Algebras of reduced E-Fountain semigroups - generalizing the right ample identity

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Outline

- Reduced E-Fountain semigroups (with the congruence condition)
- The generalized right ample property.
- Examples.
- Application to representation theory.

Preliminaries - generalized Green's relations

- Let S be a semigroup and let $E \subseteq S$ be a subset of idempotents.
- $a\tilde{\mathcal{L}}_E b \iff \forall e \in E \quad ae = a \iff be = b$
- $a\tilde{\mathcal{R}}_E b \iff \forall e \in E \quad ea = a \iff eb = b.$
- $\mathcal{L} \subseteq \tilde{\mathcal{L}}_E \qquad \mathcal{R} \subseteq \tilde{\mathcal{R}}_E$

Preliminaries - examples

- S inverse semigroup, $E = E(S)$. $\tilde{\mathcal{L}}_E = \mathcal{L}$, $\tilde{\mathcal{R}}_E = \mathcal{R}$
- S monoid, $E = \{1\}$. $\tilde{\mathcal{L}}_E = \tilde{\mathcal{R}}_E = S \times S$
- $S = \mathcal{B}_n$ (binary relations)\ $S = \mathcal{PT}_n$ (partial functions).
  E =partial identities.
  $f \tilde{\mathcal{L}}_E g \iff \mathrm{dom}(f) = \mathrm{dom}(g)$, $\quad f \tilde{\mathcal{R}}_E g \iff \mathrm{im}(f) = \mathrm{im}(g)$.
  (composition right to left)

Preliminaries - reduced E-Fountain semigroups

- S is called E-**Fountain** if every $\tilde{\mathcal{L}}_E$ -class and every $\tilde{\mathcal{R}}_E$-class contains an idempotent from E.
- S is called **reduced** E-Fountain if in addition

$$\forall e, f \in E \quad ef = e \iff fe = e$$

  - In this case, every $\tilde{\mathcal{L}}_E$ -class and every $\tilde{\mathcal{R}}_E$-class contains a unique idempotent from E (denoted a^* and a^+ for $a \in S$).
  - a^* (a^+) is the minimal right (resp. left) identity of a from E (with respect to the natural partial order on idempotents).
- S satisfies the congruence condition if $\tilde{\mathcal{L}}_E$ is a right congruence and $\tilde{\mathcal{R}}_E$ is a left congruence.
  Equivalently: $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$.
  - In this case we can associate with S a category $C(S)$.
    Objects: E. Morphisms: S. $\mathrm{dom}(a) = a^*, \mathrm{range}(a) = a^+$.

Preliminaries - reduced E-Fountain semigroups

- We consider the class of reduced E - Fountain semigroups which satisfy the congruence condition.
- Subclass: If E is a subsemilattice (commutative subsemigroup) then S is called E-Ehresmann.
- Inverse semigroups, $\mathcal{B}_n$ and $\mathcal{PT}_n$ are all Ehresmann.

reduced E - Fountain + congruence condition

E-Ehremsmann

$\mathcal{B}_n$

$\mathcal{PT}_n$

Inverse semigroup

Right ample identity

- Let S be a reduced E-Fountain semigroup + congruence condition. S satisfies the **right ample identity** if $ea = a(ea)^*$ for every $e \in E$ and $a \in S$.
- Equivalent: $Ea \subseteq aE$ for every $a \in S$.
- Equivalent: S can be embedded in $\mathcal{PT}_n$ as an unary semigroup (with $\cdot, *$)

Right ample identity

Fact

If S satisfies the right ample identity then it is E-Ehresmann (aka E-Ehresmann right restriction).

Proof.

Right ample $\implies$ E subband:

For $e, f \in E$, note that $eff = ef$ so $(ef)^* \leq f$

$$ef = f(ef)^* = (ef)^* \in E$$

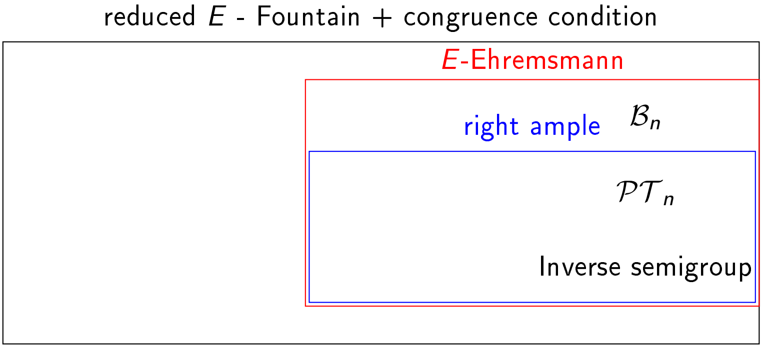
E subband $\implies$ E subsemilattice:

$eff = ef \implies fef = ef$

$ffe = fe \implies fef = fe$ so

$$ef = fe$$

Right ample identity



Generalized right ample identity

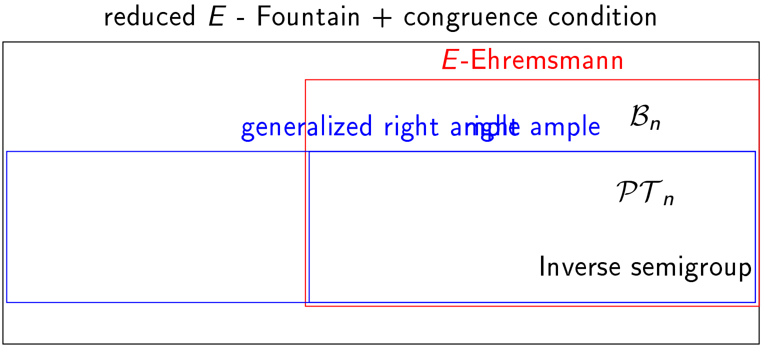
- Let S be a reduced E-Fountain semigroup + congruence condition. We say that S satisfies the **generalized right ample identity** if for every $e, f \in E$ and $a \in S$

$$\left(e\left(a(eaf)^{*}\right)^{+}\right)^{*} = \left(a(eaf)^{*}\right)^{+}$$

Generalized right ample identity

Claim

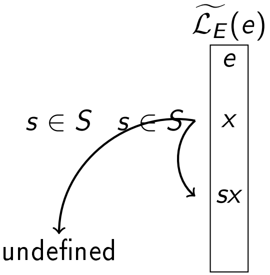
If S is Ehresmann then: generalized right ample $\iff$ right ample



Generalized right ample identity - What does it mean??

Let $\widetilde{\mathcal{L}}_E(e)$ be the $\widetilde{\mathcal{L}}_E$-class of $e \in E$. Then $\widetilde{\mathcal{L}}_E(e)$ is a partial left S-act (= S acts on the left of $\widetilde{\mathcal{L}}_E(e)$ by partial functions)

$$s * x = \begin{cases} sx & sx \in \widetilde{\mathcal{L}}_E(e) \\ \text{undefined} & sx \notin \widetilde{\mathcal{L}}_E(e) \end{cases} \quad (x \in \widetilde{\mathcal{L}}_E(e), \quad s \in S)$$



Generalized right ample identity - What does it mean??

Fact

Every $\alpha \in S$ *induces a function* $r_\alpha : \widetilde{\mathcal{L}}_E(\alpha^+) \to \widetilde{\mathcal{L}}_E(\alpha^*)$ *defined by* $r_\alpha(x) = x\alpha$.

Proof.

Assume $x \in \widetilde{\mathcal{L}}_E(\alpha^+) \implies x^* = \alpha^+$. Then $(x\alpha)^* = (x^*\alpha)^* = (\alpha^+\alpha)^* = \alpha^*$ so $x\alpha \in \widetilde{\mathcal{L}}_E(\alpha^*)$. $\square$

Proposition (IS)

S is generalized right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of partial left S-acts.

Corollary

Assume S is E-Ehresmann. S is right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of partial left S-acts.

Generalized right ample identity - What does it mean??

This gives a concrete interpretation of the associated category $C(S)$.

Proposition

Let S be a reduced E-Fountain semigroup + congruence condition + generalized right ample. The category $C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left S-acts of the form $\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are all homomorphisms of left partial S-acts between them.

Modules and homomorphisms

- Let $\mathbb{k}$ be a field and let $\mathbb{k}S$ be the associated semigroup algebra.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

- Let $\widetilde{\mathcal{L}}_E(e)$ be the $\widetilde{\mathcal{L}}_E$-class of $e \in E$. Let $\mathbb{k}\widetilde{\mathcal{L}}_E(e)$ be a $\mathbb{k}$-vector space of formal linear combinations with basis $\widetilde{\mathcal{L}}_E(e)$. Then $\mathbb{k}\widetilde{\mathcal{L}}_E(e)$ is a left $\mathbb{k}S$-module according to

$$s * x = \begin{cases} sx & sx \in \widetilde{\mathcal{L}}_E(e) \\ 0 & sx \notin \widetilde{\mathcal{L}}_E(e) \end{cases} \quad (x \in \widetilde{\mathcal{L}}_E(e), \quad s \in S)$$

- Now, r_α is a linear transformation $r_\alpha : \mathbb{k}\widetilde{\mathcal{L}}_E(\alpha^+) \to \mathbb{k}\widetilde{\mathcal{L}}_E(\alpha^*)$, defined on basis elements $r_\alpha(x) = x\alpha$.

Modules and homomorphisms

Proposition

Let S be a reduced E-Fountain semigroup + congruence condition. S is generalized right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of left $\Bbbk S$-modules.

Corollary

In this case, the category $C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left $\Bbbk S$-modules of the form $\Bbbk\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are homomorphisms of left $\Bbbk S$-modules of the form r_α.

Proposition (95% sure)

If S is finite. The category $\Bbbk C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left $\Bbbk S$-modules of the form $\Bbbk\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are all homomorphisms of left $\Bbbk S$-modules between them.

At last, some examples

Remark

Non interesting examples: $\mathcal{PT}_n$, inverse semigroups. These are also right ample. We want generalized right ample semigroups which are not right ample (E is not a subsemilattice).

- The Catalan monoid - $\mathcal{C}_n$.

$$\{f : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid i \leq f(i), \quad i \leq j \implies f(i) \leq f(j)\}$$

  $E = E(\mathcal{C}_n)$. Idempotents does not commute.
  $\mathcal{C}_n$ satisfies the generalized right ample condition - Margolis& Steinberg (2018).
  r_α are $\Bbbk S$-module homomorphisms - Steinberg (2016).
- Order preserving functions with a fixed point - $\mathcal{OPF}_n$.

$$\{f : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid f(n) = n, \quad i \leq j \implies f(i) \leq f(j)\}$$

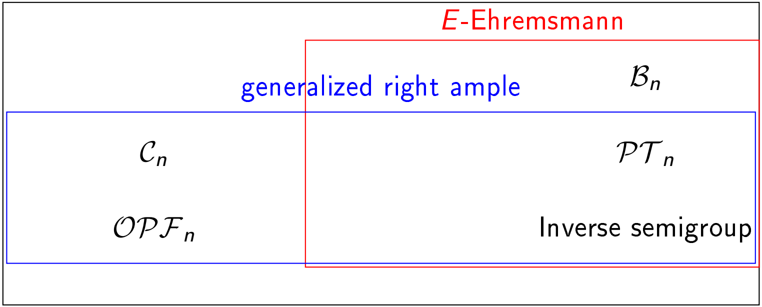
  $E = E(\mathcal{C}_n)$ (Note that $\mathcal{C}_n \subseteq \mathcal{OPF}_n$).

At last, some examples

- Let V be a Hilbert space (e.g. $V = \mathbb{R}^n$).
  Let $S = \{T : V \to V \mid T \text{ bounded}\}$.
  For every closed subspace $U \subseteq V$, let P_U be the associated orthogonal projection $P_U : V \to U$.Choose $E = \{P_U \mid U \subseteq V, \quad U \text{ closed}\}$. Note that E is not a subsemilattice.

At last, some examples

reduced E - Fountain + congruence condition



Applications to representation theory

- Fix S a finite reduced E-Fountain + congruence condition. Define a relation $a \trianglelefteq_l b$ if $a = be$ for some $e \in E$. (Beware! $\trianglelefteq_l$ is not even a pre-order).
- Let C be a category and let $\Bbbk$ be a field. The category algebra $\Bbbk C$ is the $\Bbbk$-Vector space of all linear combinations

$$\{\sum_{i=1}^{n} k_i m_i \mid k_i \in \Bbbk,\ m_i \text{ morphism}\}$$

with multiplication being extension of

$$m' \cdot m = \begin{cases} m'm & \mathrm{r}(m) = \mathrm{d}(m') \\ 0 & \mathrm{r}(m) \neq \mathrm{d}(m') \end{cases}$$

Applications to representation theory

- Let $\varphi : \Bbbk S \to \Bbbk C(S)$ be the linear transformation defined (on basis elements) by

$$\varphi(\alpha) = \sum_{\beta \trianglelefteq_l \alpha} \beta$$

Theorem (IS)

φ is a homomorphism of $\Bbbk$ - algebras if and only if S satisfies the generalized right ample identity.
If $\trianglelefteq_l$ is contained in a partial order then φ is an isomorphism.

Quite often $\Bbbk C(S)$ much easier to handle than $\Bbbk S$!

History of $\Bbbk S \simeq \Bbbk C(S)$

Reduced E-Fountain semigroup
+ generalized right ample (IS)

E-Ehresmann semigroups
+ right ample (IS 2017)

Ample semigroups (Guo&Chen 2012)

Inverse semigroups (Steinberg 2006)

Semilattices (Solomon 1967)

P-Ehresmann semigroups
(Wang 2017)

strict right ample semigroups
(Guo&Guo 2018)

Remark

The case of the Catalan monoid $\mathcal{C}_n$ has 5 different proofs! (Hivert&Thiéry, Grensing, SteinbergX2, Margolis&Steinberg)

Application: Projective modules

Proposition (IS)

Let S be a reduced E-Fountain semigroup + congruence condition + generalized right ample + $\trianglelefteq_l$ contained in a partial order. Then for every $e \in E$, $\Bbbk\widetilde{\mathcal{L}}_E(e)$ is a projective module. (Not necessarily indecomposable).

- The E-Ehresmann case was proved by Margolis&IS (2021) - with description of the indecomposables in certain cases.
- It is known that $\Bbbk\widetilde{\mathcal{L}}_E(e)$ is indecomposable projective for every $\mathcal{R}$-trivial monoid (with $E = E(S)$). Example: $\mathcal{C}_n$.

Question

Find a theorem that implies both the reduced E-Fountain case and the $\mathcal{R}$-trivial case.

Thank you!