Building more automatic structures for groups

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Plan for today

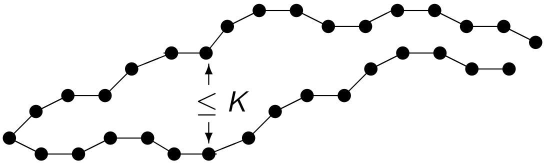
- My focus: some new composition theorems for automatic groups (joint work with Hermiller, Holt, Susse), relating to HNN extensions, amalgamated products and, more generally, graphs of groups, making use of coset automatic structures.
- I'll give some background on automatic groups, give definitions and some motivation, explain what is known, and some questions that remain open.
- Then I'll give some details of the proofs of our recent results, and include some examples of groups that we can now handle that weren't previously known to be automatic.
- $\pi_1(\mathcal{M})$ for $\mathcal{M}$ a compact 3-manifold without pieces built on $\textsc{Nil}$ or $\textsc{Sol}$ geometries was already proved automatic (Epstein et al.); our results give clean, constructive proofs via graph of groups decompositions (cf. Shapiro).

Introducing automatic groups

Thurston et al. introduced automatic groups in the 1980s, as a generalisation of hyperbolic groups, with similar algorithmic properties, in particular, hoping to find these properties in $\pi_1(\mathcal{M})$, $\mathcal{M}$ a cmpct 3-mfld.

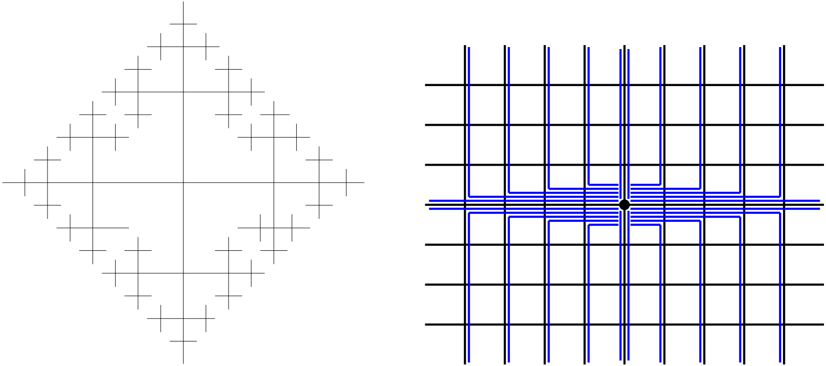
$G = \langle X \rangle$ is **automatic** if $\exists$ a set of paths from the vtx 1 in the Cayley graph $\Gamma(G, X)$, a corresponding set $L \twoheadrightarrow G$ of words, and a constant K, st

- L is a **regular set** (i.e. can be recognised by a finite state automaton with alphabet $X^{\pm}$)
- if $v, w \in L$ satisfy $v =_G w$ or $vx =_G w$ for $x \in X$, then the corresponding paths ${}_1w, {}_1v$ in Γ **K-fellow travel** ($w \sim_K v$).



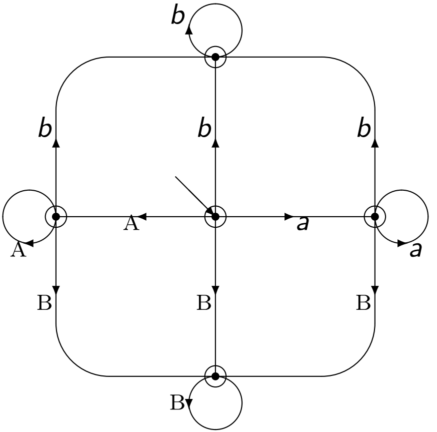
If the paths only fellow travel asynchronously, then G is called **asynchronously** automatic

The free group F_2 and $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$ are automatic.



In F_2 we select all geodesic paths/words.
In $\mathbb{Z}_2$ we also use geodesics, but have to restrict to a subset of those, such as $\{a^i b^j : i, j \in \mathbb{Z}\}$ (the 'shortlex' language), in order to get fellow travelling; note that geodesics $a^i b^i$ and $b^i a^i$ diverge to distance $2i$.

FSA accepting language $\{a^i b^j\}$ for $\mathbb{Z}^2$



An FSA can be represented by a directed, edge-labelled graph, whose vertices are its states; a word is in the language of the FSA if it corresponds to a path from the initial state to an accepting state.

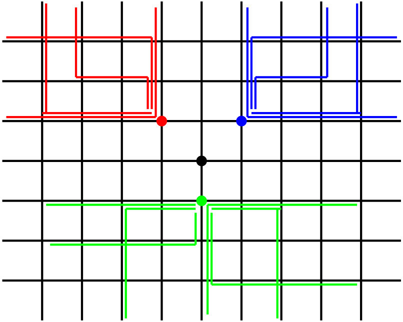
This FSA has **six** states, but we can only see **five**. All transitions that are not shown are to the sixth (non-accepting) **failure** state, and all transitions from it return to it.

All the other five states are accepting (and so ringed).

Regularity of L

The regularity of L corresponds to its association with some finite aspect of the Cayley graph.

For any word hyperbolic group (also for $\mathbb{Z}^n$, any Coxeter group, and more ...), the **cones** in the Cayley graph fall into finitely many types, forming the states of an FSA that recognises the geodesics.



There are 9=4+4+1 cone types for $\mathbb{Z}^2$.

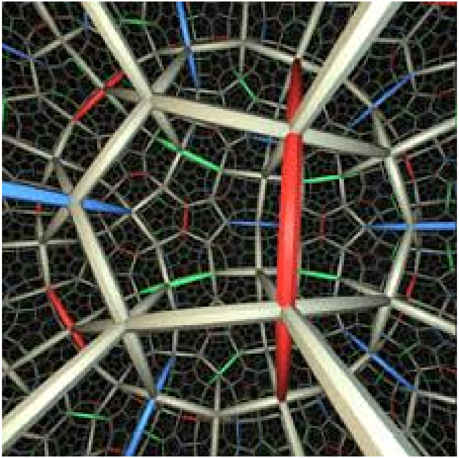
But we don't have finitely many cone types for many groups. And even when we do, we can only get an automatic structure from a finite set of cone types if the geodesics also fellow travel. In general L needs to be selected more carefully, to ensure both regularity and fellow travelling.

Some facts about automaticity

- Automatic groups are all finitely presented, have word problem soluble in quadratic time (sim. for async. automatic, except that in this case `WP` is exponential). Biautomatic groups (which satisfy an additional `ft` condition: $xv =_G w \Rightarrow {}_x v \sim_K {}_1 w$) have soluble conjugacy problem: $?\exists g,\ u_1^g =_G u_2$.
- Use of FSA gives easy algorithms for some basic properties (e.g. finiteness) in any automatic group.
- A group G is hyperbolic $\iff$ its set $\text{Geo}(G)$ of all geodesics gives automatic structure $\iff$ $\text{Geo}(G)$ gives biautomatic structure. Both word and conjugacy problem can be solved in linear time.
- $\exists$ many further examples of automatic groups, **E.g.** $\pi_1(\mathcal{X})$ for most compact 3-manifolds $\mathcal{X}$, all Coxeter groups, many Artin groups.
- It is unknown whether $\exists$ an automatic gp that is not biautomatic.
- It is unknown whether $\exists$ an automatic gp with insoluble conjugacy problem.

π_1(complement of Borromean rings) is automatic

So we can use its automatic structure to draw pictures. I don't think this still from the video 'Not knots' actually used the automatic groups software, but it could have done:



How do we get closure results for automatic groups?

If $G = \langle X \rangle$ is automatic, with language L, it's straightforward to build an automatic structure

- for G over any other finite generating set, or
- for any finite index subgroup or supergroup of G, or
- for any quotient of G by a finite normal subgroup, or
- for any extension of a finite group by G.

The results use languages for the new structure such as $\phi^{-1}(L)$ or LL_F, where ϕ is a monoid homomorphism, L_F is finite; regularity follows from basic properties of the class of regular languages. Deriving a fellow traveller property for the new language (with a new constant) is not difficult.

And we have asynchronous versions of the results. And biautomatic versions (where we have an extra 'left' `ft` condition holding between paths); except that here we lose the 'supergroup' result.

Combination results: direct and free products

Suppose that $G_1 = \langle X_1 \rangle$ and $G_2 = \langle X_2 \rangle$ are automatic with languages L_1, L_2, over disjoint finite generating sets X_1, X_2, `ft` constants K_1, K_2.

It's straightforward to build automatic structures for $G_1 \times G_2$ and $G_1 * G_2$ over $X = X_1 \cup X_2$, each with `ft` constant $K = \max\{K_1, K_2\}$.

For $G_1 \times G_2$, we use the regular language $L = L_1 L_2 = \{w_1 w_2 : w_i \in L_i\}$.
For if $w_i, w_i' \in L_i$, and $x \in X^{\pm 1} \cup \{1\}$, then

$$w_1 w_2 x =_G w_1' w_2' \Rightarrow (w_1 x =_{G_1} w_1' \wedge w_2 =_{G_2} w_2') \vee (w_1 =_{G_1} w_1' \wedge w_2 x =_{G_2} w_2')$$

For $G_1 * G_2$, we define $L_i' := \{w \in L_i : w \neq_{G_i} 1\}$ and then let L be the language consisting of ϵ together with all strings of the form $u_1 \cdots u_n$ with successive u_i taken from alternate L_i'.
For if $u = u_1 u_2 \cdots u_m, u' = u_1' u_2' \cdots u_n' \in L$, and $x \in X^{\pm} \cup \{1\}$, then

$$\begin{aligned} ux =_G u' \quad \Rightarrow \quad & m \in \{n, n-1, n+1\} \wedge (u_j =_{G_i} u_j', j < \max\{m, n\}) \\ & \wedge ((u_m x =_{G_i} u_m') \vee (x =_{G_i} u_n') \vee (u_m x =_{G_i} 1)). \end{aligned}$$

So, in both cases, L inherits fellow travelling from its components.

Going further with combination theorems

Given G_1, G_2 with $H_i \subseteq G_i$ and an isom. $\phi : H_1 \to H_2$, we can define:

$$\text{amalgamated free prod. } G_1 *_{H_1} G_2 := \langle G_1, G_2 \mid \phi(h) = h, \forall h \in H_1 \rangle,$$

$$(\text{if } G_1 = G_2 = G) \quad \text{HNN ext. } G*_{H_1,t} := \langle G_1, t \mid t^{-1}ht = \phi(h), \forall h \in H_1 \rangle.$$

Previous combination theorems for $G_1 *_{H_1} G_2$ and $G*_{H_1,t}$ dealt with

- (Epstein et al.)
  - the case H_1 finite;
  - $H_1 = \mathbb{Z}^2$, given specific automatic structures for $\pi_1(\mathcal{X}_i)$, each $\mathcal{X}_i$ based on one of 6 3-D geometries, as necessary to prove $\pi_1(\mathcal{X})$ automatic for $\mathcal{X}$ a cmpct 3-mfld without pieces based on NIL or SOL geometries,
- (Baumslag et al.) restrictions such as rationality of H_i, so that in particular, amalgamated products:
  - $F_r *_H F_s$, H f.g., are async. automatic,
  - of two fg abelian groups are automatic,
  - of two neg. curved groups, over $\mathbb{Z}$ are automatic.
- (Shapiro) found results sim. to Baumslag et al. for graphs of groups, suggesting automaticity proof for $\pi_1(\mathcal{X})$ relating to JSJ decomp.

Free product with amalgamation over finite H

Suppose that $G = G_1 *_H G_2$, with G_1, G_2 automatic, H finite,
Let $T_i = T_i' \cup \{1\}$ be a left transversal (set of unique left coset reps) for H in G_i. Then each element of G has a unique decomposition as a product $t_1 \ldots t_m h$, with $h \in H$, $m \geq 0$, successive t_j chosen from alternate T_i'.

We can adapt the automatic structures we have for G_1, G_2, and find regular sets of words L_1', L_2', such that, for each i, $L_i' \cup \{1\}$ contains a left transversal for H in G_i, and $L_i := H \cup L_i' H$ is the language of an automatic structure for G_i over $X_i \cup H$.

Now let L be the language consisting of ϵ together with all strings of the form $u_1 \cdots u_m h$ with successive u_i taken from alternate L_i'; then $L \twoheadrightarrow G$.
It's straightforward to see that L is regular. And the fellow travelling property for L is a consequence of the fellow travelling properties of L_1, L_2 together with the finiteness of H.
NB: $\exists$ a similar structure based on any right transversal, with H on the left.

We **needed** $|H| < \infty$ for this result. But we might manage without that condition if we had an **automatic coset system**.

Coset languages, and strong automatic coset systems

Let $H = \langle Y \rangle \subseteq G = \langle X \rangle$, with $|X|, |Y| < \infty$.

A **coset language** for (G, H) is a set L^H of words over X containing at least one representative of each right coset Hg of H in G.

A **strong asynchronous automatic coset system** for (G, H) is a coset language L^H together with a constant K, such that

(i) L^H is a regular language,

(ii) if $v, w \in L^H$ and $h \in H$ with $d_{\Gamma(G)}(v, hw) \leq 1$, then the paths ${}_1v$ and ${}_hw$ in $\Gamma(G)$ asynchronously K-fellow travel; in particular, $|h| \leq K$. (Holt&Hurt, 1999)

If (G, H) has a strong asynchronous automatic coset system, then (G, H) is called **strongly asynchronously coset automatic**.

If the fellow traveller condition is synchronous, then (G, H) is called **strongly (synchronously) coset automatic**.

Some applications of coset automaticity.

A strong sync. automatic coset system provides a quadratic algorithm for putting elements of cosets of H in G into a normal form and, in particular, a quadratic solution to the problem of membership of elements of G in H.

In addition, Holt&Hurt developed an algorithm to build subgroup presentations.

Automaticity of G does **not**, in general, imply coset automaticity of pairs (G, H); but if G is hyperbolic, then (G, H) is strongly sync. coset automatic when H is a quasi-convex subgroup (Redfern, thesis, 1993).

A subgroup H is **quasi-convex** in G if $\exists C$ st any geodesic in $\Gamma(G)$ joining two vtces of H stays within a C-nbd of the subgraph corresponding to H.

It turns out that coset automaticity is what we need in order to build combination theorems for free products amalgamated over subgroups, as well as for the more general constructions of graphs of groups.

From coset automatic to automatic

Theorem

Let $G = \langle X \rangle$ be a group, and $H = \langle Y \rangle$ a subgroup of G, $|X|, |Y| < \infty$. Suppose that (G, H) is strongly asynchronously coset automatic with language L^H over X, and H is asynchronously automatic with language L_H over Y.
Then the group G is asynchronously automatic over $X \cup Y$, with language $L := L_H L^H$ (the concatenation of L_H and L^H),
If (G, H) is strongly **synchronously** *coset automatic, then G is synchronously automatic.*

We construct automatic structures for composite groups by first building composite coset structures, and then applying the above theorem.

Of course the coset structures are useful too. Because we can use them to answer questions about the subgroup H.

A coset automatic structure for $G_1 *_H G_2$: how do we start?

Let $(G_1, H), (G_2, H)$ be coset automatic, with coset languages $L_1^H, L_2^H \ni \epsilon$. For each i, we define a right transversal $T_i = T_i' \cup \{1\} \subset G_i$, for H in G_i, represented by a subset of L_i

Recall that every element g of $G := G_1 *_H G_2$ has a unique representation $\tau(g)$ as a product $ht_1 \dots t_m$, with $h \in H$, and successive t_j chosen from alternate T_i'; m is called its **length**. We can rewrite any expression $g_1 \dots g_n$ for g to the word $\tau(g)$, working from the right, applying equations $g' = h't$. If successive g_j are taken from alternate sets $G_i \setminus H$, then $m = n$.

We derive L^H from the normal form $\tau(g)$. Specifically, we define L^H to consist of ϵ together with all strings of the form $u_1 \cdots u_n$ with successive u_j non-empty, taken from alternate L_i^H.

It's straightforward to see that L^H is regular, containing a right transveral for H in G. And we observe that n is the length of $\tau(g)$.

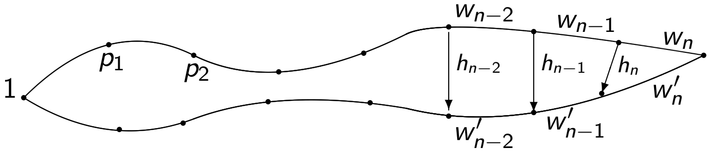
But can we verify fellow travelling for L^H? Sometimes, but not always. We need some conditions to make it work, which we need to investigate.

Trying to verify fellow travelling for L^H

Let $\Gamma = \Gamma(G, X)$, $\Gamma_i = \Gamma(G_i, X_i)$, K be a common `ft` constant for L_1, L_2.
(Case 1) Let $w = w_1 \cdots w_n$, $w' = w'_1 \ldots w'_{n'} \in L^H$ st $Hw = Hw' = Hg$, and $\tau(g) = ht_1 \ldots t_m$. Then $n = m = n'$ and (consider rewriting) $Hw_n = Ht_n = Hw'_n$.
We write $w_n = h_n w'_n$, and for $j = 1, \ldots n-1$ let p_j be the element of G represented by $w_1 \ldots w_j$. Applying coset automaticity, we observe that

$$_1w_n \sim_K {}_{h_n}w'_n \text{ in } \Gamma_i, \quad \text{and hence } |h_n|_{X_i} \leq K, \quad \text{and } {}_{p_{n-1}}w_n \sim_K {}_{p_{n-1}h_n}w'_n \text{ in } \Gamma.$$

If $n > 1$, then from $Hw_1 \cdots w_n = Hw'_1 \cdots w'_n$ and $w_n =_G h_n w'_n$ we deduce $Hw_1 \cdots w_{n-1}h_n = Hw'_1 \cdots w'_{n-1}$, and (as above) $Hw_{n-1}h_n = Hw'_{n-1}$.
Continuing, we find $h_{n-1}, \ldots, h_1 \in H$, with $w_j h_{j+1} =_G h_j w'_j$ for each j:



We have $|h_{n-j}| < K^{j+1}$, but we need more to get an `ft` property ...
NB: We get a similar picture in Case 2: Hwx = Hw'.

Crossover, and an amalgamated product result

Definition (Crossover)

Given finite subsets Y, Z of G and $H = \langle Y \rangle$, We say that the coset language L^H for (G, H) has λ-*limited crossover* wrt (Y, Z) if,

$$\forall g \in \langle Z \rangle, \text{st}, |g|_Z \leq \lambda, \text{and} \, \forall u, v \in L^H, \text{st} \, ug \in Hv, \quad |ugv^{-1}|_Y \leq \lambda.$$

Note that λ-crossover implies $K\lambda$-crossover $\forall K \in \mathbb{N}$.

Given crossover, we have fellow travelling, so a combination theorem for amalgamated products (naturally asynchronous; we can't be sure that $|w_j| = |w'_j|$ for components of words w, w' with $Hwx =_G Hw'$) :

Theorem

Suppose that $G_1 = \langle X_1 \rangle$, $G_2 = \langle X_2 \rangle$, $H \langle Y \rangle \leq G_1 \cap G_2$, and $\exists$ strong asynchronously automatic coset systems for (G_1, H) and (G_2, H), both with limited crossover with respect to Y, Then $(G_1 *_H G_2, H)$ has a strong asynchronously automatic coset system.

Stablity and HNN-extensions

Definition

Let $\phi : H_1 = \langle Y_1 \rangle \to H_2 = \langle Y_2 \rangle$ be an isomorphism. We say that (H_1, H_2, ϕ) is *μ-stable* if

$$\forall h \in H_1, \quad |h|_{Y_1} \leq \mu \Rightarrow |\phi(h)|_{Y_2} \leq \mu.$$

Stability is the condition we need for HNN-extensions:

Theorem

Let $G = \langle X \rangle$ with $H_i = \langle Y_i \rangle \subseteq G$, $\phi : H_1 \to H_2$ an isomorphism.

(i) the pairs (G, H_j) are strongly asynchronously coset automatic,

(ii) each L^{H_j} has limited crossover wrt (Y_i, Y_j),

(iii) (H_1, H_2, ϕ) and (H_2, H_1, ϕ^{-1}) are stable,

Then $(G*_{H_1,t}, H_1)$ has a strong asynchronously automatic coset system.

The full result for graphs of groups

Each of $G_1 *_H G_2$ and $G*_{H,t}$ can be found as $\pi_1(\mathcal{G})$ for a graph of groups
$$\mathcal{G} = \{G_v = \langle X_v \rangle, G_e = \langle Y_e \rangle : v \in V, e \in E, G_e \subseteq G_{\tau(e)}, G_e \cong_{\phi_e} G_{\bar{e}}\}.$$

Theorem A

Let $G = \pi_1(\mathcal{G})$ for a graph of groups $\mathcal{G}$, Suppose that, for $e, f \in E(\mathcal{G})$,

(i) $(G_{\tau(e)}, G_e)$, is strongly asynch. coset automatic, $\epsilon \in L^e_{\tau(e)}$,

(ii) $(G_e, G_{\bar{e}}, \phi_e)$ is stable,

(iii) if $\tau(e) = \tau(f)$, $L^e_{\tau(e)}$ has limited crossover wrt (Y_e, Y_f).

Then, $\forall e_0 \in E(\mathcal{G})$, (G, G_{e_0}) is strongly asynch. coset automatic.

The proof is very much like the proof for free products with amalgamation.

The coset language for (G, G_{e_0}) is derived from the **Higgins normal form** for the graph of groups G in much the same way as the coset language for $(G_1 *_H G_2, G_i)$ is derived from the normal form for $G_1 *_H G_2$.

The conditions of crossover and stability ensure fellow travelling.

The Higgins normal form

We define the **Higgins normal form** for $G = \pi_1(\mathcal{G})$ wrt a subtree $\mathcal{T} \subseteq \mathcal{G}$, vertex v_0, language L_{v_0} for the vertex group G_{v_0}.

It's the set of all words $w_0 s_1 u_1 \cdots s_m u_m$, where $q = e_1 \cdots e_m$ is a path in $\mathcal{G}$ with $\iota(e_1) = v_0$, $w_0 \in L_{v_0}$, $u_i \in L^{e_i}_{\tau(e_i)}$, and $s_i = s_{e_i}$ if $e_i \notin \mathcal{T}$, $s_i = \epsilon$, otherwise (+ certain u_i, u_{i+1} pairs disallowed).

We derive our coset language L^{e_0} for (G, G_{e_0}) similarly wrt $\mathcal{T}$, e_0; it's the set of all $u_0 s_1 u_1 \cdots s_m u_m$ as above, except that $\iota(e_1) = \tau(e_0)$, $u_0 \in L^{e_0}_{\tau(e_0)}$.

Corollary

Suppose that $G = \pi_1(\mathcal{G})$ satisfies the conditions of Theorem A, and that e_0 is an edge of $\mathcal{G}$ for which G_{e_0} is asynchronously automatic. Then G has an asynchronously automatic structure whose language is the Higgins normal form, based on any tree $\mathcal{T}$, and the vertex $v_0 = \tau(e_0)$.

We form the concatenation $L_{e_0} L^{e_0}$ of languages for G_{e_0} and (G, G_{e_0}), and note that this gives the Higgins normal form wrt $v_0 = \tau(e_0)$.

Deducing a synchronous result

The results we've seen so far give us **asynchronously** automatic coset systems for (G, H), from which we can only hope to build **asynchronously** automatic structures for G.

But often we can do better. Let Geo^H be the coset language of all words w that are of minimal lengths within the coset Hw. We have:

Proposition

If L^H is a strong asynchronously automatic coset system for which $L^H \cap \mathrm{Geo}^H$ is also a coset language. Then $L^H \cap \mathrm{Geo}^H$ is a strong synchronously automatic coset system for (G, H).

And if we impose extra conditions on our component coset automatic systems, we can ensure the appropriate conditions on the structures we build in our combination theorems.

In that case, we have synchronous substructures.

Applying our combination theorems

We can construct strong synch. aut. coset systems satisfying appropriate conditions of crossover/stability for various (G, H), $G \supseteq H$, such as:

- G is abelian,
- G is hyperbolic rel. to a collection of abelian subgps that includes H,
- G is shortlex automatic, and its geodesics 'concatenate up' from H, i.e.

  $$w \in \mathrm{Geo}(H), v \in \mathrm{Geo}(G, H) \Rightarrow wv \in \mathrm{Geo}(G).$$

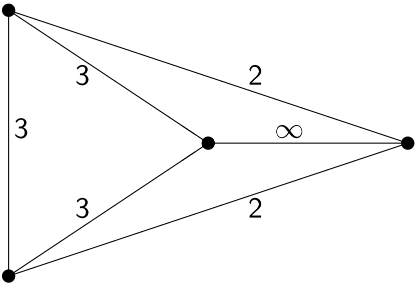
  e.g. G is a Coxeter group, or a sufficiently large Artin group, and H is a parabolic subgroup.

Then, for graphs of groups $\mathcal{G}$ built out of each of these three types of groups (module some conditions in the first case), we can construct strong **synchronous** coset automatic systems and prove that $\pi_1(\mathcal{G})$ is automatic.

Similarly, we have results about amalgamated products such as $G_1 *_H G_2$ with G_1 abelian, G_2 rel. hyperbolic wrt a collection of abelian groups that includes H.

In particular, . . .

We find synchronous automatic structures for various Artin groups not previously proved to be automatic, such as the group with diagram:



That its word problem was soluble was already clear from its expression as an amalgamated product.

Synchronous structures for 3-manifold groups

Corollary

Let $\mathcal{M}$ be an orientable, connected, compact 3-manifold with incompressible toral boundary, whose prime factors have JSJ decompositions containing only hyperbolic pieces. Then $\pi_1(\mathcal{M})$ is automatic, wrt a Higgins language of normal forms.

We have $\pi_1(\mathcal{M}) = \pi_1(M_1) * \pi_1(\mathcal{M}_2) * \cdots * \pi_k(\mathcal{M}_k)$, where each $\mathcal{M}_i$ is prime.

Then each $\pi_1(\mathcal{M}_i)$ is $\pi_1(\mathcal{G}_i)$ for a graph of groups $\mathcal{G}_i$ that are hyperbolic relative to (free) abelian subgroups. We apply our results to build automatic structures for each $\pi_1(\mathcal{G}_i)$.

We can verify that the conditions are satisfied that make the automatic structures synchronous.