Semigroups of Left I-quotients

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Background

- Ore (1940)

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- Gould and Ghroda (2010)

Left I-order

Definition

A subsemigroup S of an inverse semigroup Q is a *left I-order* in Q or Q is a semigroup of *left I-quotients* of S if every element of Q can be written as $a^{-1}b$ where a and b are elements of S and a^{-1} is the inverse of a in the sense of inverse semigroup theory.

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Left ample semigroups

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for all $x, y \in S^1$

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- A semigroup S is a left ample if and only if
  (i) $E(S)$ is a semilattice.
  (ii) every $\mathcal{R}^*$-class contains an idempotent $(a\,\mathcal{R}^*\,a^+)$.
  (iii) for all $a \in S$ and all $e \in E(S)$,

$$(ae)^+ a = ae.$$

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$$(ae)^+ a = ae.$$

- $\phi : S \longrightarrow \mathcal{I}_S$ defined by

$$a\phi = \rho_a$$

where $\rho_a : Sa^+ \longrightarrow Sa$ defined by $x\rho_a = xa$

Left ample semigroups

Theorem

If S is a left ample semigroup then,
S is a left I-order in its inverse hull $\Sigma(S) \iff S$ satisfies (LC) condition.

Extension of homomorphisms

- Let S be a subsemigroup of Q and let $\phi : S \to P$ be a morphism from S to a semigroup P. If there is a morphism $\overline{\phi} : Q \to P$ such that $\overline{\phi}|_S = \phi$, then we say that ϕ *lifts to* Q. If ϕ lifts to an isomorphism, then we say that Q and P are *isomorphic over* S.

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- On a straight left I-order semigroup S in a semigroup Q we define a relation $\mathcal{T}_S^Q$ on S as follows:

$$(a, b, c) \in \mathcal{T}_S^Q \iff ab^{-1}Q \subseteq c^{-1}Q.$$

Theorem

Let S be a straight left I-order in Q and let T be a subsemigroup of an inverse semigroup P. Suppose that $\phi : S \to T$ is a morphism. Then ϕ lifts to a (unique) morphism $\overline{\phi} : Q \to P$ if and only if for all $(a, b, c) \in S$:
(i) $(a, b) \in \mathcal{R}_S^Q \Rightarrow (a\phi, b\phi) \in \mathcal{R}_T^P$;
(ii) $(a, b, c) \in \mathcal{T}_S^Q \Rightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_T^P$.
If (i) and (ii) hold and $S\phi$ is a left I-order in P, then $\overline{\phi} : Q \to P$ is onto.

Corollary

Let S be a straight left I-order in Q and let $\phi : S \to P$ be an embedding of S into an inverse semigroup P such that $S\phi$ is a straight left I-order in P. Then Q is isomorphic to P over S if and only if for any $a, b, c \in S$:
(i) $(a, b) \in \mathcal{R}_S^Q \Leftrightarrow (a\phi, b\phi) \in \mathcal{R}_{S\phi}^P$; *and*
(ii) $(a, b, c) \in \mathcal{T}_S^Q \Leftrightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_{S\phi}^P$.

Corollary

Let S be a straight left I-order in semigroups Q and P and φ be the embedding of S in P. Then $Q \cong P$ if and only if for all $a, b \in S$,

$$a \,\mathcal{R}\, b \text{ in } Q \iff a\varphi \,\mathcal{R}\, b\varphi$$

and

$$(a, b, c) \in \mathcal{T}_{\mathcal{S}}^{\mathcal{Q}} \iff (a\varphi, b\varphi, c\varphi) \in \mathcal{T}_{\mathcal{S}\varphi}^{\mathcal{P}}.$$

Semilattice of semigroups

Definition

Let Y be a semilattice. A semigroup S is called a *semilattice* Y of semigroups $S_\alpha, \alpha \in Y$, if $S = \bigcup_{\alpha \in Y} S_\alpha$ where $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$.

Strong semilattices of semigroups

Definition

Let Y be a semilattice. Suppose to each $\alpha \in Y$ there is associated semigroup S_α and assume that $S_\alpha \cap S_\beta \neq \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ be a homomorphism such that the following conditions hold:
1) $\varphi_{\alpha,\alpha} = \iota_{S_\alpha}$,
2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$,
On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$a * b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})$$

if $a \in S_\alpha, b \in S_\beta$.

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- $$a_\alpha^{-1} b_\alpha c_\beta^{-1} d_\beta = (ta_\alpha)^{-1}(rd_\beta) \text{ where } S_{\alpha\beta} b_\alpha \cap S_{\alpha\beta} c_\beta = S_{\alpha\beta} w$$

  and $tb = rc = w$ for some $t, r \in S_{\alpha\beta}$.

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  and $tb = rc = w$ for some $t, r \in S_{\alpha\beta}$.
- Q is a semigroup
- The multiplication on Q extends the multiplication on S.
- S is a left I-order in $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$.

Left I-orders in semilattices of inverse semigroups

Theorem

(Gantos) *Let S be a semilattice of right cancellative monoids S_α. Suppose that S, and S_α, has (LC) condition. Then $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$ is a semilattice of bisimple inverse monoids (Σ_α inverse hulls of S_α) and the multiplication in Q is defined by*

$$a_\alpha^{-1} b_\alpha c_\beta^{-1} d_\beta = (ta_\alpha)^{-1}(rd_\beta) \text{ where } S_{\alpha\beta} b_\alpha \cap S_{\alpha\beta} c_\beta = S_{\alpha\beta} w$$

and $tb = rc = w$ for some $t, r \in S_{\alpha\beta}$.

Corollary

The semigroup S defined as above is a left I-order in $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$.

Theorem

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice of right cancellative monoids with (LC) condition and S has (LC) condition. Let $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$ where Σ_α is inverse hull of S_α. Then Q is a strong semilattice of monoids Σ_α and S is a left I-order in Q.

Left I-orders in semilattices of inverse semigroups

- we say that a $(2,1)$-morphism $\phi : S \to T$, where S and T are left ample semigroups with Condition (LC) is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

Left I-orders in semilattices of inverse semigroups

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$$T(b\phi) \cap T(c\phi) = T(w\phi).$$

Lemma

Let S_α be a left ample semigroup with (LC) condition and $\varphi_{\alpha,\beta}, \alpha \geq \beta$, is (LC)-preserving. *Then $S = \bigcup_{\alpha \in Y} S_\alpha$ is a left I-order in a strong semilattice $Q = \bigcup_{\alpha \in Y} \Sigma_\alpha$ where Σ_α is an inverse hull of S_α.*

Primitive inverse semigroups of left I-quotients

Theorem

A semigroup S is a left I-order in a primitive inverse semigroup Q if S satisfies the following conditions;
A) S *is categorical at* 0,
B) S *is 0-cancellative,*
C) *For any* $a, b \in S^*$ $a \mathcal{R}^* b \Longleftrightarrow xa \neq 0,\ xb \neq 0$ *for some* $x \in S^*$,
D) λ *is transitive,* $(\ a \lambda b \Longleftrightarrow a = b = 0 \ \text{ or } \ Sa \cap Sb \neq 0\ )$
E) $Sa \neq 0$ *for all* $a \in S$.

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- Let S satisfies the conditions (A)-(E). Define $\Sigma = \{(a, b) \in S \times S; a \mathcal{R}^* b\}$

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- $(a, b) \sim (c, d) \iff a = b = c = d = 0$, or there exist $x, y \in S$ such that $xa = yc \neq 0,\ xb = yd \neq 0$.
- Let $Q = \Sigma/\sim$ define

$$[a, b][c, d] = \begin{cases} [xa, yd] & \text{if } b\,\lambda\,c \quad \text{and} \quad xb = yc \neq 0 \\ 0 & \text{else} \end{cases}$$

and $0[a, b] = [a, b]0 = 00 = 0$ where $0 = [0, 0]$.

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- Q is primitive.
- S is embedded in Q.
- S is a left I-order in Q.

Primitive inverse semigroups of left I-quotients

Theorem

A semigroup S is a left I-order in a primitive inverse semigroup Q if and only if S satisfies the following conditions;

A) S *is categorical at* 0,

B) S *is 0-cancellative,*

C) For any $a, b \in S^*$ $a\,\mathcal{R}^*\,b \Longleftrightarrow xa \neq 0,\; xb \neq 0$ *for some* $x \in S^*$,

D) λ *is transitive,* $(\, a\,\lambda\, b \Longleftrightarrow a = b = 0$ *or* $Sa \cap Sb \neq 0\,)$

E) $Sa \neq 0$ *for all* $a \in S$.

Brandt semigroups of left I-quotients

Lemma

A semigroup S is a left I-order in a primitive inverse semigroup Q. Then Q is a Brandt semigroup if and only if for all $a, b \in S$ there exist $c, d \in S$ such that $ca \, \mathcal{R}^* \, d \, \lambda \, b$.