit in [13] in the unordered case, taken up and made rather more explicit in [11]. Here we aim for an even directer presentation for S-posets, noting that we have difficulties to overcome due to the partial orders involved.

Lemma 3.1. *Let S be a pomonoid and let $\underline{a} = (a_1, a_2, \ldots)$ be a sequence of elements of S. Let*

$$F = Sx_1 \cup Sx_2 \cup \ldots$$

be the free left S-poset on $\{x_i : i \in \mathbb{N}\}$ and let

$$H = \{(x_i, a_i x_{i+1}) : i \in \mathbb{N}\} \subseteq F \times F.$$

(i) For any $sx_m, tx_n \in F$,

$$sx_m \leq_H tx_n \text{ if and only if } sa_m a_{m+1} \ldots a_w \leq ta_n a_{n+1} \ldots a_w$$

for some $w \geq m, n$. Further,

$$sx_m \equiv_H tx_n \text{ if and only if } sa_m a_{m+1} \ldots a_v = ta_n a_{n+1} \ldots a_v$$

for some $v \geq m, n$.

(ii) The S-poset $F(\underline{a}) = F/\equiv_H$ is the direct limit of the directed sequence

$$Sx_1 \to Sx_2 \to \ldots$$

where $\alpha_i : Sx_i \to Sx_{i+1}$ is given by $x_i \alpha_i = a_i x_{i+1}$.

(iii) The S-poset $F(\underline{a})$ is strongly flat.

Proof. (i) Suppose that

$$sx_m \leq_H tx_n;$$

then there exist $h \in \mathbb{N}^0$ and $s_i \in S$ and $(y_i, z_i) \in H \cup H^{-1}$, $1 \leq i \leq h$ such that

$$sx_m \leq s_1 y_1, s_1 z_1 \leq s_2 y_2, \ldots, s_h z_h \leq tx_n.$$

We proceed by induction on h. If $h = 0$, then

$$sx_m \leq tx_n \text{ in } F$$

so that $m = n$ and $s \leq t$ in S. Certainly

$$sa_m \leq ta_m = ta_n.$$

Suppose inductively that from

$$ux_i = s_1 z_1 \leq s_2 y_2, \ldots, s_h z_h \leq tx_n$$

we can deduce that

$$ua_i \ldots a_o \leq ta_n \ldots a_o$$

for some $o \geq \max\{i, n\}$.

Case (I): $(y_1, z_1) = (x_j, a_j x_{j+1})$.
From $sx_m \leq s_1 y_1 = s_1 x_j$ we have that $m = j$ and $s \leq s_1$; from $ux_i = s_1 z_1 = s_1 a_j x_{j+1}$ we deduce that $i = j+1$ and $u = s_1 a_j$. Hence

$$\begin{array}{rcl} sa_m \dots a_o & = & sa_j a_{j+1} \dots a_o \\ & \leq & s_1 a_j a_{j+1} \dots a_o \\ & = & ua_{j+1} \dots a_o \\ & = & ua_i \dots a_o \\ & \leq & ta_n \dots a_o \end{array}$$

and $o \geq \max\{i, n\} \geq \max\{m, n\}$.

Case (II): $(y_1, z_1) = (a_j x_{j+1}, x_j)$. From $sx_m \leq s_1 y_1 = s_1 a_j x_{j+1}$ we have that

$$m = j+1, s \leq s_1 a_j$$

and from $ux_i = s_1 z_1 = s_1 x_j$ we have that

$$i = j \text{ and } u = s_1.$$

Hence $s \leq ua_i$, so that if $i = o$,

$$s \leq ua_i \leq ta_n \dots a_o$$

giving that

$$sa_m \leq ta_n \dots a_o a_m$$

where $m > i = o \geq n$. On the other hand, if $i < o$, so that $o \geq m$,

$$\begin{array}{rcl} sa_m \dots a_o & \leq & ua_i a_m \dots a_o \\ & = & ua_i a_{i+1} \dots a_o \\ & \leq & ta_n \dots a_o \end{array}$$

where $o \geq \max\{m, n\}$.

Conversely, suppose that $sa_m \dots a_w \leq ta_n \dots a_w$ where $w \geq \max\{m, n\}$. Then

$$sx_m \leq_H sa_m x_{m+1} \leq_H \dots \leq_H sa_m \dots a_w x_{w+1}$$

$$\leq ta_n \dots a_w x_{w+1} \leq_H ta_n \dots a_{w-1} x_w \leq_H \dots \leq_H tx_n$$

so that $sx_m \leq_H tx_n$ as required.

Clearly if $sa_m \dots a_w = ta_n \dots a_w$ for some $w \geq \max\{m, n\}$, then $sx_m \leq_H tx_n \leq_H sx_m$, so that $sx_m \equiv_H tx_n$.

On the other hand, if $sx_m \equiv_H tx_n$, then from $sx_m \leq_H tx_n \leq_H sx_m$ we have that

$$sa_m \dots a_w \leq ta_n \dots a_w, ta_n \dots a_v \leq sa_m \dots a_v$$

for some $v, w \geq \max\{m, n\}$. Without loss of generality assume that $v \geq w$. Then

$$sa_m \dots a_w a_{w+1} \dots a_v \leq ta_n \dots a_w a_{w+1} \dots a_v \leq sa_m \dots a_v$$

so that $sa_m \dots a_v = ta_n \dots a_v$ as required.

(ii) Define $\beta_i : Sx_i \to F(\underline{a})$ by $x_i \beta_i = [x_i]$. Notice that if $i < j$ then

$$x_i \alpha_i \dots \alpha_{j-1} \beta_j = (a_i a_{i+1} \dots a_{j-1} x_j) \beta_j = [a_i a_{i+1} \dots a_{j-1} x_j] = [x_i] = x_i \beta_i.$$

Now let P be an S-poset, and $\gamma_i : Sx_i \to P$, $i \in \mathbb{N}$, a set of S-pomorphisms such that for any $i < j$ we have that $\gamma_i = \alpha_i \dots \alpha_{j-1} \gamma_j$.

Define $[ux_i]\delta$ to be $(ux_i)\gamma_i$. If $[ux_i] \leq [vx_j]$, then from (i) we know that there exists $k \geq i, j$ such that

$$ua_i \dots a_k \leq va_j \dots a_k.$$

It follows that

$$\begin{array}{rcl}[ux_i]\delta & = & (ux_i)\gamma_i \\ & = & ux_i\alpha_i\dots\alpha_{k-1}\gamma_k \\ & = & (ua_i\dots a_{k-1}x_k)\gamma_k \\ & \leq & (va_j\dots a_{k-1}x_k)\gamma_k \\ & = & (vx_j\alpha_i\dots\alpha_{k-1})\gamma_k \\ & = & (vx_j)\gamma_j \\ & = & [vx_j]\delta\end{array}$$

so that δ is well-defined, order preserving and clearly compatible with the action of S. We also have that for each $i \in \mathbb{N}$, $\beta_i\delta = \gamma_i$, and δ is unique with respect to the latter property. Hence $F(\underline{a})$ is indeed the direct limit of the given system.

(iii) This follows from Theorem 2.6. □

We remark that the above is a special case of a more general result concerning direct limits of free S-acts and S-posets; for the details, see [20].

The equivalence of (i) and (iv) in the next lemma is implicit in [13]. We note that in (i) and (ii) it is clear that

$$Sb_1 \subseteq Sb_2 \subseteq \dots.$$

Proposition 3.2. *Let S be a pomonoid, let $\underline{a} = (a_1, a_2, \dots)$ be a sequence of elements of S and let $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) *for every left S-act A and for every sequence of elements $b_1, b_2, \dots$ of A such that $b_i = a_ib_{i+1}$ for all $i \in \mathbb{N}$,*

$$Sb_n = Sb_{n+1} = \dots;$$

(ii) *for every left S-poset A and for every sequence of elements $b_1, b_2, \dots$ of A such that $b_i = a_ib_{i+1}$ for all $i \in \mathbb{N}$,*

$$Sb_n = Sb_{n+1} = \dots;$$

(iii) *in $F(\underline{a})$ we have that*

$$S[x_n] = S[x_{n+1}] = \dots;$$

(iv) *for all $i \geq n$ there exists $j_i \geq i+1$ such that*

$$Sa_ia_{i+1}\dots a_{j_i} = Sa_{i+1}\dots a_{j_i}.$$

Proof. It is clear that (i) implies (ii) and that (ii) implies (iii).

We suppose now that (iii) holds. Let $i \geq n$, so that $S[x_i] = S[x_{i+1}]$. Then $[x_{i+1}] = u[x_i]$ for some $u \in S$, so that by Lemma 3.1 there exists $j_i \geq i+1$ such that

$$a_{i+1}a_{i+2}\dots a_{j_i} = ua_ia_{i+1}a_{i+2}\dots a_{j_i}.$$

Then

$$\begin{array}{rcl}Sa_i\dots a_{j_i} & \subseteq & Sa_{i+1}\dots a_{j_i} \\ & = & Sua_ia_{i+1}\dots a_{j_i} \\ & \subseteq & Sa_i\dots a_{j_i},\end{array}$$

so that $Sa_i\dots a_{j_i} = Sa_{i+1}\dots a_{j_i}$ as required.

Finally, assume that (iv) is true, let A be an S-act and let $b_i \in A$ be such that $b_i = a_i b_{i+1}$ for $i \in \mathbb{N}$. Then for any $i \geq n$ we have that

$$\begin{aligned} Sb_i &\subseteq Sb_{i+1} \\ &= Sa_{i+1} \dots a_{j_i} b_{j_i+1} \\ &= Sa_i a_{i+1} \dots a_{j_i} b_{j_i+1} \\ &= Sb_i, \end{aligned}$$

so that $Sb_n = Sb_{n+1} = \dots$ as claimed. □

Our next corollary is now immediate.

Corollary 3.3. *A pomonoid S has Condition (A) if and only if it has Condition (A^O).*

We say that a left S-poset A over a pomonoid S is *locally cyclic* if every finitely generated S-subposet of A is contained in a cyclic S-poset [13].

Lemma 3.4. *(c.f* [13, Result 1.2]*) The following are equivalent for a pomonoid S;*
(i) for any sequence $\underline{a} = (a_1, a_2, \dots)$ of elements of S, $F(\underline{a})$ is cyclic;
(ii) any direct limit of a sequence of copies of the left S-poset S is cyclic;
(iii) S satisfies Condition (A^O) (or equivalently, Condition (A));
(iv) any locally cyclic left S-poset is cyclic.

Proof. The equivalence of (i) and (ii) is clear, since any direct limit of a sequence of copies of S must be constructed in the manner of $F(\underline{a})$.

Suppose now that (i) holds. Let A be an S-poset and suppose that

$$Sb_1 \subseteq Sb_2 \subseteq \dots$$

is an ascending chain of cyclic S-subposets of A. Let $\underline{a} = (a_1, a_2, \dots)$ be a sequence of elements of S such that $b_i = a_i b_{i+1}$ for each $i \in \mathbb{N}$. It is clear that in $F(\underline{a})$ we have that

$$S[x_1] \subseteq S[x_2] \subseteq \dots$$

so that as $F(\underline{a})$ is cyclic,

$$S[x_1] \subseteq S[x_2] \subseteq \dots \subseteq Su[x_n]$$

for some $u \in S$ and $n \in \mathbb{N}$. It now follows that for any $j \geq n$,

$$S[x_j] \subseteq Su[x_n] \subseteq S[x_n] \subseteq S[x_j]$$

so that

$$S[x_n] = S[x_{n+1}] = \dots .$$

From (iii) implies (ii) of Proposition 3.2 we have

$$Sb_n = Sb_{n+1} = \dots$$

so that Condition (A^O) holds.

To show that (iii) implies (iv), let S have Condition (A^O) and let B be a locally cyclic S-poset. Let $b_1 \in B$; if B is not cyclic then $Sb_1 \subset B$, so there exists $b_1' \notin Sb_1$. Now B is locally cyclic, so that $Sb_1 \cup Sb_1' \subseteq Sb_2$ for some $b_2 \in B$, and clearly, $Sb_1 \subset Sb_2$. Continuing in this manner we obtain an infinite ascending chain of cyclic S-subposets of B, contradicting the existence of Condition (A^O). Hence B is cyclic.

Finally, assume that (iv) is true. Since $F(\underline{a})$ is the union of an ascending chain of cyclic S-subposets, it is clear that $F(\underline{a})$ is locally cyclic, hence cyclic by assumption. □

We now focus on the question of when $\mathcal{SF} = \mathcal{P}$.

Lemma 3.5. *Let S be a pomonoid such that every left S-poset $F(\underline{a})$ is projective. Then S satisfies Condition (A^O) (or equivalently, Condition (A)).*

Proof. As $F(\underline{a})$ is a union of an ascending chain, if projective it must therefore be cyclic. The result now follows from Lemma 3.4. □

Every S-poset $F(\underline{a})$ is strongly flat from Lemma 3.1.

Corollary 3.6. *Let S be a pomonoid such that every strongly flat left S-poset is projective. Then S satisfies Condition (A^O) (or equivalently, Condition (A)).*

The following argument is essentially that of [11]; we include it here for completeness, since all the preliminaries are set up.

Lemma 3.7. *Let S be a pomonoid such that every left S-poset $F(\underline{a})$ is projective. Then S satisfies (M_R).*

Proof. Let
$$a_1S \supseteq b_1S \supseteq b_2S \supseteq \cdots$$
be a decreasing sequence of principal right ideals of S. Then there are elements $a_i, i \geq 2$ such that $b_i = b_{i-1}a_{i+1}$ (where $b_0 = a_1$). Then
$$b_1 = a_1a_2, b_2 = b_1a_3 = a_1a_2a_3, \ldots.$$

Let $\underline{a} = (a_1, a_2, \ldots)$ and let $F(\underline{a})$ be defined as in Lemma 3.1.

Let I be the identity map in $F(\underline{a})$ and let $\alpha : F \to F(\underline{a})$ be the canonical S-pomorphism. Since $F(\underline{a})$ is projective, there exists an S-pomorphism $\gamma : F(\underline{a}) \to F$ such that

$$\begin{tikzcd} & F(\underline{a}) \arrow[dl, "\gamma"'] \arrow[d, "I"] \\ F \arrow[r, "\alpha"'] & F(\underline{a}) \end{tikzcd}$$

commutes.

Suppose that for each $i \in \mathbb{N}$ we have that
$$[x_i]\gamma = c_ix_{j(i)}.$$
Then for any $i \geq 2$,
$$c_1x_{j(1)} = [x_1]\gamma = (a_1 \ldots a_{i-1}[x_i])\gamma = a_1 \ldots a_{i-1}c_ix_{j(i)},$$
so that $j(i) = j(1) = j$ say, and moreover
$$c_1 = a_1 \ldots a_{i-1}c_i$$
for all i. It follows that $c_1S \subseteq a_1 \ldots a_{i+1}S$, that is, $c_1S \subseteq b_iS$ for all $i \in \mathbb{N}$. Now
$$[x_1] = [x_1]I = [x_1]\gamma\alpha = c_1x_j\alpha = [c_1x_j],$$
so by Lemma 3.1,
$$a_1 \ldots a_n = c_1a_j \ldots a_n$$
for some $n \geq j$. Hence
$$b_{n-1}S = a_1 \ldots a_nS \subseteq c_1S$$
so that $b_{n-1}S = b_nS = \ldots$ and our descending chain terminates as required.

□

Corollary 3.8. *Let S be a pomonoid such that every strongly flat left S-poset is projective. Then S satisfies (M_R).*

Our next technical lemma has two significant uses. The strategy for the proof is again taken from the unordered case in [11], but note that that article omits the proof that c is idempotent. We say that a left po-congruence ρ on a pomonoid S is *strongly flat* if S/ρ is strongly flat.

Lemma 3.9. *Let S be a pomonoid and let ρ be a strongly flat left po-congruence on S such that the set $\{dS : d \in B\}$ has a minimal element with respect to inclusion, where $B = [1]$. Then S/ρ is projective.*

Proof. From Lemma 2.9, S/ρ is strongly flat as a left S-act. Let $c \in B$ be such that cS is minimal in $\mathcal{I} = \{dS : d \in B\}$. We will now show that c is idempotent. Since $c\,\rho\,c^2$, by the Corollary to Result 4 of [11] we have $cu = c^2u$ for some $u \in S$ with $u\,\rho\,1$. Then $c^2uS \subseteq cS$ but cS is minimal in $\mathcal{I}$, hence $c\,\mathcal{R}\,c^2u$. Hence $c = c^2ux$ for some $x \in S$ and so

$$c^2 = c^3ux = c^2ux = c.$$

Let $d \in B$, so that $d\,\rho\,c$. Exactly as in [11] we have that $dv = cv$ for some $v \in B$ and then

$$cS = cvS = dvS \subseteq dS.$$

Thus cS is the least element in $\mathcal{I}$.

Now let $\theta : S/\rho \to Sc$ be defined by $[u]\theta = uc$. Then $[u] \leq [v]$ implies that there exists $w\,\rho\,1$ such that $uw \leq vw$. Since $w \in B$ we have that $cS \subseteq wS$, so that $c = wt$ for some $t \in S$. Therefore $uwt \leq vwt$ implies that $uc \leq vc$ hence θ is well-defined and order-preserving. To check that θ preserves the S-action,

$$(s[u])\theta = [su]\theta = (su)c = s(uc) = s[u]\theta.$$

To check the injectivity let $sc \leq tc$; then

$$[s] = s[1] = s[c] = [sc] \leq [tc] = t[c] = t[1] = [t]$$

as ρ is an S-poset congruence. Thus θ is injective and clearly θ is a surjective S-pomorphism; moreover, we have also shown that the inverse of θ preserves order, so that θ is an S-poset isomorphism. As c is an idempotent, by Proposition 2.3, Sc and hence S/ρ are projective. □

Theorem 3.10. *If S satisfies (M_R), then every strongly flat cyclic left S-poset is projective.*

Proof. Let C be a strongly flat cyclic S-poset. By Corollary 2.8 of Section 2, $C \cong S/\rho$ where ρ is a strongly flat left congruence. Let $B = [1]$. Since S has (M_R), there is an element $c \in B$ such that cS is minimal in $\{dS : d \in B\}$. The result now follows from Lemma 3.9. □

We will call a generating set X of an S-poset A *independent* if for any $x, x' \in X$ such that $x \in Sx'$ we have $x = x'$.

Lemma 3.11. *Let A be a left S-poset which satisfies the ascending chain condition for cyclic subposets. If X is a set of generators for A, then X contains an independent set of generators for A.*

Proof. Regarded as an S-act, A satisfies the ascending chain condition for cyclic S-subacts (since these coincide with the cyclic S-subposets). The result now follows from that in the S-act case (Lemma 2 of [11]). $\square$

Lemma 3.12. *Let A be a strongly flat left S-poset which satisfies the ascending chain condition for cyclic S-subposets. If A is indecomposable then A is cyclic.*

Proof. This follows immediately from Lemma 2.9 and Lemma 3 of [11]. $\square$

Corollary 3.13. *If S satisfies Condition (A^O), then every strongly flat left S-poset is a disjoint union of cyclic strongly flat S-posets.*

Proof. It is clear that if A is a strongly flat S-poset, then so are its indecomposable components. It is then immediate from Lemma 3.12 that the indecomposable components are cyclic. $\square$

We now come to the main theorem of this section. We remark that the equivalence of (iii) and (v) is given in [18].

Theorem 3.14. *Let S be a pomonoid. Then the following conditions are equivalent:*
(i) *every strongly flat left S-poset is projective;*
(ii) *every left S-poset of the form $F(\underline{a})$ is projective;*
(iii) *S satisfies Condition (A^O) and (M_R);*
(iv) *S satisfies Condition (A) and (M_R);*
(v) *every strongly flat left S-act is projective.*

Proof. Since every $F(\underline{a})$ is strongly flat, clearly (i) implies (ii). If every S-poset $F(\underline{a})$ is projective, then S has (M_R) from Lemma 3.7 and $(\mathrm{A}^{\mathrm{O}})$ from Lemma 3.5, so that (ii) implies (iii).

Now suppose that (iii) holds. As S satisfies Condition $(\mathrm{A}^{\mathrm{O}})$, from Corollary 3.13, every strongly flat S-poset A is a disjoint union of strongly flat cyclic S-posets; as in addition S has (M_R), then in view of Theorem 3.10, these are all projective, and it follows that A is projective and (iii) implies (i).

The remainder of the result follows from Theorem 1.2 and Corollary 3.3. $\square$

4. Poperfect Pomonoids

We recall that a pomonoid is *left poperfect* if every left S-poset has a *projective cover*, that is, a cover that is projective.

Lemma 4.1. *(cf.* [26]*) A cover of a cyclic left S-poset is cyclic.*

Proof. Suppose that $A = Sa$ is a cyclic left S-poset and suppose that $\beta : B \to A$ is a coessential S-po-epimorphism. Let $b \in B$ be such that $b\beta = a$; then $\beta' = \beta|_{Sb} : Sb \to A$ is an S-po-epimorphism. Since β is coessential we must have that $B = Sb$ and B is cyclic as required. $\square$

We now wish to identify those subpomonoids of S that are the congruence classes of the identity, for any left po-congruence. This will enable us to find conditions under which cyclic S-posets have projective covers.

Definition 4.2. A subpomonoid P of a pomonoid S is *right po-unitary* if for any

$$p, a_1, b_1, \cdots, a_n, b_n, q \in P, s_1, \ldots, s_n \in S,$$

if

$$p \leq s_1\, a_1,\ s_1\, b_1 \leq s_2\, a_2,\ \cdots,\ s_n\, b_n \leq q,$$

then

$$s_1, s_2, \cdots, s_n \in P.$$

We recall that a submonoid U of a monoid M is *right unitary* if $a, ba \in U$ implies that $b \in U$.

Lemma 4.3. *Let S be a pomonoid. If U is a right po-unitary subpomonoid, then U is right unitary.*

Proof. Suppose that $a, ba \in U$. Then as

$$ba \leq b \cdot a,\ \ b \cdot a \leq ba$$

the definition of po-unitarity gives us that $b \in U$. □

The following fact concerning right unitary submonoids is useful.

Lemma 4.4. *Let U be a right unitary submonoid of a monoid S. Then for $a, b \in U$,*

$$Ua \subseteq Ub \text{ if and only if } Sa \subseteq Sb.$$

Proof. If $Ua \subseteq Ub$, then certainly $Sa \subseteq Sb$.

Conversely, if $Sa \subseteq Sb$, then $a = ub$ for some $u \in S$, but as U is right unitary, $u \in U$ so that $Ua \subseteq Ub$ as required. □

Notice that a right unitary submonoid need not be right po-unitary. For an example, take that of $\mathbb{N}^0 = \{0, 1, 2, \ldots\}$ under $+$, with the usual ordering. Then $\mathbb{E} = \{2n : n \in \mathbb{N}^0\}$ is (right) unitary. Notice that

$$0 \leq 1 + 0, 1 + 0 \leq 2$$

but $1 \notin \mathbb{E}$.

Lemma 4.5. *Let S be a pomonoid and let $P \subseteq S$. Then $P = [1]$ for a left po-congruence on S if and only if P is a right po-unitary subpomonoid of S.*

Proof. Let ρ be a left po-congruence on S and let $P = [1]$. Then P is a subpomonoid of S, as if $p_1, p_2 \in P$, then

$$p_1 p_2 \,\rho\, p_1\, 1 \,\rho\, p_1 \,\rho\, 1.$$

Suppose now that $p, a_1, b_1, \cdots, a_n, b_n, q \in P$ and $s_1, \ldots s_n \in S$ are such that

$$p \leq s_1 a_1,\ \ s_1 b_1 \leq s_2 a_2, \cdots, s_n b_n \leq q.$$

As ρ is a left po-congruence, we have in S/ρ that

$$[1] = [p] \leq [s_1 a_1] = s_1[a_1] = [s_1] = s_1[b_1] = [s_1 b_1]$$
$$\leq [s_2 a_2] \ldots = [s_n b_n] \leq [q] = [1]$$

which implies that

$$[1] \leq [s_1] \leq [s_2] \ldots [s_n] \leq [1]$$

so that

$$[1] = [s_1] = \ldots = [s_n] = [1]$$

as required.

Conversely, let P be a left po-unitary subpomonoid of S. Let ρ be $\equiv_{P\times P}$, the S-po-congruence generated by $P \times P$ (note that $P \times P = (P \times P) \cup (P \times P)^{-1}$). From the construction of $\equiv_{P\times P}$, we have that $P \times P \subseteq \rho$ so that as $1 \in P$ we have $P \subseteq [1]$.

Let $w \in [1]$. Then there are elements

$$s_1, \ldots, s_n, t_1, \ldots, t_m \in S$$

and

$$(u_1, v_1), \ldots, (u_n, v_n), (x_1, y_1), \ldots, (x_m, y_m) \in P \times P$$

such that

$$1 \leq s_1u_1, s_1v_1 \leq s_2u_2, \ldots, s_nv_n \leq w = w\,1,$$

$$w\,1 \leq t_1x_1, t_1y_1 \leq t_2x_2, \ldots, t_my_m \leq 1$$

so that as P is left po-unitary we have that $w \in P$ and $P = [1]$ as required.

□

We note that the result below also appears without proof in [26], but the preceding lemma in that article, characterising congruence classes of identities, is incorrect if applied to S-poset congruences.

Proposition 4.6. *(cf.* [26] *and* [16, Proposition III 17.22]*) Let ρ be a left po-congruence on a pomonoid S. The cyclic left S-poset S/ρ has a projective cover if and only if the subpomonoid $R = [1]$ contains a minimal left ideal generated by an idempotent.*

Proof. Suppose that the cyclic S-poset S/ρ has a projective cover; from Lemma 4.1 this must be cyclic. Without loss of generality, let $\alpha : Se \to S/\rho$ be a coessential S-po-epimorphism. Then for some $u \in S$,

$$(ue)\alpha = [1] = u(e\alpha).$$

Since α is coessential, $Sue = Se$ so that $e = que$ for some $q \in S$; we can assume that $q = eq$. Calculating, we have that

$$(uq)^2 = (uq)(uq) = u(qu)eq = u(que)q = ueq = uq,$$

so that $uq \in E(S)$. Moreover,

$$[1] = (ue)\alpha = (uque)\alpha = uq(ue)\alpha = (uq)[1] = [uq]$$

so that $uq \in R = [1]$.

Suppose now that $w \in R$ and $Rw \subseteq Ruq$. Then $w = wuq$ and

$$(wue)\alpha = w(ue)\alpha = w[1] = [w],$$

so that

$$Swue = Se.$$

We then have that

$$Sw = Swuq = Swueq = Seq = Sq$$

and so

$$Sq = Sw = Swuq \subseteq Suq \subseteq Sq.$$

By Lemma 4.3 and 4.4, $Rw = Ruq$ so that Ruq is a minimal left ideal in R.

Conversely, suppose that $R = [1]$ contains a minimal left ideal $Re, e \in E(R)$. Define $\theta : Se \to S/\rho$ by $(se)\theta = [s]$. If $se \leq te$ then as ρ is a po-congruence, we have that

$$[s] = s[1] = s[e] = [se] \leq [te] = t[e] = t[1] = [t],$$

so that θ is well defined and order preserving. It is now easy to see that θ is an onto S-pomorphism. Notice that $e\theta = [1]$.

If $Spe \subseteq Se$ and $\theta|_{Spe} : Spe \to S/\rho$ is onto, then we must have that $(rpe)\theta = [1]$ for some $r \in S$. It follows that $rp \in R$ so that $Rrpe = Re$ and consequently, $Srpe = Se$. We then have that $Spe = Se$ so that θ is coessential as required.

□

Our next corollary follows immediately from Proposition 4.6 and the comment following Definition 2.4.

Corollary 4.7. *A pomonoid S satisfies Condition (D^O) if and only if every cyclic left S-poset has a projective cover.*

Lemma 4.8. *(cf.* [26]*) If a left S-poset A is the union of an infinite strictly ascending chain of cyclic S-subposets then A does not have a projective cover.*

Proof. Suppose $A = \cup_{n\in\mathbb{N}} Sa_n$ and

$$Sa_1 \subset Sa_2 \subset \cdots Sa_n \subset \cdots$$

where all inclusions are strict, is an ascending chain of cyclic S-subposets of A and assume that A has a projective cover P with coessential S-po-epimorphism $\alpha : P \to A$.

Now $P = \cup_{i\in I} P_i$ and we can assume that each $P_i = Se_i$ for some idempotent e_i in S. If $|I| > 1$, take $i \in I$; then if $e_i\alpha \in Sa_n$ for some $n \in \mathbb{N}$, we have that $P_i\alpha \subseteq Sa_n$. Then $\alpha|_{P\setminus P_i}$ is still an S-po-epimorphism and thus P cannot be a cover for A. Finally if $|I| = 1$, say $I = \{1\}$, then if $e_1\alpha \in Sa_m$, the image of α is contained in Sa_m, a contradiction. □

Theorem 4.9. *Let S be a pomonoid. Then S is left poperfect if and only if S satisfies Conditions (A^O) and (D^O).*

Proof. Suppose S is left poperfect. Then Condition ($\mathrm{A^O}$) and Condition ($\mathrm{D^O}$) follow from Lemma 4.8 and Corollary 4.7, respectively.

Conversely, suppose that S satisfies Conditions ($\mathrm{A^O}$) and ($\mathrm{D^O}$). By Corollary 4.7, every cyclic S-poset has a projective cover.

Let A be an arbitrary S-poset. From Lemma 3.11, A has an independent set X of generators. For each $x \in X$, let $\alpha_x : Se_x \to Sx$ be a coessential S-po-epimorphism, where $e_x \in E(S)$. Let $G = \bigcup_{\bar{x}\in X} Se_x\bar{x}$ be the S-subposet of the free left S-poset on $\overline{X} = \{\bar{x} : x \in X\}$ and define $\alpha : G \to A$ by $(se_x\bar{x})\alpha = (se_x)\alpha_x$. Clearly, α is an S-po-epimorphism.

Suppose that α is not coessential. Then there exists $y \in X$ and a strict left ideal I of Se_y such that

$$\alpha : \bigcup_{x\in X\setminus\{y\}} Se_x\bar{x} \ \cup I\bar{y} \to A$$

is onto. Consequently, $y = (ue_x\bar{x})\alpha_x \in Sx$ for some $x \neq y$, a contradiction, or $y = (pe_y\bar{y})\alpha$ for some $pe_y \in I$ and so $\alpha_y : I \to Sy$ is onto, contradicting the coeesentiality of α_y. Hence α is coessential. □

5. Right collapsible subpomonoids

In this section we consider Condition (K) for a pomonoid S, introduced by Kilp for monoids in [14]. In [15], it is proved that a monoid is left perfect if and only if it satisfies Condition (A) and (K). Similar techniques are employed in the article of Renshaw [19]. Our aim here is to show the ordered analogue.

Our first result follows immediately from Lemma 2.9 and Lemma 2.2 of [14].

Lemma 5.1. *Let ρ be a left po-congruence on S such that S/ρ is strongly flat and let $P = [1]$. Then P is a right collapsible subpomonoid.*

Lemma 5.2. *Let $P \subseteq S$ be a right collapsible subpomonoid and let ρ be the relation $\equiv_{P\times P}$ on S. Then*
(i) *ρ is a left po-congruence;*
(ii) *$P \subseteq [1]$*
and
(iii) *S/ρ is strongly flat.*

Proof. (i) and (ii) are clear from the definition of $\equiv_{P\times P}$.

(iii) Suppose now that $[s] \leq [t]$ in S/ρ. Then

$$s \leq u_1p_1,\ u_1q_1 \leq u_2p_2, \dots, u_nq_n \leq t$$

for some $p_1, q_1, \dots, p_n, q_n \in P$ and $u_1, \dots, u_n \in S$. Since P is right collapsible, we can find $z_1 \in P$ with $p_1z_1 = q_1z_1$. Then

$$sz_1 \leq u_1p_1z_1 = u_1q_1z_1.$$

If $n = 1$, we then have that $sz_1 \leq tz_1$. Otherwise, $sz_1 \leq u_2p_2z_1$ and we pick $z_2 \in P$ with $p_2z_1z_2 = q_2z_1z_2$. Then

$$sz_1z_2 \leq u_2p_2z_1z_2 = u_2q_2z_1z_2.$$

If $n = 2$ we obtain that $sz_1z_2 \leq tz_1z_2$, if not we continue in this manner, until we obtain that $sz_1 \dots z_n \leq tz_1 \dots z_n$. As $z_1 \dots z_n \in P$, and $P \subseteq [1]$, we have that S/ρ is strongly flat by Corollary 2.8. □

We can now verify the ordered analogue of Theorem 2.3 of [14].

Lemma 5.3. *Let S be a pomonoid. All strongly flat cyclic left S-posets are projective if and only if S satisfies Condition (K).*

Proof. Suppose that all strongly flat cyclic S-posets are projective. Let $P \subseteq S$ be a right collapsible subpomonoid. By the above lemma we can construct a left po-congruence ρ on S such that S/ρ is strongly flat and $P \subseteq [1]$. By assumption, S/ρ is projective, and so there exists an idempotent $e \in S$ with $e\,\rho\,1$ and such that for all $s, t \in S$, if $[s] \leq [t]$ then $se \leq te$.

As in Lemma 5.2, we know that if $s\,\rho\,t$, then there exists $z \in P$ with $sz \leq tz$. We have that $1\,\rho\,e$ and so $z \leq ez$ for some $z \in P$. Now $ez\,\rho\,z$, and so there exists $w \in P$ with $ezw \leq zw$. We therefore have

$$ezw \leq zw \leq ezw$$

and so $ezw = zw$. Let $x \in P$; since $1\,\rho\,x$ for all $x \in P$, we will have $e = xe$ from Lemma 2.5.

Now let $x \in P$ be an arbitrary element and let $l = zw$. Then

$$xl = xel = el = l,$$

so that l is a right zero for P.

Conversely, suppose that (K) holds. Let ρ be a left po-congruence on S such that S/ρ is strongly flat; we must show that S/ρ is isomorphic to some Se, where $e \in E(S)$, *as an S-poset*. Let $P = [1]$; then P is a right collapsible subpomonoid of S by Lemma 5.1. By assumption there exists a right zero say $e \in P$. Then e is an idempotent and $1 \,\rho\, e$.

Suppose $[s] \leq [t]$ for some $s, t \in S$. As S/ρ is strongly flat, there exists $u \in S$ such that $u \,\rho\, 1$ and $su \leq tu$. Note that

$$se = s(ue) \leq t(ue) = te,$$

hence S/ρ is projective by Lemma 2.5. □

6. Left poperfect pomonoids and $\mathcal{SF} = \mathcal{P}$

The aim of this section is to show that a pomonoid is left poperfect if and only if $\mathcal{SF} = \mathcal{P}$. The same result has recently appeared in [18]. For completeness, we provide a full proof. In view of Corollary 3.13 and Lemma 6.1 this amounts to showing that in the presence of Condition ($\mathrm{A^O}$), Condition ($\mathrm{D^O}$) is equivalent to (M_R). It will then follow immediately that a pomonoid is left poperfect if and only if it is left perfect.

Lemma 6.1. *If S satisfies Condition (D^O), then every strongly flat cyclic left S-poset is projective.*

Proof. As in Theorem 3.10 a strongly flat cyclic S-poset is isomorphic to S/ρ where ρ is some strongly flat left po-congruence and $B = [1]$ is a left po-unitary subpomonoid of S. Condition ($\mathrm{D^O}$) gives that B has a minimal left ideal say Be generated by an idempotent e. By Lemma (8.12) in [7], eB is a minimal right ideal of B.

Suppose now that $d \in B$ and $dS \subseteq eS$. Then $d = ed$, so that $dB \subseteq eB$ and so the minimality of eB gives that $dB = eB$. Consequently, $eS = dS$, so that eS is minimal in $\mathcal{I} = \{dS : d \in B\}$. The result now follows from Lemma 3.9. □

Let S be a pomonoid. Given that we proved in Section 3 that Conditions (A) and ($\mathrm{A^O}$) are equivalent, the proof of the next result could essentially be taken from [13]. However, a significant part of the proof of Result 1.7 of that article relies on categorical techniques that we have avoided below. Our argument is in some sense a clarification of that in [13]. As stated above, it also follows from the strategy given in [18].

Theorem 6.2. *Let S be a pomonoid such that S satisfies Condition (M_R) and Condition (A^O). Then S has Condition (D^O).*

Proof. If S has (M_R) and ($\mathrm{A^O}$), then as every strongly flat S-poset is projective, it follows from Theorem 2.6 of Section 2, that every direct limit of copies of S, regarded as a left S-poset, is projective.

Let S/ρ be a cyclic S-poset; to avoid ambiguity in this proof we denote the ρ-class of $a \in S$ by $[a]_\rho$. Let $B = [1]_\rho$.

Suppose $v \in E(S) \cap B$ and $t \in B$ with $St \subseteq Sv$. As $t \in B$ and B is a submonoid it is clear that $t^n \in B$.

Let

$$Sx_1 \to Sx_2 \to \dots$$

be a direct sequence of copies of S, where $x_i\alpha_i = tx_{i+1}$ for all $i \in \mathbb{N}$. Put

$$\underline{t} = (t, t, \dots)$$

so that by Lemma 3.1, the direct limit is $F(\underline{t})$. By assumption, $F(\underline{t})$ is projective, so as it is indecomposable, $F(\underline{t}) = S[px_i]$ for some px_i where $[px_i]$ is ordered right e-cancellable for some $e \in E(S)$.

Let $\nu_i : Sx_i \to S/\rho$ be defined by $x_i\nu_i = [1]_\rho$.

$$\begin{tikzcd}
Sx_1 \arrow[r, "\nu_1"] \arrow[d, "\alpha_1"] \arrow[ddddd, bend right=60, "\beta_1"] & S/\rho \\
Sx_2 \arrow[ur, "\nu_2"] \arrow[d, "\alpha_2"] \arrow[dddd, bend right=50, "\beta_2"] & \\
Sx_3 \arrow[uur, "\nu_3"] \arrow[d, "\alpha_3"] \arrow[ddd, bend right=40, "\beta_3"] & \\
Sx_i \arrow[uuur, "\nu_i"] \arrow[d, "\alpha_i"] \arrow[dd, bend right=30, "\beta_i"] & \\
{} \arrow[d, dashed] & \\
S[px_i] \arrow[uuuuur, "\gamma"] &
\end{tikzcd}$$

We note that

$$x_i\,\alpha_i\,\nu_{i+1} = (tx_{i+1})\nu_{i+1} = t[1]_\rho = [t]_\rho = [1]_\rho = x_i\nu_i$$

which implies that $\alpha_i\nu_{i+1} = \nu_i$. By definition of direct limit, there exists an S-pomorphism $\gamma : S[px_i] \to S/\rho$ such that $\beta_i\gamma = \nu_i$ for all $i \in \mathbb{N}$.

Define $\tau : S[px_i] \to Sx_i$ by $(u[px_i])\tau = uepx_i$. As $[px_i]$ is ordered right e-cancellative, it follows that τ is well defined. It is easy to see τ is an S-pomorphism.

Now

$$[px_i]\tau\beta_i = (epx_i)\beta_i = [epx_i] = e[px_i] = [px_i],$$

so that

$$\tau\,\beta_i = I_{S[px_i]}.$$

Put $\psi = \beta_{i+1}\tau\alpha_i : Sx_{i+1} \to Sx_{i+1}$; then

$$\psi^2 = (\beta_{i+1}\tau\alpha_i)(\beta_{i+1}\tau\alpha_i) = \beta_{i+1}\tau(\alpha_i\beta_{i+1})\tau\alpha_i = \beta_{i+1}\tau\beta_i\tau\alpha_i = \beta_{i+1}I_{S[px_i]}\tau\alpha_i = \beta_{i+1}\tau\alpha_i = \psi.$$

It is then easy to see that

$$x_{i+1}\psi = hx_{i+1}$$

for some $h \in E(S)$.

Calculating,

$$hx_{i+1} = x_{i+1}\psi = x_{i+1}\beta_{i+1}\tau\alpha_i = (wx_i)\alpha_i = wtx_{i+1}$$

for some $w \in S$ and therefore $h = wt$, giving that $Sh \subseteq St$.

We check that

$$\beta_{i+1}\,\tau\,\alpha_i\,\nu_{i+1}=\beta_{i+1}\,\tau\,\alpha_i\,\beta_{i+1}\,\gamma=\beta_{i+1}\,\gamma=\nu_{i+1}$$

and

$$[h]_\rho=(h\,x_{i+1})\nu_{i+1}=x_{i+1}\psi\nu_{i+1}=x_{i+1}\nu_{i+1}=[1]_\rho$$

thus $h\in B$.

Suppose now that

$$Se_1\supseteq Se_2\supseteq Se_3\cdots$$

is a desending chain of principal left ideals generated by idempotents $e_i\in S$. From Lemma 1.2.10 of [12], there are idempotents $g_1,g_2,\dots$ such that for all $i\in\mathbb{N}$, we have that $Sg_i=Se_i$ and

$$g_1\geq g_2\geq\dots$$

under the natural partial order on $E(S)$. Higgins remarks on [12, page 28] that if S is regular and satisfies M_R, then it also satisfies M_L. Here we do not know that S is regular, but certainly

$$g_1S\supseteq g_2S\supseteq\dots$$

and as S has M_R we deduce that for some $n\in\mathbb{N}$,

$$g_nS=g_{n+1}S=\dots$$

and hence $g_n=g_{n+1}=\dots$. Consequently,

$$Se_n=Se_{n+1}=\dots.$$

Certainly $1\in B$ and we have shown that every principal left ideal St where $t\in B$ contains a principal left ideal Sh where $h\in E(S)\cap B$. It follows from the above that there is an idempotent $e'\in B$ such that Se' is minimal with respect to being generated by an element of B. By Lemma 4.4, Be' is a minimal left ideal of B. Hence S satisfies Condition $(\mathrm{D}^{\mathrm{O}})$. □

We can now give our final result. Some of the equivalences appear in Section 3 of [18].

Theorem 6.3. *For a pomonoid S, the following are equivalent:*
(i) every strongly flat left S-poset is projective;
(ii) S satisfies Conditions (A^O) and (M_R);
(iii) S satisfies Conditions (A^O) and (D^O);
(iv) S is left poperfect;
(v) S satifies Conditions (A^O) and (K);
(vi) every strongly flat left S-act is projective;
(vii) S satisfies Conditions (A) and (M_R);
(viii) S satisfies Conditions (A) and (D);
(ix) S is left perfect;
(x) S satifies Conditions (A) and (K).

Proof. In view of Theorems 1.2, 3.14, 4.9, 6.2 and Corollary 3.3, we need only to show that (ii) and (iii) are equivalent.

If (iii) holds, by Corollary 3.13, every strongly flat S-poset can be written as a disjoint union of cyclic strongly flat S-posets which are projective by Lemma 6.1 as S satisfies Condition $(\mathrm{D}^{\mathrm{O}})$, hence every strongly flat S-poset is projective. By Theorem 3.14, S satisfies (ii).

Conversely, suppose that (*ii*) holds, then (*iii*) follows from Theorem 6.2.

□

We remark that it is clear that Condition (D) implies (D^O), and in view of Lemma 6.1, (D^O) implies (K). It is known [15] that (K) does not imply (D), and the same example (of the free monogenic monoid) with length as partial order, shows that (K) does not imply (D^O). It remains to show whether (D) and (D^O) are equivalent.

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