CONSTELLATIONS WITH RANGE

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Abstract. Constellations are asymmetric generalisations of categories. Although they are not required to possess a notion of range, most of the commonly occurring ones do. Here we show that constellations with a well-behaved range operation are equivalent to ordered categories with restrictions. As a special case, we show that the constellation analogs of inverse semigroups are equivalent to ordered groupoids.

1. Introduction

Constellations are "one-sided" versions of categories, in which there is a notion of domain but in general no notion of range. They were first defined by Gould and Hollings in [2]. Their purpose was to provide the means for obtaining a variant of the so-called Ehresmann-Schein-Nambooripad (ESN) Theorem relating inverse semigroups to inductive groupoids (see [1] and [11]) which applied to left restriction semigroups.

Following the establishment of the ESN Theorem, Lawson provided a correspondence along similar lines in [7] that connected so-called Ehresmann semigroups (bi-unary semigroups with both and domain and range-like operations) to certain types of ordered categories. Two-sided restriction semigroups are Ehresmann semigroups, and are covered as a special case in [7]. However, one-sided restriction semigroups are perhaps even more natural than two-sided ones since they provide the algebraic model for partial functions, so there was interest in obtaining an analogous result for them. The problem is the lack of a range operation, hence the need to develop the notion of a constellation as in [2].

Following their introduction in [2], where they were essentially used as a tool to model left restriction semigroups, constellations were studied further for their own sake in [4], where it was shown that every category can be obtained from a "canonically simple" constellation (which is a quotient of the category) by means of a straightforward extension process that can be viewed as a one-sided variant of idempotent completion. For example, the category of sets may be obtained via this extension process from the constellation of sets, where only surjective functions are considered and composition of two functions is defined if and only if the image of the first is contained in the domain of the second.

Every category has a constellation reduct, and the resulting constellation has an inherent notion of range. In fact much more general sorts of constellations often

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admit a well-defined range operation satisfying some natural laws. In this work we define constellations with range, give many examples, and explore their basic properties. Thus constellations with range may be thought of as mid-way between constellations and categories, and as such have a definite utility.

A strategy for many types of semigroup with domain and range-like operations is to try to represent them very economically, using partial operations in which as few products as possible are retained, for example as a small category. As briefly discussed above, this was the approach of the ESN theorem for inverse semigroups, which led to the work of Lawson who established an equivalence between Ehresmann semigroups and inductive categories in [7]. It was also the approach used by Gould and Hollings in [2], where they showed that left restriction semigroups correspond to what they called inductive constellations. We adopt a similar approach to constellations with range. By limiting to certain products, it is possible to obtain a category with additional order structure, in which there is a notion of left (but not generally right) restriction satisfying some natural conditions that appear in [7]. We then show it is possible to reverse the process by beginning with such an ordered category and constructing a constellation with range from it. The constructions are shown to be inverse, and as for the ESN Theorem and the cases considered in [7] and [2], a category isomorphism is established between such constellations with range and such ordered categories with restrictions.

A special case arises from ordered groupoids. The ESN Theorem asserts that the category of inductive groupoids (ordered groupoids in which the domain elements are a meet semilattice under the given order), is essentially just that of inverse semigroups. We show that ordered groupoids in general are nothing but constellations having a natural notion of inverse, and hence are constellations with range. A family of examples is given that generalise those based on the partial isometries in an involuted semigroup as in [8].

2. Algebraic Preliminaries

Throughout, we generally write functions on the right of their arguments rather than the left, so "xf" rather than "$f(x)$". Correspondingly, we write function compositions left to right, so that "fg" is "first f, then g", rather than the other way around. An exception to this is unary operation application; if D is a unary operation on the set S, we write $D(s)$ for $s \in S$ rather than sD.

Next we give the definitions and some of the basic properties of constellations and categories that we use in what follows.

Let C be a class (usually a set) with a partial binary operation. Recall that $e \in C$ is a *right identity* if it is such that, for all $x \in C$, if $x \cdot e$ is defined then it equals x; left identities are defined dually. An *identity* is both a left and right identity. (Note we are not assuming that $e \cdot e$ exists in any of these cases.)

Following [4] and [7] where an object-free formulation was also used, recall that a *category* $(C, \circ)$ is a class C with a partial binary operation $\circ$ satisfying the following:

(Cat1) $x \circ (y \circ z)$ exists if and only if $(x \circ y) \circ z$ exists, and then the two are equal;
(Cat2) if $x \circ y$ and $y \circ z$ exist then so does $x \circ (y \circ z)$;
(Cat3) for each $x \in P$, there are identities e, f such that $e \circ x$ and $x \circ f$ exist.

The identities e, f in (Cat3) are easily seen to be unique, and we write $D(x) = e$ and $R(x) = f$. (Note that this is the opposite of the convention often used, and corresponds to the fact that we view a composition of functions fg as "first f, then g", discussed earlier.)

It also follows easily that for every identity e, $e \circ e = e$, and $D(e) = e = R(e)$. Moreover, the collection of domain elements $D(x)$ (equivalently, range elements $R(x)$) is precisely the collection of identities in the category. We will often view a category as a partial algebra $(C, \circ, D, R)$. If x, y are elements of a category C, then the product $x \circ y$ exists if and only if $R(x) = D(y)$, and if $x \circ y$ exists then $D(x \circ y) = D(x)$ and $R(x \circ y) = R(y)$.

Following an earlier equivalent definition given in [2], we make use of a result in [4] and define a *constellation* $(P, \cdot)$ to be a class P with a partial binary operation $\cdot$ satisfying the following:

(Const1) if $x \cdot (y \cdot z)$ exists then $(x \cdot y) \cdot z$ exists, and then the two are equal;
(Const2) if $x \cdot y$ and $y \cdot z$ exist then so does $x \cdot (y \cdot z)$;
(Const3) for each $x \in P$, there is a unique right identity e such that $e \cdot x = x$.

Since e in (Const3) is unique given $x \in P$, we call it $D(x)$. It follows that $D(P) = \{D(s) \mid s \in P\}$ is the set of right identities of P, a set we call the *projections* of P. We adopt the usual convention of referring to the constellation $(P, \cdot)$ simply as P if there is no ambiguity. However, because D can be viewed as a unary operation, we also often view constellations as partial algebras $(P, \cdot, D)$. As shown in [4], every category becomes a constellation when the operation R is ignored. We say a constellation $(P, \cdot)$ is *small* if P is a set.

The following are some useful basic facts about constellations, to be found in [2] or [4].

Result 2.1. *For s, t elements of the constellation P, $s \cdot t$ exists if and only if $s \cdot D(t)$ exists, and then $D(s \cdot t) = D(s)$.*

A constellation is *categorial* if it arises from a category as a reduct (obtained by dropping R).

Result 2.2. *Let P be a constellation. Then P is categorial if and only if for all $s \in P$ there is a unique $e \in D(P)$ such that $s \cdot e$ exists, and then $R(s) = e$ when P is viewed as a category.*

Recall P is *normal* if for all $e, f \in D(P)$, if $e \cdot f$ and $f \cdot e$ exist, then $e = f$. Our next result highlights the fact that constellations are intimately associated with orders.

Result 2.3. *If P is a constellation, define the relation $\leq$ on $D(P)$ by $e \leq f$ if and only if $e \cdot f$ exists. Then $\leq$ is a quasiorder we call the* standard quasiorder *on $D(P)$. Defining $s \leq t$ for $s, t \in P$ whenever $s = e \cdot t$ for some $e \in D(P)$ (equivalently, $s = D(s) \cdot t$) makes $\leq$ a quasiorder on all of P that agrees with the standard quasiorder on $D(P)$, called the* natural quasiorder *on P. In both these cases, the quasiorder is a partial order if and only if P is normal, and then we use "order" rather than "quasiorder" to describe them.*

An important example of a small normal constellation, introduced in [2], is $\mathcal{C}_X$, consisting of partial functions on the set X, in which $s \cdot t$ is the usual composite of s followed by t provided $\mathrm{Im}(s) \subseteq \mathrm{Dom}(t)$, and undefined otherwise, and $D(s)$ is the restriction of the identity map on X to $\mathrm{Dom}(s)$.

A *subconstellation* Q of a constellation P is a subset of P that is closed under the constellation product wherever it is defined, and closed under D; Q is then a constellation in its own right as shown in [4], where it was also shown that every small normal constellation embeds as a subconstellation in the (normal) constellation $\mathcal{C}_X$ for some choice of X.

For constellations, the notion of morphism is as follows. If P, Q are constellations, a function $\rho : P \to Q$ is a *radiant* if for all $s, t \in P$ for which $s \cdot t$ exists, so does $(s\rho) \cdot (t\rho)$ and indeed $(s \cdot t)\rho = (s\rho) \cdot (t\rho)$, and $D(s\rho) = D(s)\rho$. As observed in [2], the class of constellations is a category in which the morphisms are radiants.

3. Range in a constellation

3.1. **Constellations with range.** In the examples of constellations considered in [4], most had a notion of range, which in all cases had the property that the range $R(s)$ of an element s was an element of $D(P)$ with the property that $s \cdot R(s)$ exists, and $R(s)$ was the smallest $e \in D(P)$ with respect to the standard quasiorder on $D(P)$ for which $s \cdot e$ exists. More precisely, P was such that for all $s \in P$, the set

$$s_D = \{e \in D(P) \mid s \cdot e \text{ exists, and for all } f \in D(P), s \cdot f \text{ exists implies } e \leq f\}$$

contains a single element. (In a general constellation, s_D could be empty, or even have more than one element if the standard quasiorder on $D(P)$ is not a partial order, depending on the choice of s.)

Definition 3.1. *A* constellation with range *is a constellation $(P, \cdot)$ in which for all $s \in P$ the set s_D has a single element (which we call $R(s)$).*

Again, we often view a constellation with range as a partial binary algebra having two unary operations, $(P, \cdot, D, R)$. As usual, we may at times simply write P when the partial binary and unary operations are understood.

Proposition 3.2. *Suppose P is a constellation with range. Then*

- *(R1) $D(R(s)) = R(s)$ for all $s \in P$;*
- *(R2) $s \cdot R(s)$ exists for all $s \in P$;*
- *(R3) if $s \cdot t$ exists then so does $R(s) \cdot t$, for all $s, t \in P$;*
- *(R4) $R(e) = e$ for all $e \in D(P)$;*

- *(R5) P is normal.*

Proof. (R1) and (R2) are immediate. If $s \cdot t$ exists then $s \cdot D(t)$ exists by Result 2.1, so $R(s) \leq D(t)$, and so $R(s) \cdot D(t)$ exists, so $R(s) \cdot t$ exists, again by Result 2.1, which proves (R3). If $e \in D(P)$, and if $e \cdot f$ exists for some $f \in D(P)$, then $e \leq f$ by definition, so $e \in e_D$, and then $R(e) = e$, giving that (R4) holds. Finally, if $e, f \in D(P)$ with $e \cdot f$ and $f \cdot e$ both existing, then $e \leq f \leq e$ and $e, f \in e_D$ so that $e = f$ and (R5) holds. □

Conversely, we have the following.

Proposition 3.3. *Suppose $(P, \cdot, D)$ is a constellation with additional unary operation R satisfying the laws (R1)–(R5) in Proposition 3.2. Then $(P, \cdot, D, R)$ is a constellation with range.*

Proof. For all $s \in P$, $R(s) \in D(P)$ by (R1). (R2) shows that $R(s)$ is one $e \in D(P)$ for which $s \cdot e$ exists, while if also $s \cdot f$ exists for some $f \in D(P)$, then $R(s) \cdot f$ exists by (R3), so $R(s) \leq f$ under the standard order on $D(P)$, which is a partial order by (R5) and Result 2.3. So $R(s)$ is the smallest $e \in D(S)$ for which $s \cdot e$ exists. □

There is a kind of dual version of Result 2.1 applying to constellations with range.

Lemma 3.4. *Suppose P is a constellation with range. Then for all $s, t \in P$, $s \cdot t$ exists if and only if $R(s) \cdot t$ exists. Hence for all $s \in P$, $R(s)$ is the smallest $e \in D(P)$ such that $s \cdot e = s$, and if $s \cdot t$ exists, then $R(s \cdot t) \leq R(t)$.*

Proof. (R3) asserts one direction. Conversely, suppose that $R(s) \cdot t$ exists. Now $s \cdot R(s)$ exists by (R2), so it follows from (Const2) that $s \cdot (R(s) \cdot t)$ exists and hence by (Const1) so does $(s \cdot R(s)) \cdot t = s \cdot t$. Hence if $s \cdot e$ exists for some $s \in P$ and $e \in D(P)$, then $R(s) \cdot e$ exists, so $R(s) \leq e$. So if $s \cdot t$ exists, then it equals $s \cdot (t \cdot R(t)) = (s \cdot t) \cdot R(t)$, and so $R(s \cdot t) \leq R(t)$. □

There is a further order-theoretic characterisation of when $s \cdot t$ exists in a constellation with range.

Proposition 3.5. *In the constellation with range P, $s \cdot t$ exists if and only if $R(s) \leq D(t)$.*

Proof. From Result 2.1 and Lemma 3.4, the following are equivalent: $s \cdot t$ exists; $R(s) \cdot t$ exists; $R(s) \cdot D(t)$ exists; $R(s) \leq D(t)$. □

If P is a constellation, define the equivalence relation ϵ_D on P such that $(x, y) \in \epsilon_D$ if and only if $D(x) = D(y)$. Note that if $(x, y) \in \epsilon_D$ and $z \cdot x$ exists then so does $z \cdot D(x) = z \cdot D(y)$ by Result 2.1, whence so does $z \cdot y$ by the same result, and moreover $D(z \cdot x) = D(z) = D(z \cdot y)$. So ϵ_D is a strong (in the sense of [5]) left congruence on P.

Similarly we may define ϵ_R on P by setting $(x, y) \in \epsilon_R$ if and only if $R(x) = R(y)$. In general this need not be a right congruence, but if it is, it is strong.

Proposition 3.6. *Let P be a constellation with range. Then ϵ_R is a right congruence if and only if $R(x \cdot y) = R(R(x) \cdot y)$ for all $x, y \in P$, in which case it is strong.*

Proof. Suppose $R(x \cdot y) = R(R(x) \cdot y)$ for all $x, y \in P$. Let $(s, t) \in \epsilon_R$. If $s \cdot x$ exists then $R(s) \cdot x = R(t) \cdot x$ exists by Lemma 3.4, whence so does $t \cdot x$ by the same result. Moreover, $R(s \cdot x) = R(R(s) \cdot x) = R(R(t) \cdot x) = R(t \cdot x)$. So $(s \cdot x, t \cdot x) \in \epsilon_R$, which is therefore a (strong) right congruence.

Conversely, suppose ϵ_R is a right congruence on P. For all $s, t \in P$ for which $s \cdot t$ (equivalently by Lemma 3.4, $R(s) \cdot t$) exists, then since also $R(s) = R(R(s))$, we have that $(s, R(s)) \in \epsilon_R$, and so $R(s \cdot t) = R(R(s) \cdot t)$. □

Both ϵ_D and ϵ_R are variants of relations defined on semigroups considered by Lawson in [7], and elsewhere, where their being left or right congruences (neither of which is necessary) is called the left/right congruence condition. Recall that $D(s \cdot D(t)) = D(s \cdot t)$ $(= D(s))$ whenever $s \cdot t$ exists.

Definition 3.7. *The constellation with range $(P, \cdot, D, R)$ satisfies* the congruence condition *if $R(R(s) \cdot t) = R(s \cdot t)$ whenever $s \cdot t$ exists (which is whenever $R(s) \cdot t$ exists by Lemma 3.4).*

Proposition 3.8. *The constellation with range $\mathcal{C}_X$ satisfies the congruence condition.*

Proof. Suppose $s, t \in \mathcal{C}_X$. If $s \cdot t$ exists in $\mathcal{C}_X$, then $R(s) \cdot t$ exists by Lemma 3.4, and in this case, $R(s \cdot t) = R(st) = R(R(s)t) = R(R(s) \cdot t)$ as computed in the bi-unary semigroup of partial functions on X $(PT_X, \times, D, R)$. This is because the law $R(st) = R(R(s)t)$ is known to hold there (see [12] for example). □

As shown in [4], every normal constellation embeds in $\mathcal{C}_X$ for suitable X. It follows from the above result that every normal constellation embeds in a constellation with range satisfying the congruence condition.

Clearly, every category $(C, \circ, D, R)$ is a constellation with range satisfying the congruence condition. Indeed categories are easy to characterise within constellations with range.

Proposition 3.9. *Let $(P, \cdot, D, R)$ be a constellation with range. The following are equivalent.*

(1) *$(P, \cdot, D, R)$ is a category.*
(2) *For $s \in P$, if $s \cdot e$ exists then $e = R(s)$.*
(3) *For all $s, t \in P$ for which $s \cdot t$ exists, $R(s \cdot t) = R(t)$.*

Proof. The equivalence of the first two conditions is immediate from Result 2.2. If P satisfies the third condition and $s \cdot e$ exists for some $s \in P$ and $e \in D(P)$ then it equals s, so $R(s) = R(s \cdot e) = R(e) = e$, so the second condition is satisfied. That the first condition implies the third is clear. □

Definition 3.10. *The constellation P is* right cancellative *if whenever $a \cdot c = b \cdot c$, we have $a = b$.*

This agrees with the category definition if P is a category. The subconstellation of one-to-one partial functions $\mathcal{I}_X$ of $\mathcal{C}_X$ is right cancellative and normal (recalling that for us, $s \cdot t$ is to be interpreted as "first s then t). Conversely we have the following, which follows easily from the proof for the general case given in [4].

Proposition 3.11. *If the small normal constellation $(P, \cdot, D)$ is right cancellative, then P embeds in $(\mathcal{I}_P, \cdot, D)$.*

Proof. It was shown in [4] that the mapping $\phi : P \to \mathcal{C}_P$ taking $s \in P$ to $\psi_s \in \mathcal{C}_P$ given by $x\psi_s := x \cdot s$ for all $x \in P$ for which the latter is defined, is an embedding of P in $\mathcal{C}_P$ as a constellation. It is easy to see that each ψ_s is injective if (and only if) P is right cancellative, and so the image of ϕ lies within $\mathcal{I}_P$. □

Definition 3.12. *If $(P, \cdot, D, R)$ is a constellation with range, we say it is* strongly right cancellative *if $(P, \cdot, D)$ is right cancellative as a constellation and satisfies the condition that for all $e, f \in D(P)$ and $s \in P$, $R(e \cdot s) = R(f \cdot s)$ implies $e = f$.*

Example 3.13. *A partial function example.*

Let $X = \{x, y, z\}$ and $P = \{f, i, g, a, b\} \subseteq \mathcal{C}_X$, with $f = \{(x, x)\}$, $i = \{(x, x), (y, y)\}$, $g = \{(z, z)\}$, $a = \{(x, z)\}$ and $b = \{(x, z), (y, z)\}$. It is easy to see that $(P, \cdot, D, R)$ is a subconstellation of $(\mathcal{C}_X, \cdot, D, R)$ that is also closed under range and hence is a constellation with range itself, moreover one that satisfies the congruence condition since $\mathcal{C}_X$ does. Moreover if $s \cdot u = t \cdot u$ for any $s, t, u \in P$, if $u \in D(P)$ then obviously $s = t$, but if not then $u \in \{a, b\}$ and $s, t \in D(P)$ and so $s = D(s) = D(s \cdot u) = D(t \cdot u) = D(t) = t$. However, $R(f \cdot b) = R(a) = g = R(b) = R(i \cdot b)$, yet $f \neq i$. So even for constellations with range satisfying the congruence condition, the second part of the definition of being strongly right cancellative is independent of the right cancellative law.

The left cancellative property is not a natural one in constellations. However, there is a closely related property which is.

Definition 3.14. *The constellation with range $(P, \cdot, D, R)$ is* left pre-cancellative *if, for all $a, b, c \in P$, $R(a) \cdot b = R(a) \cdot c$ whenever $a \cdot b = a \cdot c$.*

We see that $(\mathcal{C}_X, \cdot, D, R)$ is left pre-cancellative as a constellation with range, whence so is $(P, \cdot, D, R)$ in the above example.

For a category, "strongly right cancellative" is easily seen to be the same as "right cancellative", and similarly for "left pre-cancellative" and "left cancellative". Note that because we are interpreting $f \circ g$ in a category as "first f, then g", for us the left cancellative property is associated with the epic property, and right cancellative with monic, which is the opposite of the more common approach.

In the example $(\mathcal{C}_X, \cdot, D, R)$ of a constellation with range satisfying the congruence condition, the usual category product of elements in $\mathcal{C}_X$ may be defined as follows: if $s, t \in \mathcal{C}_X$, define $s \circ t := s \cdot t$ if $R(s) = D(t)$, and undefined otherwise. This generalises.

Theorem 3.15. *If $(P, \cdot, D, R)$ is a constellation with range, then $(P, \circ, D, R)$ is a category under the new partial operation*

$$s \circ t = s \cdot t \text{ if } R(s) = D(t), \text{ and undefined otherwise}$$

if and only if $(P, \cdot, D, R)$ satisfies the congruence condition. Moreover if $(P, \cdot, D, R)$ is right cancellative, then the category $(P, \circ, D, R)$ is right cancellative, and $(P, \cdot, D, R)$ is left pre-cancellative if and only if $(P, \circ, D, R)$ is left cancellative.

Proof. Suppose $(P, \cdot, D, R)$ is a constellation with range that satisfies the congruence condition, and define $s \circ t$ as above; this makes sense because if $R(s) = D(t)$ then $R(s) \cdot t = D(t) \cdot t$ exists, and so $s \cdot t$ exists by Lemma 3.4. Also, if $x \circ y$ exists then of course $D(x \circ y) = D(x)$, but also $R(x) = D(y)$ and so

$$R(x \circ y) = R(x \cdot y) = R(R(x) \cdot y) = R(D(y) \cdot y) = R(y).$$

Hence for $x, y, z \in P$, the following are equivalent: $(x \circ y) \circ z$ exists; $R(x) = D(y)$ and $R(x \circ y) = D(z)$; $R(x) = D(y)$ and $R(y) = D(z)$; $x \circ y$ and $y \circ z$ exist; $R(y) = D(z)$ and $R(x) = D(y \circ z)$; $x \circ (y \circ z)$ exists. When both are defined, we have

$$x \circ (y \circ z) = x \cdot (y \cdot z) = (x \cdot y) \cdot z = (x \circ y) \circ z.$$

All of this establishes (Cat1). This argument also shows that if $x \circ y$ and $y \circ z$ are defined then $x \circ (y \circ z)$ exists, which establishes (Cat2).

Suppose $e \in D(P)$, with $s \in P$. If $s \circ e$ exists, then it equals $s \cdot e = s$. Also, $e \circ s$ exists if and only if $e = R(e) = D(s)$, and then $e \circ s = e \cdot s = D(s) \cdot s = s$, so e is both a right and left identity under $\circ$, hence an identity, and of course $e \circ s$ exists for $e = D(s)$. Moreover $s \circ R(s)$ exists since $R(s) = D(R(s))$. So (Cat3) holds.

Conversely, suppose $(P, \cdot, D, R)$ is a constellation with range for which the above definition makes it a category. Suppose $x \cdot y$ exists. Then $z = R(x) \cdot y$ exists by Lemma 3.4, and clearly $R(x) = D(z)$, so $x \circ z$ exists, and

$$R(x \cdot y) = R((x \cdot R(x)) \cdot y) = R(x \cdot (R(x) \cdot y)) = R(x \cdot z) = R(x \circ z) = R(z) = R(R(x) \cdot y).$$

This shows that $(P, \cdot, D, R)$ satisfies the congruence condition.

If $(P, \cdot, D, R)$ is right cancellative, then of course $(P, \circ, D, R)$ is right cancellative as a category.

If $(P, \cdot, D, R)$ is left pre-cancellative, and $x \circ y = x \circ z$, then $x \cdot y = x \cdot z$ and $R(x) = D(y) = D(z)$, so $y = D(y) \cdot y = R(x) \cdot y = R(x) \cdot z = D(z) \cdot z = z$. So $(P, \circ, D, R)$ is left cancellative as a category. Conversely, suppose the category $(P, \circ, D, R)$ is left cancellative. If $x \cdot y = x \cdot z$ then $R(x) \cdot y$ and $R(x) \cdot z$ exist by Lemma 3.4, and so $x \cdot (R(x) \cdot y) = (x \cdot R(x)) \cdot y = x \cdot y = x \cdot z = \cdots = x \cdot (R(x) \cdot z)$, so because $D(R(x) \cdot y) = D(R(x) \cdot z) = D(R(x)) = R(x)$, $x \circ (R(x) \cdot y) = x \circ (R(x) \cdot z)$ and so $R(x) \cdot y = R(x) \cdot z$. Hence the constellation $(P, \cdot, D, R)$ is left pre-cancellative. □

We call the category $(P, \circ, D, R)$ obtained from the constellation with range $(P, \cdot, D, R)$ satisfying the congruence condition as in the previous result $\mathrm{Cat}(P)$.

Note that the right cancellative property of the constellation with range P is not equivalent to $\mathrm{Cat}(P)$ being right cancellative. For example, let $X = \{x, y\}$ and then let $P = \{e, f, 1, a, b\}$ be the subconstellation with range of $\mathcal{C}_X$ equipped with its usual domain and range, in which e is the identity map on x, f is the identity map on y, 1 is the identity map on all of X, $a = \{(x, y)\}$ and $b = \{(x, y), (y, y)\}$. Then it is not hard to check that the category $\mathrm{Cat}(P)$ is right cancellative (as relatively few products are defined), although in P, $e \cdot b = a = a \cdot b$, yet $e \neq a$, so P is not right cancellative.

3.2. **Some examples and non-examples of constellations with range.** Aside from the first example to follow, all are "concrete" in the sense that they are set-based: the elements are functions between sets (perhaps with additional structure). In each case, the standard order on $D(P)$ corresponds to the relation of being a substructure in the appropriate sense. All appeared in [4] as examples of constellations.

Example 3.16. *The constellation determined by a partially ordered set.*

Also introduced in [2], a normal constellation arises from any partially ordered set $(Q, \leq)$: simply define $e = e \cdot f$ whenever $e \leq f$ in $(Q, \leq)$, and let $D(e) = e$ for all $e \in Q$. Note that Q also has range (given by $R(e) = e$ for all $e \in Q$), and indeed is strongly right cancellative, left pre-cancellative and satisfies the congruence condition.

Example 3.17. *The constellation of sets.*

Let S be the class of sets. There is a familiar category structure SET associated with S, consisting of sets as the objects and maps between them as the arrows. Taking the "arrow only" point of view, the category consists of all possible maps between all possible sets, with the operations D and R given by specifying $D(f)$ to be the identity map on the domain of f and $R(f)$ the identity map on its codomain, with the partial operation of category composition defined if and only if domains and codomains coincide. Note that for $f, g \in SET$, it is possible for $D(f) = D(g)$ and $xf = xg$ for all $x \in dom(f) = dom(g)$ yet $R(f) \neq R(g)$.

In [4], the constellation $CSET$ is defined from SET by analogy with $\mathcal{C}_X$ as above, by taking the elements to be the *surjective* functions, with D defined as in the category, but with composition of functions $f \cdot g$ defined if and only if $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$. This normal constellation has range, with $R(f)$ the identity map on the image of f. Note that $D(s \cdot t) = D(s)$, but $R(s \cdot t)$ is not simply $R(t)$, so there is no left/right symmetry as in the category. This example is also both left pre-cancellative and satisfies the congruence condition. In contrast to SET, if $f, g \in CSET$ are such that $D(f) = D(g)$ and $xf = xg$ for all $x \in dom(f) = dom(g)$, then $f = g$ and so $R(f) = R(g)$.

Example 3.18. *The constellation of groups.*

Let $\mathcal{G}$ be the class of groups. In the constellation $CGRP$, the elements are the surjective homomorphisms between groups, and composition and D are as in the constellation of sets $CSET$. The constellation $CGRP$ has range, as in $CSET$, giving another left pre-cancellative example that satisfies the congruence condition.

The example generalises widely: any class of algebras of the same type, such as rings, modules, semigroups and so on, will give rise to a constellation with range in a similar way.

Example 3.19. *The constellation of rings with additive homomorphisms.*

Let $RING^+$ be the category of rings but with arrows being additive abelian group homomorphisms. (That this is a category is easily checked.) Again there is an associated constellation $CRING^+$, consisting of mappings of the form $f : R \to S$ where S is "as small as possible", which here means generated as a ring by $\mathrm{Im}(f)$. This example has range: for each element $f : R \to S$, $R(f) = 1_S$, the identity map on S. However, it does not satisfy the congruence condition. Consider the maps $f : \mathbb{Z}[x] \to \mathbb{Z}[x]$ given by $f(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x$ and $g : \mathbb{Z}[x] \to \mathbb{Z}[x^2]$ given by $g(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_2x^2$, where $\mathbb{Z}[x^2]$ is the ring of integer polynomials in which only even powers of x appear. Both f, g are additive homomorphisms, $\mathrm{Im}(f)$ generates $\mathbb{Z}[x]$ and $\mathrm{Im}(g)$ generates $\mathbb{Z}[x^2]$, so both $f, g \in CRING^+$. Now fg, the composite f followed by g, maps $a_0 + a_1x + \cdots + a_nx^n$ to a_0, and $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$, so $f \cdot g : \mathbb{Z}[x] \to \mathbb{Z}$ is an element of $CRING^+$ and $R(f \cdot g)$ is the identity map on $\mathbb{Z}$. But $R(f)$ is the identity map on $\mathbb{Z}[x]$, so $R(R(f) \cdot g) = R(g)$, which is the identity map on $\mathbb{Z}[x^2]$.

Example 3.20. *The constellation of partial maps with infinite domain.*

Suppose X is an infinite set, and denote by $\mathcal{C}_X^\infty$ the set of all partial maps in $\mathcal{C}_X$ that have infinite domains, a subconstellation of $\mathcal{C}_X$. It was noted in [4] that there is no analogous category, and indeed the normal constellation $\mathcal{C}_X^\infty$ does not have range: for if $s \in \mathcal{C}_X$ has finite image, then there is no smallest $e \in D(\mathcal{C}_X^\infty)$ such that $s \cdot e$ exists.

4. Constellations with range are ordered categories with restrictions

If the category $\mathrm{Cat}(P)$ obtained from the constellation with range $(P, \cdot, D, R)$ is to remain capable of capturing the information present in the original constellation with range, it is sufficient that it retain information about constellation products of the form $e \cdot s$, where $e \in D(P)$, $s \in P$ and $e \leq D(s)$; with such products in hand, we may recover the general constellation product as $s \cdot t = s \circ (R(s) \cdot t)$. This is because if $s \cdot t$ exists (or equivalently by Lemma 3.4), $R(s) \cdot t$ exists, then $R(s) = D(R(s)) = D(R(s) \cdot t)$, so $s \circ (R(s) \cdot t)$ exists and equals $s \cdot (R(s) \cdot t) = (s \cdot R(s)) \cdot t = s \cdot t$. To achieve this we equip the category with both a partial order and a (left) restriction operation. Note that in general there is no natural (quasi)order in a category, whereas constellations always have their natural quasiorder.

Definition 4.1. *We say $(P, \cdot, D, \leq)$ is a* weakly quasiordered constellation *if $(P, \cdot, D)$ is a constellation and $(P, \leq)$ is a quasiorder satisfying, for all $a, b, c, d \in P$:*

- *(O1) if $a \leq b$ and $c \leq d$ with both $a \cdot c$ and $b \cdot d$ existing, then $a \cdot c \leq b \cdot d$;*

- *(O2) if $a \leq b$ then $D(a) \leq D(b)$;*
- *(O3) if $e \in D(P)$ and $e \leq D(a)$, then there exists $e|a \in P$, the unique $x \in Q$ for which $x \leq a$ and $D(x) = e$.*

If the constellation with range $(P, \cdot, D, R)$ is equipped with a quasiorder $\leq$ such that $(P, \cdot, D, \leq)$ is a weakly quasiordered constellation and for all $a, b \in P$,

- *(O2′) if $a \leq b$ then $R(a) \leq R(b)$,*

then we say $(P, \cdot, D, R, \leq)$ is a weakly quasiordered constellation with range*.*

In all cases above, if the quasiorder $\leq$ is a partial order, we drop the prefix "quasi". Note that we insert "weakly" into the above definition, because an ordered constellation is defined in [2] using rather stronger properties than those used here.

Proposition 4.2. *Let $P = (P, \cdot, D)$ be a constellation, with $\leq$ the natural quasiorder on it. Then $(P, \cdot, D, \leq)$ is weakly quasiordered in which $e|a = e \cdot a$ whenever $e \leq D(a)$. If $P = (P, \cdot, D, R)$ is a constellation with range, then $(P, \cdot, D, R, \leq)$ is a weakly ordered constellation with range.*

Proof. By Proposition 3.3 in [14], $(P, \cdot, D, \leq)$ is weakly quasiordered in which $e|a = e \cdot a$ whenever $e \leq D(a)$. If P has range then the natural quasiorder is a partial order by (R5) in Proposition 3.2, and if $a, b \in P$ are such that $a \leq b$ under the natural order, then $a = D(a) \cdot b$, so because $b \cdot R(b)$ exists, so does $(D(a) \cdot b) \cdot R(b) = a \cdot R(b)$. Hence $R(a) \cdot R(b)$ exists and $R(a) \leq R(b)$. □

The case of weakly ordered constellations of most interest to us here is that of categories.

Definition 4.3. *We say that $(Q, \circ, D, R, \leq)$ is an* ordered category with restrictions *if $(Q, \circ, D, R)$ is a category and $(Q, \circ, D, \leq)$ is weakly ordered as a constellation with range.*

In the category case, laws (O1), (O2) and (O2′) define Ω-structured categories, considered in Section 2 of [7] following their introduction in [1], while (O3) corresponds to (OC8) (i) in [7]. So it follows from what was observed by Lawson in Lemma 2.6 of [7] that ordered categories with restrictions as defined here are indeed ordered categories with restrictions in the sense of [7], since they are easily seen to satisfy the implication

$$D(x) = D(y),\ x \leq y \ \Rightarrow\ x = y$$

(see the proof of Proposition 2.8 in [7]).

Definition 4.4. *The ordered category with restrictions $(Q, \circ, D, R, \leq)$ is* fully right cancellative *if it is right cancellative as a category and satisfies the condition that for all $e, f \in D(Q)$ and $s \in Q$, if $e|s$ and $f|s$ are defined and $R(e|s) = R(f|s)$, then $e = f$.*

The partial order on an ordered category with restrictions $(Q, \circ, D, R, \leq)$ can be expressed purely in terms of restriction. This is noted in [7] for the similar cases considered there, but we include the easy proof here for completeness.

Proposition 4.5. *Let $(Q, \circ, D, R, \leq)$ be an ordered category with restrictions. For $s, t \in Q$, $s \leq t$ if and only if $D(s) \leq D(t)$ and $s = D(s)|t$.*

Proof. If $s \leq t$ then $D(s) \leq D(t)$, so $D(s)|t$ exists and is the unique $x \leq t$ with $D(x) = D(s)$; thus $x = s$. The converse is immediate. □

Lemma 4.6. *Let $(Q, \circ, D, R, \leq)$ be an ordered category with restrictions. Then for $e \in D(Q)$ and $s, t \in Q$ for which $e \leq D(s)$ and $R(s) \leq D(t)$:*

1. *$R(e|s)|t = R(e|s)|(R(s)|t)$ (and all exist), and*
2. *if $s \circ t$ exists then $e|(s \circ t) = (e|s) \circ R(e|s)|t$.*

Proof. Since $e|s \leq s$, $R(e|s) \leq R(s) \leq D(t)$. Hence $R(e|s)|t$ and $R(e|s)|(R(s)|t)$ both exist (since $R(s)|t$ has domain $R(s) \geq R(e|s)$).

Since both $R(e|s)|t$ and $R(e|s)|(R(s)|t)$ have domain $R(e|s)$ and because of the facts that $R(e|s)|(R(s)|t) \leq R(s)|t \leq t$ and $R(e|s)|t \leq t$, by uniqueness we have $R(e|s)|(R(s)|t) = R(e|s)|t$.

Suppose $s \circ t$ exists. Now $R(e|s)|t$ exists and $R(e|s)|t \leq t$, so by (O1), $(e|s) \circ R(e|s)|t \leq s \circ t$. Moreover, $D((e|s) \circ R(e|s)|t) = D(e|s) = e$, so by (O3), $e|(s \circ t) = (e|s) \circ R(e|s)|t$. □

We can now obtain a characterisation of those categories that arise from constellations with range satisfying the congruence condition as in Theorem 3.15.

Theorem 4.7. *If $(P, \cdot, D, R)$ is a (strongly right cancellative, left pre-cancellative) constellation with range that satisfies the congruence condition, then $(P, \circ, D, R, \leq)$ is an ordered (fully right cancellative, left cancellative) category with restrictions under the new partial operation $s \circ t$, that is defined and equal to $s \cdot t$ precisely when $R(s) = D(t)$ and undefined otherwise, $\leq$ is the natural order on P as a (normal) constellation, and for $e \leq D(s)$, $e|s = e \cdot s$.*

Conversely, if $(Q, \circ, D, R, \leq)$ is a (fully right cancellative, left cancellative) ordered category with restrictions, then setting $s \cdot t$ equal to $s \circ (R(s)|t)$ whenever $R(s) \leq D(t)$ makes $(Q, \cdot, D, R)$ into a (strongly right cancellative, left pre-cancellative) constellation with range satisfying the congruence condition, and the given partial order is nothing but the natural order on the constellation.

Proof. Suppose $(P, \cdot, D, R)$ is a constellation with range satisfying the congruence condition and define $s \circ t$ as above. Theorem 3.15 shows that $(P, \circ, D, R)$ is a category, moreover one that is right cancellative (left cancellative) as a category if $(P, \cdot, D, R)$ is strongly right cancellative (left pre-cancellative). By Proposition 4.2, $(P, \cdot, D, R, \leq)$ is a weakly ordered constellation with range if $\leq$ is the natural order and $e|s$ is as described. Because $\circ$ is just a restricted form of $\cdot$, it follows immediately that $(P, \circ, D, R, \leq)$ is an ordered category with restrictions.

Suppose the constellation with range $(P, \cdot, D, R)$ satisfying the congruence condition is strongly right cancellative. If $s \circ u = t \circ u$ then $s = t$ as in Theorem 3.15. If $R(e|s) = R(f|s)$ then $R(e \cdot s) = R(f \cdot s)$, so $e = f$. So $(P, \circ, D, R, \leq)$ is fully right cancellative as an ordered category with restrictions. Suppose $(P, \cdot, D, R)$ is left pre-cancellative. If $s \circ t = s \circ u$ then $s \cdot t = s \cdot u$, $R(s) = D(t) = D(u)$ and by the left pre-cancellative property, $t = D(t) \cdot t = R(s) \cdot t = R(s) \cdot u = D(u) \cdot u = u$, and so $(P, \circ, D, R)$ is left cancellative.

Conversely, suppose $(C, \circ, D, R, \leq)$ is an ordered category with restrictions, so that (Cat1)–(Cat3) hold (for $\circ$), $D(C) = \{D(s) \mid s \in C\}$ is the set of identities (equivalently right identities) of $(C, \circ)$, and all of (O1), (O2), (O2′) and (O3) hold for $\circ$. Define the new product $s \cdot t$ on C as in the theorem statement; this is well-defined since $D(R(s)|t) = R(s)$. Suppose $x, y, z \in C$ are such that $x \cdot (y \cdot z)$ exists. Then $R(y) \leq D(z)$, $y \cdot z = y \circ (R(y)|z)$, $R(x) \leq D(y \cdot z) = D(y)$ and $x \cdot (y \cdot z) = x \circ (R(x)|(y \circ (R(y)|z)))$. Since $R(x) \leq D(y)$, $x \cdot y$ exists, and equals $x \circ (R(x)|y)$. Also, $R(x \cdot y) = R(R(x)|y) \leq R(y) \leq D(z)$, so $(x \cdot y) \cdot z$ exists and $R(y)|z$ exists. Hence

$$\begin{aligned}
(x \cdot y) \cdot z &= (x \circ (R(x)|y)) \circ (R(R(x)|y)|z) \\
&= x \circ ((R(x)|y) \circ R(R(x)|y)|z) \\
&= x \circ ((R(x)|y) \circ R(R(x)|y)|(R(y)|z)) \text{ by Lemma 4.6 (1)} \\
&= x \circ (R(x)|(y \circ (R(y)|z)) \text{ by Lemma 4.6 (2)} \\
&= x \cdot (y \cdot z).
\end{aligned}$$

So (Const1) holds.

Now suppose $x \cdot y$ and $y \cdot z$ exist. Then $R(x) \leq D(y) = D(y \circ (R(y)|z)) = D(y \cdot z)$, so $x \cdot (y \cdot z)$ exists, establishing (Const2).

Pick $x \in C$. Then $R(D(x)) = D(x)$, so $D(x) \cdot x = D(x) \circ (D(x)|x) = D(x) \circ x = x$. If also $e \cdot x = x$ for some $e \in D(C)$, then $D(x) = D(e \cdot x) = D(e \circ (e|x)) = D(e) = e$, so (Const3) holds. Hence $(C, \circ, D)$ is a constellation.

Supppose $e, f \in D(C)$, and $e \cdot f, f \cdot e$ both exist. Then $e = R(e) \leq f$ under the standard quasiorder on $D(C)$ and similarly $f \leq e$, so $e = f$. Hence $(C, \cdot, D)$ is normal.

Turning to range, Propositions 3.2 and 3.3 between them show that establishing (R1)–(R5) for R as given suffices to establish that $(C, \cdot, D, R)$ is a constellation with range. Law (R1) is immediate. For all $s \in C$, $R(s) = D(R(s))$, so $s \cdot R(s)$ exists, establishing (R2). Also, if $s \cdot t$ exists then $R(s) \leq D(t)$, and so $R(R(s)) \leq D(t)$, so $R(s) \cdot t$ exists, which establishes (R3). (R4) is trivial and (R5) is normality, shown above. So $(C, \cdot, D, R)$ is a constellation with range. Finally, if $s \cdot t$ exists then $R(s \cdot t) = R(s \circ (R(s)|t)) = R(R(s)|t) = R(R(s) \circ (R(s)|t)) = R(R(s) \cdot t)$, so $(C, \cdot, D, R)$ satisfies the congruence condition.

If $s \leq t$ under the given partial order, then $s = D(s)|t$ by Proposition 4.5, so $D(s) \cdot t = D(s) \circ (D(s)|t) = D(s) \circ s = s$. Conversely, if $s = e \cdot t$ for some $e \in D(P)$, then $s = e \circ (e|t) = e|t \leq t$. So the given partial order is the natural order on $(C, \cdot, D)$.

Suppose $(C, \circ, D, R, \leq)$ is fully right cancellative as an ordered category with restrictions. If $s \cdot u = t \cdot u$, then $s \circ (R(s)|u) = t \circ (R(t)|u)$. So applying R to both sides gives $R(R(s)|u) = R(R(t)|u)$ (since $R(x \circ y) = R(y)$ in a category). Because $(C, \circ, D, R, \leq)$ is fully right cancellative, $R(s) = R(t)$, so $R(s)|u = R(t)|u$, and so

$s = t$, again because $(C, \circ, D, R, \leq)$ is fully right cancellative. If $R(e \cdot s) = R(f \cdot s)$, then $R(e|s) = R(f|s)$, so $e = f$. Hence $(C, \cdot, D, R)$ is strongly right cancellative.

Suppose $(C, \circ, D, R)$ is left cancellative. If $s \cdot t = s \cdot u$ then $s \circ (R(s)|t) = s \circ (R(s)|u)$, so $R(s)|t = R(s)|u$, $(C, \circ, D, R)$ is left cancellative, and so $R(s) \cdot t = R(s) \cdot u$. So $(C, \cdot, D, R)$ is left pre-cancellative. □

The constructions in the previous theorem are mutually inverse, as we now show. If $\mathcal{P} = (P, \cdot, D, R)$ is a constellation with range satisfying the congruence condition and $\mathcal{C} = (C, \circ, D, R, \leq)$ is an ordered category with restrictions, denote by $\mathbf{C}(\mathcal{P})$ the ordered category with restrictions $(P, \circ, D, R, \leq)$ obtained as in Theorem 4.7 from $\mathcal{P}$, and denote by $\mathbf{P}(\mathcal{C})$ the constellation with range $(C, \circ, D, R)$ obtained as in that result from the ordered category with restrictions $\mathcal{C}$.

Proposition 4.8. *Let $\mathcal{C}$ be an ordered category with restrictions and $\mathcal{P}$ a constellation with range satisfying the congruence condition. Then $\mathbf{C}(\mathbf{P}(\mathcal{C})) = \mathcal{C}$ and $\mathbf{P}(\mathbf{C}(\mathcal{P})) = \mathcal{P}$.*

Proof. The constellation operation on $\mathbf{P}(\mathcal{C})$ is given by $x \cdot y = x \circ (R(x)|y)$ providing $R(x) \leq D(y)$, so the category operation on $\mathbf{C}(\mathbf{P}(C))$ is given by $x \odot y = x \circ (R(x)|y)$ providing $R(x) = D(y)$. But $D(y)|y = y$, so $x \odot y = x \circ y$ when each is defined, which is exactly when $R(x) = D(y)$. Of course D and R are unchanged in the passage from C to $\mathbf{C}(\mathbf{P}(\mathcal{C}))$. Now $x \leq y$ under the given partial order of C if and only if $x = D(x) \cdot y$ in $\mathbf{P}(\mathcal{C})$ by Proposition 4.5, which is nothing but the partial order on $\mathbf{C}(\mathbf{P}(\mathcal{C}))$. Restriction is of course determined by $\leq$ and D.

Conversely, the category operation on $\mathbf{C}(\mathcal{P})$ is $x \circ y = x \cdot y$ providing $R(x) = D(y)$. Then the constellation operation on $\mathbf{P}(\mathbf{C}(\mathcal{P}))$ is $x : y = x \circ (R(x)|y) = x \circ (R(x) \cdot y) = x \cdot (R(x) \cdot y) = (x \cdot R(x)) \cdot y = x \cdot y$, and is defined exactly when $R(x) \leq D(y)$, so that $R(x) \cdot D(y)$ exists, that is, $x \cdot D(y)$ exists by Lemma 3.4, which by Result 2.1 is to say that $x \cdot y$ exists. Again, D and R are unchanged in the passage from P to $\mathbf{P}(\mathbf{C}(\mathcal{P}))$. □

So every ordered category with restrictions may equivalently be viewed as a constellation with range satisfying the congruence condition, and vice versa. Moreover, the relevant notion of morphism does not depend on which viewpoint is taken, as we show next.

Definition 4.9. *A* range radiant *is a radiant $\rho : P \to Q$ between constellations with range P, Q additionally satisfying $(R(s))\rho = R(s\rho)$ for all $s \in P$.*

The class of constellations with range is a category if arrows are taken to be range radiants, as is easily seen. Likewise, the class of ordered categories with restrictions is itself a category in which the arrows are functors ρ that are order-preserving, meaning that $s \leq t$ implies $s\rho \leq t\rho$.

Lemma 4.10. *If $f : \mathcal{C}_1 \to \mathcal{C}_2$ is an order-preserving functor between ordered categories with restrictions $\mathcal{C}_1 = (C_1, \circ, D, R), \mathcal{C}_2 = (C_2, \circ, D, R)$, then for all $e \in D(C_1)$, $e\rho \in D(C_2)$, and if $e \leq D(s)$ then $e\rho \leq D(s\rho)$, and $(e|s)\rho = e\rho|s\rho$.*

Proof. Suppose $\rho : \mathcal{C}_1 \to \mathcal{C}_2$ is an order-preserving functor as described. Then for $e \in D(C_1)$, $e\rho = D(e)\rho = D(e\rho) \in D(C_2)$. If $e \leq D(s)$ then $e\rho \leq D(s)\rho = D(s\rho)$, so $e\rho|s\rho$ exists. Moreover since $e|s \leq s$, we have $(e|s)\rho \leq s\rho$, and so because $D((e|s)\rho) = (D(e|s))\rho = e\rho$, by uniqueness we must have $(e|s)\rho = e\rho|s\rho$. □

Theorem 4.11. *The category of ordered categories with restrictions is isomorphic to the category of constellations with range satisfying the congruence condition.*

Proof. Let $\mathcal{P} = (P, \cdot, D, R)$ and $\mathcal{Q} = (Q, \cdot, D, R)$ be constellations with range satisfying the congruence condition, also viewed as in Theorem 4.7 as ordered categories with restrictions $\mathcal{P} = (P, \circ, D, R, \leq)$ and $\mathcal{Q} = (Q, \circ, D, R, \leq)$ respectively.

Suppose $\rho : P \to Q$ is a range radiant between $\mathcal{P}, \mathcal{Q}$ viewed as constellations with range. For $s \in P$, we have $(D(s))\rho = D(s\rho)$ and $(R(s))\rho = R(s\rho)$. Also, for $s, t \in P$ for which $s \circ t$ exists, we have $s \circ t = s \cdot t$ and $R(s) = D(t)$, so $(s \circ t)\rho = (s \cdot t)\rho = s\rho \cdot t\rho = s\rho \circ t\rho$, since $R(s\rho) = R(s)\rho = D(t)\rho = D(t\rho)$. So ρ is a functor. Furthermore, for $s, t \in P$ for which $s \leq t$, we have $s = D(s) \cdot t$ and so $s\rho = (D(s) \cdot t)\rho = D(s)\rho \cdot t\rho = D(s\rho) \cdot t\rho$ and so $s\rho \leq t\rho$. So ρ is an order-preserving functor if $\mathcal{P}, \mathcal{Q}$ are instead viewed as ordered categories with restriction.

Conversely, let $\rho : P \to Q$ be an order-preserving functor between $\mathcal{P}, \mathcal{Q}$ viewed as ordered categories with restriction. Again, $D(s)\rho = D(s\rho)$ and $R(s)\rho = R(s\rho)$ for all $s \in P$. If $s, t \in P$ with $s \cdot t$ existing, then $R(s) \leq D(t)$, so $R(s)\rho \leq D(t)\rho$, and so $R(s\rho) \leq D(t\rho)$, so $s\rho \cdot t\rho$ exists, and so using Lemma 4.10, we obtain $(s \cdot t)\rho = (s \circ (R(s)|t))\rho = s\rho \circ (R(s)|t)\rho = s\rho \circ R(s)\rho|t\rho = s\rho \circ R(s\rho)|t\rho = s\rho \cdot t\rho$. So ρ is a range radiant between $\mathcal{P}$ and $\mathcal{Q}$ viewed as constellations with range. □

We saw earlier that every normal constellation embeds in a constellation with range satisfying the congruence condition. It follows from the above results that every normal constellation arises as a subreduct of an ordered category with restrictions.

5. Inverse constellations

We now consider a special case of the above. Recall that a *groupoid* is a structure $(Q, \circ, ', D, R)$ such that $(Q, \circ, D, R)$ is a category and $'$ is a unary operation on Q such that for all $s \in Q$, $s \circ s' = D(s)$ and $s' \circ s = R(s)$.

Recall also that an *ordered groupoid* $(Q, \circ, ', D, R, \leq)$ is a groupoid $(Q, \circ, ', D, R)$ equipped with a partial order $\leq$ such that:

- for all $s, t, u, v \in Q$, if $s \leq t$ and $u \leq v$ then $s' \leq t'$ and if $s \circ u, t \circ v$ exist, then $s \circ u \leq t \circ v$;
- if $e \in D(Q)$ with $e \leq D(s)$, then there exists a unique $e|s \in Q$ called the *restriction of s to e*, such that $e|s \leq s$ and $D(e|s) = e$.

Note that although (O2) and (O2′) in the definition of an ordered category with restrictions given earlier are not part of the definition of an ordered groupoid, they follow easily from (O1) and the law for inverses. It follows that there is a notion of co-restriction as well, defined as $s|e = (e|s')'$ whenever $e \leq R(s)$, satisfying an axiom dual to the second one above. Hence an ordered groupoid $(Q, \circ, D, R, ', \leq)$

is an ordered category with restrictions. In this section we identify the types of constellation with range they correspond to.

In [6], the notion of a "true inverse" of an element in a left restriction semigroup S was considered; $s \in S$ has *true inverse* t if $st = D(s)$ and $ts = D(t)$. It was shown that true inverses are unique if they exist, and that if every element has a true inverse then S is an inverse semigroup in which s' is the true inverse of s. On the other hand, the concept of a groupoid is a familiar one in category theory, and recovers the notion of a group in the single object case. We now consider the natural variant of this concept in the constellation setting.

Definition 5.1. *The constellation $(P, \cdot, D)$ is* D-regular *if for all $a \in P$ there is $b \in P$ for which $a \cdot b = D(a)$.*

Proposition 5.2. *Every D-regular constellation is right cancellative.*

Proof. Suppose $(P, \cdot, D)$ is D-regular. If $a \cdot c = b \cdot c$ then since there exists d for which $c \cdot d = D(c)$, both $a \cdot (c \cdot d)$ and $b \cdot (c \cdot d)$ exist, and $a \cdot (c \cdot d) = (a \cdot c) \cdot d = (b \cdot c) \cdot d = b \cdot (c \cdot d)$, so $a \cdot D(c) = b \cdot D(c)$, and so $a = b$. □

A more general concept than D-regularity is regularity.

Definition 5.3. *The constellation $(P, \cdot, D)$ is* regular *if for every $a \in P$ there exists $b \in P$ for which $(a \cdot b) \cdot a = a$.*

Clearly, a D-regular constellation is regular and, by Proposition 5.2, right cancellative. Indeed we have the following.

Proposition 5.4. *The constellation $(P, \cdot, D)$ is regular and right cancellative if and only if it is D-regular.*

Proof. Suppose P is regular and right cancellative. Then for all $a \in P$ there is $b \in P$ for which $(a \cdot b) \cdot a$ exists and equals $a = D(a) \cdot a$, so by the right cancellative property, $a \cdot b = D(a)$, giving that P is D-regular. The converse was just dealt with. □

Definition 5.5. *If P is a constellation, we say the element $s \in P$ has* D-inverse *$t \in P$ if $s \cdot t = D(s)$ and $t \cdot s = D(t)$.*

If t is a D-inverse for s in a constellation, then obviously then s is a D-inverse for s also.

We have the following generalisation of a familiar fact about inverses in categories (which are normal when viewed as constellations).

Proposition 5.6. *In a constellation P, every $e \in D(P)$ is a D-inverse for itself, and each element of P has at most one D-inverse if and only if P is normal.*

Proof. For $e \in D(P)$, $e \cdot e = e = D(e)$ so e is a D-inverse of itself.

Suppose P is normal and s' and t are both D-inverses of $s \in P$. So $s' \cdot s$ and $s \cdot s'$ exist, whence so does $s' \cdot (s \cdot s') = s' \cdot D(s) = s'$ by (Const2). So by (Const1),

$$s' = s' \cdot D(s) = s' \cdot (s \cdot t) = (s' \cdot s) \cdot t = D(s') \cdot t,$$

so $s' \leq t$ under the natural order on P. Similarly, $t \leq s'$, and so $s' = t$ by Result 2.3.

Conversely, if D-inverses are unique, suppose $e \cdot f$ and $f \cdot e$ both exist where $e, f \in D(P)$. Then $e \cdot f = D(e)$ and $f \cdot e = D(f)$, and so f is a D-inverse of e, whence $f = e$ by uniqueness. This establishes normality. □

Definition 5.7. *We say the constellation P is an* inverse constellation *if it is normal and every element has a D-inverse; equivalently, it is a constellation in which every element has a unique D-inverse.*

Clearly, any inverse constellation is D-regular, and so right cancellative by Proposition 5.2.

Proposition 5.8. *Every inverse constellation is a constellation with range satisfying the congruence condition in which $R(s) = s' \cdot s = D(s')$ for all $s \in P$, and is both strongly right cancellative and left pre-cancellative as a constellation with range.*

Proof. Let P be an inverse constellation and define $R(s)$ as in the proposition statement, for any $s \in P$. Now $R(s) = s' \cdot s = D(s') \in D(P)$ and $s \cdot R(s) = s \cdot (s' \cdot s) = (s \cdot s') \cdot s = D(s) \cdot s = s$. Suppose $s \cdot e$ exists for some $e \in D(S)$. Then $s' \cdot (s \cdot e) = (s' \cdot s) \cdot e = R(s) \cdot e$ exists, and so $R(s) \leq e$. Since P is normal, $\leq$ is a partial order, and so by definition P is a constellation with range.

If $s \cdot t$ (equivalently by Lemma 3.4, $R(s) \cdot t$) exists, then $s' \cdot (s \cdot t) = (s' \cdot s) \cdot t$ exists. Hence by Lemma 3.4,

$$R(s{\cdot}t) = R((s{\cdot}R(s)){\cdot}t) = R(s{\cdot}(R(s){\cdot}t)) \leq R(R(s){\cdot}t) = R((s'{\cdot}s){\cdot}t) = R(s'{\cdot}(s{\cdot}t)) \leq R(s{\cdot}t),$$

so by normality, all are equal and so in particular $R(s \cdot t) = R(R(s) \cdot t)$. Hence P satisfies the congruence condition.

Now P is right cancellative by an earlier remark. Suppose $s \in P$ and $e, f \in D(P)$ are such that $R(e \cdot s) = R(f \cdot s)$. So $(e \cdot s)' \cdot (e \cdot s) = (f \cdot s)' \cdot (f \cdot s)$. But $s \cdot s' = D(s)$ exists, so $(e \cdot s) \cdot s' = e \cdot (s \cdot s') = e \cdot D(s) = e$, and similarly $(f \cdot s) \cdot s' = f$. Hence $(e \cdot s)' \cdot ((e \cdot s) \cdot s') = (e \cdot s)' \cdot e = (e \cdot s)'$ and similarly $(f \cdot s)' \cdot ((f \cdot s) \cdot s') = (f \cdot s)'$. So

$$\begin{aligned} (e \cdot s)' &= (e \cdot s)' \cdot ((e \cdot s) \cdot s') \\ &= ((e \cdot s)' \cdot (e \cdot s)) \cdot s' \\ &= ((f \cdot s)' \cdot (f \cdot s)) \cdot s' \\ &= (f \cdot s)' \cdot ((f \cdot s) \cdot s') \\ &= (f \cdot s)'. \end{aligned}$$

So $e \cdot s = f \cdot s$, and by the right cancellative property, $e = f$. So P is strongly right cancellative as a constellation with range.

If $a, b, c \in P$ and $a \cdot b = a \cdot c$ then since $a' \cdot a = R(a)$ where a' is the D-inverse of a, we have that $R(a) \cdot b = (a' \cdot a) \cdot b$ exists and equals $a' \cdot (a \cdot b) = a' \cdot (a \cdot c) = (a' \cdot a) \cdot b = R(a) \cdot c$. So P is left pre-cancellative as a constellation with range. □

An inverse constellation that is a category is nothing but a groupoid, as is easily seen. A monoid is an inverse constellation if and only if it is a group. Every poset P gives an inverse constellation if we define $e \cdot f = e$ if and only if $e \leq f$; this is a constellation in which $D(e) = e$ for all $e \in P$, as noted first in [2], and normal as noted in [4], and since $e \cdot e = e = D(e)$ for all $e \in P$, e is a D-inverse for itself.

Another example of an inverse constellation is the constellation $\mathcal{I}_X$ of one-to-one partial functions on the set X, equipped with the additional operation of inversion: this is because for all $f \in \mathcal{I}_X$, if f' is the inverse of f in the usual partial function sense, then $f \cdot f' = D(f)$ and $f' \cdot f = D(f')$, so f' is the D-inverse of f in the constellation sense also. If $D(P)$ has one element then the small inverse constellation P is a group, hence is isomorphic to a group of permutations. More generally, we have the following.

Proposition 5.9. *If X is a small inverse constellation, then it embeds as an inverse constellation in the inverse constellation $\mathcal{I}_X$.*

Proof. Consider the radiant $\rho : X \to \mathcal{C}_X$ as in the proof of Theorem 4.11 in [2], which maps X into $\mathcal{C}_X$. Now if $s \cdot u = t \cdot u$ then since $u \cdot u'$ exists, we have $s = s \cdot D(u) = s \cdot (u \cdot u') = (s \cdot u) \cdot u' = (t \cdot u) \cdot u' = t \cdot (u \cdot u') = t \cdot D(u) = t$. So the radiant $\rho : X \to \mathcal{C}_X$ maps into $\mathcal{I}_X$. It remains to show that for $s \in X$, $s'\rho = (s\rho)'$. Now for $x \in \text{Dom}(\rho_s)$, we have that $x\rho_s = x \cdot s$, and then since $s \cdot (s' \cdot s)$ exists, we have that

$$x \cdot s = x \cdot (s \cdot (s' \cdot s)) = (x \cdot s) \cdot (s' \cdot s) = (x \cdot s) \cdot D(s'),$$

so $x \cdot s \in \text{Dom}(s'\rho)$, and indeed

$$x\rho_s\rho_{s'} = (x \cdot s) \cdot s' = x \cdot (s \cdot s') = x \cdot D(s) = x.$$

This shows that $\rho_s \cdot \rho_{s'} = \rho_{D(s)} = D(\rho_s)$. Similarly, $\rho_{s'} \cdot \rho_s = D(\rho_{s'})$. All of this shows that $\rho_{s'} = (\rho_s)'$, as required. □

As in the proof of Proposition 5.8, in the inverse constellation P, $s \cdot (s' \cdot s) = (s \cdot s') \cdot s = s$ for all $s \in P$. Other familiar facts of inverse semigroup theory such as $(st)' = t's'$ do not carry over since the existence of $s \cdot t$ does not ensure that $t' \cdot s'$ exists (even if both s, t have D-inverses). For example, let $X = \{a, b\}$, with $s, t \in \mathcal{I}_X$ defined as follows: $s = \{(x, x)\}, t = 1$. Then in $\mathcal{I}_X$, $s' = s$, $t' = t$, $s \cdot t = s$ yet $t' \cdot s' = t \cdot s$ does not exist. However, we do have the following.

Proposition 5.10. *If P is an inverse constellation and $s, t \in P$ are such that both $s \cdot t$ and $t' \cdot s'$ exist, then $(s \cdot t)' = t' \cdot s'$.*

Proof. Since $s \cdot t$, $t \cdot t'$ and $t' \cdot s'$ all exist, so does $t \cdot (t' \cdot s') = (t \cdot t') \cdot s'$ and hence $s \cdot (t \cdot (t' \cdot s')) = (s \cdot t) \cdot (t' \cdot s')$ which also equals $s \cdot ((t \cdot t') \cdot s') = (s \cdot (t \cdot t')) \cdot s' = (s \cdot D(t)) \cdot s' = s \cdot s' = D(s) = D(s \cdot t)$. So $(s \cdot t) \cdot (t' \cdot s') = D(s \cdot t)$. Similarly, on interchanging the role of s, t' and s', t, we have $(t' \cdot s') \cdot (s \cdot t) = D(t' \cdot s')$. So $(s \cdot t)' = t' \cdot s'$. □

In the small case, the above result also follows from the embeddabity of P in $\mathcal{I}_X$ as in Proposition 5.9.

Every inverse constellation is D-regular as a constellation, as well as being a constellation with range satisfying the congruence condition. Conversely, we have the following.

Theorem 5.11. *Let P be a D-regular constellation with range satisfying the congruence condition. Then P is an inverse constellation, and $R(s) = D(s')$.*

Proof. For $s \in P$, let $t \in P$ be such that $s \cdot t = D(s)$, and let $t' = R(s) \cdot t$, which exists by Lemma 3.4. Let u be such that $t \cdot u = D(t)$, which exists by assumption. Then there exists $s \cdot (t \cdot u) = s \cdot D(t) = s$, which must also equal $(s \cdot t) \cdot u = D(s) \cdot u$. Since $D(s) \cdot s$ exists, $(s \cdot t) \cdot s$ exists, hence so too does $R(s \cdot t) \cdot s$ by Lemma 3.4. By the congruence condition, $R(t') = R(R(s) \cdot t) = R(s \cdot t)$, so $R(t') \cdot s$ and hence $t' \cdot s$ exists by Lemma 3.4. Then

$$\begin{aligned} t' \cdot s &= (R(s) \cdot t) \cdot (D(s) \cdot u) \\ &= ((R(s) \cdot t) \cdot D(s)) \cdot u \\ &= (R(s) \cdot t) \cdot u \\ &= R(s) \cdot (t \cdot u) \text{ since } t \cdot u \text{ exists} \\ &= R(s) \cdot D(t) \\ &= R(s). \end{aligned}$$

So $s \cdot t' = s \cdot (R(s) \cdot t) = (s \cdot R(s)) \cdot t = s \cdot t = D(s)$, and $t' \cdot s = R(s)$, so $D(t') = D(t' \cdot s) = R(s)$. Hence t' is an (hence the) D-inverse of s. Moreover, P is normal by (R5) of Proposition 3.2. □

Since $\mathcal{C}_X$ is a constellation with range satisfying the congruence condition, it is not D-regular (since it is not even right cancellative as a constellation, as easy examples show). However, it is not hard to check that for all $a \in \mathcal{C}_X$ there exists $b \in \mathcal{C}_X$ for which $b \cdot a = R(a)$.

An example of D-regular constellation that is not an inverse constellation is as follows. Let $X = \{x, y, x', y'\}$ and let $a = \{(x, x')\}$, $b = \{(x', x), (y', y)\}$, $c = b^{-1} = \{(x, x'), (y, y')\}$, with $e = \{(x, x)\}$, $f = \{(x, x), (y, y)\}$, $g = \{(x', x'), (y', y')\}$. Then $P = \{a, b, c, e, f, g\}$ is a subconstellation of $\mathcal{C}_X$, with $a \cdot b = e = D(a)$, $b \cdot c = g = D(b)$, and $c \cdot b = f = D(c)$, so P is normal and D-regular (indeed for all $s \in P$ there is even a unique $t \in P$ for which $s \cdot t = D(s)$), and even has range, with $R(a) = R(c) = g$ and $R(b) = f$), but does not satisfy the congruence condition (since $R(a \cdot b) = R(e) = e$, while $R(R(a) \cdot b) = R(g \cdot b) = R(b) = f \neq e$), and hence is not an inverse constellation.

Of course every inverse constellation is also regular. Conversely, from Proposition 5.4 and Theorem 5.11 we have the following.

Corollary 5.12. *Let P be a regular strongly right cancellative constellation with range satisfying the congruence condition. Then P is D-regular and hence is an inverse constellation.*

Because they are constellations with range satisfying the congruence condition, inverse constellations will correspond to some types of ordered categories with restrictions by the results of Section 4. First we need to clarify the notion of radiant specific to inverse constellations; in fact it is nothing but the notion of radiant. (An analogous result is familiar for inverse semigroups.)

Proposition 5.13. *Let P, Q be inverse constellations, with $\psi : P \to Q$ a radiant. Then ψ is a range radiant between P, Q viewed as constellations with range, and also $(s\psi)' = (s')\psi$ for all $s \in P$.*

Proof. Suppose ψ is a radiant. Then for all $s \in P$, $D(s\psi) = D(s)\psi = (s \cdot s')\psi = (s\psi) \cdot (s'\psi)$, and similarly $D(s'\psi) = (s'\psi) \cdot (s\psi)$, so by definition and uniqueness, $(s\psi)' = s'\psi$. Then by Proposition 5.8,

$$R(s)\psi = D(s')\psi = D(s'\psi) = D((s\psi)') = R(s\psi).$$

So ψ is a range radiant. □

Theorem 5.14. *Under the correspondences given in Theorems 4.7 and 4.11, the categories of inverse constellations and ordered groupoids are isomorphic.*

Proof. Suppose P is an inverse constellation. Then viewing it as a constellation with range, $\mathbf{C}(P)$ is an ordered category with restrictions. Now $D(s) = s \cdot s'$ and $D(s') = s' \cdot s$ in P, where s' is the D-inverse of s in P. But $R(s) = D(s')$ and $R(s') = D(s'') = D(s)$ by uniqueness, so $s \cdot s' = s \circ s'$ and $s' \cdot s = s' \circ s$ in $\mathbf{C}(P)$.

Suppose $s, t \in \mathbf{C}(P)$, with s', t' the D-inverses of s, t in P. Suppose that $s \leq t$. Then $s = D(s)|t$ with $D(s) \leq D(t)$, so $s = D(s) \cdot t = (s \cdot s') \cdot t$ in P, and as $D(s) \leq D(t)$, so $(s \cdot s') \cdot (t \cdot t')$ exists and equals $s \cdot s'$. So

$$s \cdot s' = (s \cdot s') \cdot (t \cdot t') = ((s \cdot s') \cdot t) \cdot t' = s \cdot t',$$

which therefore exists, so $R(s) \cdot t' = D(s') \cdot t'$ exists and equals

$$(s' \cdot s) \cdot t' = s' \cdot (s \cdot t') = s' \cdot (s \cdot s') = (s' \cdot s) \cdot s' = D(s') \cdot s' = s',$$

so $s' \leq t'$. So $\mathbf{C}(P)$ is an ordered groupoid.

For the converse, suppose C is an ordered groupoid. Then it is an ordered category with restrictions, so $\mathbf{P}(C)$ is a constellation with range. Moreover for $s \in \mathbf{P}(C)$, if s' is its inverse in C, then $D(s) = R(s')$, so in $\mathbf{C}(\mathbf{P}(C))$ and hence in $\mathbf{P}(C)$, $s \cdot s' = s \circ s' = D(s)$ and $s' \cdot s = s' \circ s = R(s) = D(s')$, and so s' is a D-inverse of s in $\mathbf{P}(C)$, unique by normality (Proposition 5.6). Hence $\mathbf{P}(C)$ is an inverse constellation.

The remainder of the correspondence details now follow from Proposition 4.8 and Theorem 4.11, also using Proposition 5.13. □

In light of this result, Proposition 5.9 follows from the analogous fact for ordered groupoids; see Theorem 9 in Section 4.1 of [8] for example. Our proof is rather shorter than that of the corresponding result in [8].

It is well-known that inverse semigroups may equivalently be viewed as inductive groupoids, which are ordered groupoids in which $D(P)$ is a semilattice under its inherited order; this is the ESN Theorem. We therefore have the following immediate consequence of Theorem 5.14.

Corollary 5.15. *The category of inverse constellations in which $D(P)$ is a meet-semilattice under its natural order is isomorphic to the category of inverse semigroups.*

The ESN Theorem was generalised significantly (although with category isomorphism replaced by category equivalence) when Nambooripad showed in [9] that the category of regular semigroups is equivalent to the category of ordered groupoids in which $D(P)$ is a regular biordered set in a way compatible with its structure as an ordered groupoid. In light of Theorem 5.14, this fact could instead be stated in terms of inverse constellations rather than ordered groupoids.

Generalising in a different way, consider an involuted semigroup S, meaning a semigroup equipped with an involution satisfying $(st)^* = t^*s^*$ and $s^{**} = s$ for all $s, t \in S$. Let $E^*(S) = \{e \in S \mid e^* = e = e^2\}$, the set of *projections* in S and define $I^*(S) = \{s \in S \mid ss^*s = s\}$, the set of *partial isometries* of S. In Section 4.2 of [8], Lawson shows that $I^*(S)$ forms an ordered groupoid in which the identities are the projections, $s \circ t = st$ is defined if and only if $s^*s = tt^*$, and $s \leq t$ means $s = ss^*t$ with $ss^* \leq tt^*$ under the usual ordering of idempotents in $E^*(S) = \{ss^* \mid s \in I(S)\}$ given by $e \leq f$ whenever $e = ef(= fe)$. Next, we generalise this fact, but using inverse constellations rather than ordered groupoids. The advantage of such an approach is that the order and restriction (and even range operation) need not be considered, only the constellation and domain (partial) operations.

Let S be a semigroup with E a non-empty subset of $E(S) = \{e \in S \mid e^2 = e\}$. In [3], an element $s \in S$ was said to be E-regular if there exists $t \in S$ for which $sts = s$ with $st, ts \in E$; in this case, there is even $u \in S$ such that $sus = s$ and $usu = u$, with $su, us \in E$ (simply let $u = tst$). Call such u an *E-inverse of s*. (Semigroups in which every element has an E-inverse for some $E \subseteq E(S)$ were studied in [15].) In the case of an involuted semigroup S, if $s \in I^*(S)$, then s^* is an $E^*(S)$-inverse of s (since from $ss^*s = s$, it follows that $s^*ss^* = s^*$, and $ss^*, s^*s \in E^*(S)$).

Let $R_E(S)$ be the set of E-regular elements of S, suppose $T \subseteq R_E(S)$ with $E \subseteq T$, and let $I_E(T) = \{(s, s') \mid s, s' \in T, s' \text{ is an } E\text{-inverse of } s\}$. We define certain possible partial operations on $I_E(T)$ as follows:

- $(s, s') \circ (t, t') = (st, t's')$ providing $s's = tt'$ (equivalently, $s = stt'$ and $t = s'st$), and undefined otherwise;
- $(s, s') \cdot (t, t') = (st, t's')$ providing $s = stt'$, and undefined otherwise;
- $D((s, s')) = (ss', ss')$, $R((s, s')) = (s's, s's)$;
- $(s, s')' = (s', s)$.

Let us say that E is *right reduced* if for all $e, f \in E$, $ef = f$ whenever $fe = f$, and *reduced* if for all $e, f \in E$, $ef = f$ if and only if $fe = f$. The term "reduced" was used in [7] in this way, and subsequently the one-sided versions were used in [13] and then also [15]. The set of idempotents $E(S)$ is not usually right reduced. Indeed a band S has $S = E(S)$ right reduced if and only if it is right regular (meaning that $xyx = yx$ for all $x, y \in S$), as is easily seen.

We say E is *T-normal* if it is such that, if $e \in E$ and $s \in T$ has an E-inverse s' in T, then $s'es \in E$ whenever $e = ess'$.

For example, let $T = S$ be a semigroup in which $E(S) \neq \emptyset$, and then $E = E(S)$ is trivially T-normal since $s'es \in E(S)$ if s' is an inverse of s and e is idempotent.

Lawson's example is the special case in which S is an involuted semigroup, $E = E^*(S)$, the set of projections of S, and $T = I^*(S)$, the partial isometries of S: since ss^* and s^*s are projections, every partial isometry s is $E^*(S)$-regular with inverse s^*. But because $E^*(S)$ is reduced (since for $e, f \in E^*(S)$, $ef = f$ if and only if $fe = f^*e^* = (ef)^* = f^* = f$), s^* is the unique $E^*(S)$-inverse of S. The set $E^*(S)$ is also T-normal, since if $s \in T$ and $e \in E^*(S)$ with $e = es^*s$, then $s^*es \in E^*(S)$ as is easily seen.

Theorem 5.16. *Suppose S is a semigroup containing non-empty $E \subseteq E(S)$, with $E \subseteq T \subseteq S$. Then $(I_E(T), \circ, D, R)$ is a groupoid with (s', s) the inverse of (s, s') for all $(s, s') \in I_E(S)$.*

Moreover, $(I_E(T), \cdot, D)$ is an inverse constellation if and only if E is T-normal and right reduced, with (s', s) the D-inverse of (s, s') for all $(s, s') \in I_E(S)$.

Proof. Pick $(s, s'), (t, t') \in I_E(T)$. Since $ss', s's \in E$ $(s, s')' = (s', s) \in I_E(T)$ also, and $(e, e) \in I_E(T)$ for all $e \in E$ since e is an E-inverse of itself, so in particular, $D((s, s'))$ and $R((s, s')) \in I_E(T)$.

If $s = stt'$ in S then

$$(st)(t's')(st) = (stt')(s'st) = s(s'st) = st$$

and

$$(t's')(st)(t's') = t's'(stt')s' = t's'ss' = t's',$$

with $(st)(t's') = ss' \in E$.

If also $t = s'st$, then $(t's')(st) = t't \in E$ also. Hence $(s, s') \circ (t, t') = (st, t's') \in I_E(T)$. The verification that $I_E(T)$ is a groupoid with operations as described in the theorem statement is now routine.

Now suppose E is T-normal and E is right reduced. Again, suppose $(s, s'), (t, t') \in I_E(T)$. As just shown, if $s = stt'$ then $t's'$ and st are inverses and $(st)(t's') \in E$. Clearly $s's = s'stt'$, so by T-normality, $(t's')(st) = t'(s's)t \in E$ also (since $s's \in E$ and $t \in T$). So $st, t's'$ are E-inverse to one-another, and so $(s, s') \cdot (t, t') = (st, t's') \in I_E(T)$. It remains to show that $I_E(T)$ is a constellation under $\cdot$.

If $(s, s') \cdot ((t, t') \cdot (u, u')) = (s, s') \cdot (tu, u't') = (stu, u't's')$ exists then $t = tuu'$ and $s = s(tu)(u't') = stt'$, so $(s, s') \cdot (t, t')$ exists, and $(st)uu' = st$, so $((s, s') \cdot (t, t')) \cdot (u, u')) = (st, t's') \cdot (u, u') = (stu, u't's')$ exists. Evidently the two are equal. This establishes (Const1).

Suppose $(s, s'), (t, t'), (u, u') \in I_E(T)$. If $(s, s') \cdot (t, t')$ and $(t, t') \cdot (u, u') = (tu, u't')$ exist then $stt' = s$ and $tuu' = t$. So $s(tu)(u't') = stt' = s$, and so $(s, s') \cdot ((t, t') \cdot (u, u'))$ exists. Hence (Const2) holds.

Now if $(g, g') \in I_E(T)$ is a right identity in $(I_E(T), \cdot)$ then $g'g \in E$, and then for all $(x, x') \in I_E(T)$, $(x, x') = (x, x') \cdot (g, g') = (xg, g'x')$ whenever it exists, namely when $xgg' = x$. But $g'gg' = g'$, so letting $(x, x') = (g', g)$, we obtain $g' = g'g = g \in E$, and $(g, g') = (g, g)$. Conversely, if $e \in E$ then $(e, e) \in I_E(T)$, and if $(x, x') \cdot (e, e) = (xe, ex')$ exists for some $x \in I(S)$, then $x = xee = xe$, so

$x'x = x'xe$, and so since $x'x \in E$, we have $x'x = ex'x$ by the right reduced property of E, and so $x' = x'xx' = ex'xx' = ex'$. So $(x, x')\cdot(e, e) = (xe, ex') = (x, x')$. Hence (e, e) is a right identity. So the set of right identities of $(I_E(T), \cdot)$ is $\{(e, e) \mid e \in E\}$.

Choose $(s, s') \in I_E(T)$. Then $ss' \in E \subseteq I_E(T)$, $(ss', ss') \in I_E(T)$, and $(ss', ss')\cdot(s, s')$ exists (since $(ss')(ss') = ss'$ in S) and equals (s, s'). If $(e, e)\cdot(s, s') = (s, s')$ for some right identity (e, e), then $ess' = e$, $es = s$ and $s'e = s'$. Hence $ss' = (es)(s'e) = (ess')e = ee = e$. This establishes (Const3).

Hence, $(I_E(T), \cdot)$ is a constellation in which $D((s, s')) = (ss', ss')$ for all $s \in S$, and the set of projections is $\{(e, e) \mid e \in E\}$.

Moreover, for all $s \in I_E(T)$, $(s, s')\cdot(s', s)$ exists (since $ss's = s$ in S) and equals $(ss', ss') = D((s, s'))$, and similarly $(s', s)\cdot(s, s') =$ exists and equals $D((s', s))$. So $(I_E(T), \cdot)$ is an inverse constellation.

Conversely, suppose $(I_E(T), \cdot)$ is an inverse constellation as in the statement of the theorem. Suppose $e \in E$ and s' is an E-inverse of s for which $e = ess'$. It follows that $(e, e)\cdot(s, s')$ exists and equals $(es, s'e)$, so $s'e$ is an E-inverse of es. In particular then, $s'es = s'e(es) \in E$. Hence E is T-normal. Pick any $f, g \in E$; then (g, g) is a projection in the constellation, and if $fg = f$ then $fgg = f$ so $(f, f)\cdot(g, g)$ exists and equals (f, f), so ($fg = f$ and) $gf = f$. Hence E is right reduced. □

The fact that this construction is not left/right-symmetric (because the right reduced property of E is needed, not the left reduced property) suggests that $I_E(T)$ is more naturally viewed as an inverse constellation (a one-sided concept), rather than as an ordered groupoid (an equivalent though inherently symmetric object).

As an example, if S is a right regular band, then as noted earlier, $E(S) = S$ is right reduced and S-normal, so $I_S(S)$ is an inverse constellation. Indeed if S is a right zero semigroup (a special case of a right regular band), we obtain that $(s, t) \in I_S(S)$ for all $s, t \in S$, and $(s, t)\cdot(u, v) = (s, t) \circ (u, v) = (u, t)$ exists if and only if $s = v$, $D((s, t)) = (t, t)$ and $R((s, t)) = (s, s)$.

As shown in [15], E-inverses in the semigroup S are unique when they exist if and only if E is *pre-reduced*, meaning that for all $e, f \in E$, $ef = f$ and $fe = e$ imply $e = f$, and $ef = e$ and $fe = f$ imply $e = f$. If E is reduced then it is pre-reduced.

Corollary 5.17. *Suppose S is a semigroup with $E \subseteq E(S)$, $T \subseteq R_E(S)$ with $E \subseteq T$. If E is right reduced, pre-reduced and T-normal, then T is an inverse constellation, where we define $s \cdot t = st$ if and only if $stt' = s$, where t' is the E-inverse of t.*

Proof. The conditions on W and T ensure that any $s \in T$ has a unique E-inverse $s' \in T$. Then, $(T, \cdot) \cong (I_E(T), \cdot)$ as partial algebras, via the isomorphism $s \leftrightarrow (s, s')$, where s' is the unique E-inverse of $s \in S$, as is easily seen. The result now follows from Theorem 5.16. □

If S is a $*$-semigroup, recall that $I^*(S)$ is the set of partial isometries T in S. As noted in the discussion prior to Theorem 5.16, $E^*(S)$ is reduced, hence right reduced and pre-reduced, and is $I^*(S)$-normal. We can now apply Corollary 5.17 to recover Theorem 3 in Section 4.2 in [8]: $I^*(S)$ is an inverse constellation, where we define $s \cdot t = st$ if and only if $stt^* = s$, hence is an ordered groupoid in the way described in [8].

6. Open questions

As already observed, the normal constellation $(\mathcal{C}_X, \cdot, D)$ has range, with $R(f)$ equal to the identity map on the image of f, and indeed $(\mathcal{C}_X, \cdot, D, R)$ is left pre-cancellative and satisfies the congruence condition as noted previously. It would be of interest to axiomatize those small constellations with range that are embeddable within $\mathcal{C}_X$. A finite axiomatization exists in the case of semigroups of partial functions equipped with D and R (see [10]), the axioms having a similar form to those of left pre-cancellative constellations with range satisfying the congruence condition. Based on this, it is tempting to conjecture that every left pre-cancellative constellation with range satisfying the congruence condition is so embeddable.

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