Schreier extensions and the Grothendieck construction

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Group extensions

An extension of groups is a diagram

$$N \xhookrightarrow{k} G \xrightarrow{e} H$$

where k is the kernel of e and e exhibits H as the quotient G/N.

Examples include

$$\mathbb{Z}/2\mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$

and

$$\mathbb{Z}/3\mathbb{Z} \hookrightarrow S_3 \xrightarrow{\sigma} \mathbb{Z}/2\mathbb{Z}.$$

We would like a classification all the distinct extensions for fixed choices of N and H. Such a classification was given by Otto Schreier in 1926.

Monoid extensions

An extension of monoids is a diagram

$$N \overset{k}{\hookrightarrow} G \overset{e}{\twoheadrightarrow} H$$

where N is isomorphic to $e^{-1}(1)$ and H is isomorphic to G/E_N where E_N is the congruence generated by $k(n) \sim 1$ for $n \in N$.

General extensions of monoids are not well-behaved and so additional restrictions are usually imposed.

A Schreier extension is a monoid extension such that for every $h \in H$ there is a $u_h \in e^{-1}(\{h\})$ such that for all $g \in e^{-1}(\{h\})$ there is a unique $n \in N$ with $k(n)u_h = g$.

Note that every group extension is Schreier: for *any* choice of u_h the unique n is given by $k(n) = gu_h^{-1}$.

Classifying Schreier extensions

The classification of group extensions with specified N and H can be extended to Schreier extensions of monoids.

They can be specified by (equivalence classes of) pairs (α, χ), where $\alpha\colon H \times N \to N$ and $\chi\colon H \times H \to N$ are functions satisfying

- $\alpha(h, 1) = 1$,
- $\alpha(h, n_1 n_2) = \alpha(h, n_1)\alpha(h, n_2)$,
- $\alpha(1, n) = n$,
- $\chi(h_1, h_2)\alpha(h_1 h_2, n) = \alpha(h_1, \alpha(h_2, n))\chi(h_1, h_2)$,
- $\chi(1, h) = 1 = \chi(h, 1)$,
- $\chi(x, y)\chi(xy, z) = \alpha(x, \chi(y, z))\chi(x, yz)$.

This data appears to be quite complicated and difficult to interpret. In this talk I hope to show how it arises naturally via category theory.

Background: Categories

A category $\mathcal{C}$ consists of a collection of *objects* $\mathcal{C}_0$ and, for each pair of objects (A, B), a set $\mathcal{C}(A, B)$ of *morphisms* from A to B.

If $f \in \mathcal{C}(X, Y)$ we write $f\colon X \to Y$ and say X is the *domain* of f and Y is the *codomain* of f.

If $f\colon X \to Y$ and $g\colon Y \to Z$ then we can form the *composite* morphism $gf\colon X \to Z$. Moreover, composition is associative. Finally, for each object X there is an *identity morphism* $1_X\colon X \to X$.

A central example is the category Set whose objects are sets and whose morphisms are functions. Another important example is Mon — the category of monoids and monoid homomorphisms.

Every monoid M gives a category $\mathcal{B}M$ with a single object $*$ and with $\mathcal{B}M(*, *) = M$. Composition is given by the monoid multiplication.

Background: Functors

A functor $F\colon \mathcal{C} \to \mathcal{D}$ between categories consists of a function $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ between the collections of objects and, for each $X, Y \in \mathcal{C}_0$, a function $F_{X,Y}\colon \mathcal{C}(X, Y) \to \mathcal{D}(F_0(X), F_0(Y))$.
(We will write F for both F_0 and $F_{X,Y}$).

These must satisfy $F(1_X) = 1_{F(X)}$ and $F(fg) = F(f)F(g)$.

For example, there is a functor $U\colon \mathrm{Mon} \to \mathrm{Set}$ taking the underlying set of each monoid (and the underlying function of the homomorphisms).

We obtain a category Cat whose objects are (small) categories and whose morphisms are functors.

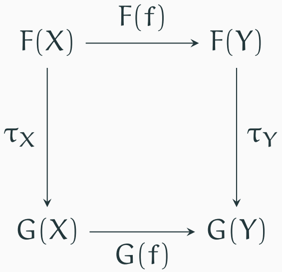
A monoid homomorphism $f\colon M \to N$ gives a functor $\mathcal{B}f\colon \mathcal{B}M \to \mathcal{B}N$ and all functors between one-object categories are of this form.

Note that $\mathcal{B}$ itself is a functor from Mon to Cat.

Background: Natural transformations

The set of functors $\mathsf{Cat}(\mathcal{C}, \mathcal{D})$ between categories is not just a set, but itself has the structure of a category. The morphisms between functors are called natural transformations.

Let $F, G \colon \mathcal{C} \to \mathcal{D}$. A natural transformation $\tau \colon F \to G$ is given by a family $(\tau_X)_{X \in \mathcal{C}_0}$ of morphisms $\tau_X \colon F(X) \to G(X)$ in $\mathcal{D}$ indexed by $\mathcal{C}_0$. For each $f \colon X \to Y$ in $\mathcal{C}$ the following square must commute.



Natural transformations between monoid homomorphisms

If monoids correspond to one-object categories and monoid homomorphisms correspond to functors between them, what are natural transformations between these functors?

Suppose $f, g \colon M \to N$. Then a natural transformation from $\mathcal{B}f$ to $\mathcal{B}g$ is given by a single morphism $t \in N$. Commutativity of the relevant square means that for all $m \in M$, we have $t\, f(m) = g(m)t$.

2-categories

Natural transformations give Cat the structure of a 2-category. We will write $\mathbf{Cat}$ for the 2-category of categories, functors and natural transformations.

A *2-category* $\mathbf{C}$ consists of a collection of objects $\mathbf{C}_0$ and, for each pair of objects (A, B), a *category* $\mathbf{C}(A, B)$, whose objects are *1-morphisms* from A to B and whose morphisms are *2-morphisms* between these. We omit the precise axioms.

Any (1-)category can be viewed as a 2-category with only identity 2-morphisms.

If we add the 2-morphisms to the category Mon from the previous slide we obtain a 2-category $\mathbf{Mon}$ — the 2-category of one-object categories.

Maps between 2-categories

There are a few notions of map between 2-categories. For simplicity, we will restrict our attention to the case where the domain is a 1-category.

The most obvious kind of map is given by strict 2-functors. A strict 2-functor $F\colon \mathcal{C} \to \mathbf{D}$ (where $\mathcal{C}$ is 1-category) is simply a functor from $\mathcal{C}$ to $\mathbf{D}$ ignoring the 2-morphisms. So it consists of a map $F_0\colon \mathcal{C}_0 \to \mathbf{D}_0$ and, for each $X, Y \in \mathcal{C}_0$, a function $F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathbf{D}(F_0(X), F_0(Y))_0$.

These must satisfy $F(1_X) = 1_{F(X)}$ and $F(fg) = F(f)F(g)$.

In category theory, we usually only care about things holding up to isomorphism. This suggests a generalisation of 2-functors called pseudofunctors where the equalities above are replaced with (specified) invertible 2-morphisms. (These must then satisfy additional 'coherence conditions' in order to be well-behaved. We omit the details.)

Oplax functors

In fact, we will want to generalise things even further by omitting the condition that the 2-morphisms are invertible.

An oplax functor $L\colon \mathcal{C} \to \mathbf{D}$ consists of

- a function $L\colon \mathcal{C}_0 \to \mathbf{D}_0$ between the collections of objects,
- a function $L_{X,Y}\colon \mathcal{C}(X,Y) \to \mathbf{D}(L(X), L(Y))_0$ for each pair of objects $X, Y \in \mathcal{C}_0$,
- a 2-morphism $\iota^L_X\colon L(1_X) \to 1_{L(X)}$ in $\mathbf{D}$ for each object $X \in \mathcal{C}_0$, called the unitors of L,
- and a 2-morphism $\gamma^L_{f,g}\colon L(f \circ g) \to L(f) \circ L(g)$ in $\mathbf{D}$ for each pair of composable 1-morphisms (f, g) in $\mathcal{C}$, called the compositors.

These must satisfy a number of coherence conditions, which we omit. We say an oplax functor is normal if its unitors are identity morphisms.

The (generalised) Grothendieck construction

An indexed family $(A_i)_{i \in I}$ is intuitively a class function $A \colon I \to \mathsf{Set}$. But this can equivalently be viewed as a map $a \colon X \to I$ where the set corresponding to $i \in I$ is given by $a^{-1}(\{i\})$. Here $X = \bigsqcup_{i \in I} A_i$.

Now suppose $\mathcal{D}$ is a category and $L \colon \mathcal{D} \to \mathbf{Cat}$ is a normal oplax functor. Analogously, we can construct a functor $F_L \colon \int L \to \mathcal{D}$.

The objects of $\int L$ are of the form $(D, \bar{D})$ where D is an object of $\mathcal{D}$ and $\bar{D}$ is an object of $L(D)$.

Morphisms from $(D, \bar{D})$ to $(E, \bar{E})$ in $\int L$ are given by pairs $(f, \bar{f})$ where $f \colon D \to E$ and $\bar{f} \colon L(f)(\bar{D}) \to \bar{E}$.

The functor $F_L \colon \int L \to \mathcal{D}$ simply projects onto the first component of the pairs.

The (generalised) Grothendieck construction: composition

What is the composite of $(f, \bar{f})\colon (D, \bar{D}) \to (E, \bar{E})$ and $(g, \bar{g})\colon (C, \bar{C}) \to (D, \bar{D})$ in $\int L$?

The first component of the composite is just fg. For the second component, we have $\bar{f}\colon L(f)(\bar{D}) \to \bar{E}$ and $\bar{g}\colon L(g)(\bar{C}) \to \bar{D}$ and want a morphism $\overline{fg}\colon L(fg)(\bar{C}) \to \bar{E}$.

Applying $L(f)$ to $\bar{g}$ we have $L(f)(\bar{g})\colon L(f)L(g)(\bar{C}) \to L(f)(\bar{D})$. Then composing with $\bar{f}$ we get $\bar{f} \circ L(f)(\bar{g})\colon L(f)L(g)(\bar{C}) \to \bar{E}$.

The domain here is $L(f)L(g)(\bar{C})$ instead of $L(fg)(\bar{C})$. But comparing these is precisely the job of the compositor $\gamma^L_{f,g}$.

Thus, we set $\overline{fg} = \bar{f} \circ L(f)(\bar{g}) \circ (\gamma^L_{f,g})_{\bar{C}}$.

Finally, the identity on $(D, \bar{D})$ is $(\mathrm{id}_D, \mathrm{id}_{\bar{D}})$.

Morphisms in $\int L$ of the form $(f, \mathrm{id}_{L(f)(\bar{D})})\colon (D, \bar{D}) \to (E, L(f)(\bar{D}))$ play a special role in that every morphism $(f, \bar{f})\colon (D, \bar{D}) \to (E, \bar{E})$ factors as $(\mathrm{id}_E, \bar{f}) \circ (f, \mathrm{id}_{L(f)(\bar{D})})$.

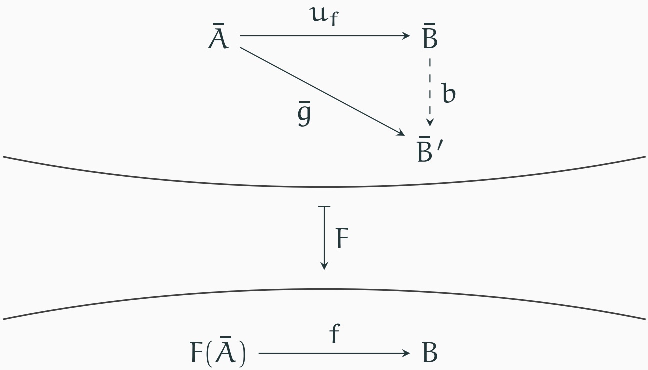
Definition

A morphism $u\colon \bar{A} \to \bar{B}$ in $\mathcal{C}$ is pre-opcartesian with respect to a functor $F\colon \mathcal{C} \to \mathcal{D}$ if for any $\bar{g}\colon \bar{A} \to \bar{B}'$ with $F(\bar{g}) = F(u)$, there exists a unique map $b\colon \bar{B} \to \bar{B}'$ with $F(b) = \mathrm{id}_{F(\bar{B})}$ such that $bu = \bar{g}$.

We say F is a pre-opfibration if for every morphism $f\colon F(\bar{A}) \to B$ there exists a pre-opcartesian lifting $u_f\colon \bar{A} \to \bar{B}$ with $F(u_f) = f$.

So F_L is a pre-opfibration.

Pre-opfibration diagram



Pre-opfibrations and Schreier extensions

Let $e\colon G \to H$ be a monoid homomorphism. What does it mean for $\mathcal{B}e$ to be a pre-opfibration?

This means that for every $h \in H$ there is a $u_h \in e^{-1}(\{h\})$ such that for all $g \in e^{-1}(\{h\})$ there is a unique $n \in e^{-1}(\{1\})$ with $nu_h = g$.

In particular, e is surjective and setting $N = e^{-1}(1)$ we obtain an extension $N \xhookrightarrow{k} G \xrightarrow{e} H$. The above condition is precisely the requirement this is a Schreier extension of monoids!

Correspondence between oplax functors and pre-opfibrations

Now suppose we have a pre-opfibration $F\colon \mathcal{C} \to \mathcal{D}$.

For each object D in $\mathcal{D}$ we consider its 'fibre' category. This is the subcategory of $\mathcal{C}$ consisting of the objects C for which $F(C) = D$ and the morphisms that are mapped by F to the identity on D.

This forms the object part of a normal oplax functor $L_F\colon \mathcal{D} \to \mathbf{Cat}$.

These two constructions are inverses (up to isomorphism).

Indeed, there is an equivalence of 2-categories between the normal oplax functors from $\mathcal{D}$ to $\mathbf{Cat}$ and the pre-opfibrations into $\mathcal{D}$.

Classifying Schreier extensions with oplax functors

We know that pre-opfibrations into $\mathcal{B}\mathrm{H}$ correspond to normal oplax functors from $\mathcal{B}\mathrm{H}$ into $\mathbf{Cat}$. Let's apply this to Schreier extensions.

Of course, for monoid extensions the fibre category $\mathrm{L}_e(*)$ is the one-object category $\mathcal{B}\mathrm{N}$. Thus, Schreier extensions with cokernel H correspond to normal oplax functors from $\mathcal{B}\mathrm{H}$ into $\mathbf{Mon}$. Moreover, such a normal oplax functor sends the single object of $\mathcal{B}\mathrm{H}$ to the kernel N of the extension.

Now we can try to unravel the definition of a normal oplax functor to obtain an explicit characterisation of Schreier extensions.

Relation to the usual characterisation: the data

A normal oplax functor $L\colon \mathcal{B}H \to \mathbf{Mon}$ consists of

- a function $L\colon \{*\} \to \mathbf{Mon}_0$ between the classes of objects,
- a function $L_{*,*}\colon H \to \mathbf{Mon}(L(*), L(*))_0$,
- a 2-morphism $\iota^L_*\colon L(1_*) \to 1_{L(*)}$ which is an identity in $\mathbf{Mon}$,
- a 2-morphism $\gamma^L_{h,h'}\colon L(hh') \to L(h) \circ L(h')$ in $\mathbf{Mon}$ for each pair of elements $h, h' \in H$.

Relation to the usual characterisation: the data

A normal oplax functor $L \colon \mathcal{B}H \to \mathbf{Mon}$ consists of

- a monoid N,
- a function from H to objects of $\mathrm{Mon}(N, N)$,
- a requirement that $\alpha(1, -) = \mathrm{id}_N$,
- a monoid 2-morphism from $\alpha(hh', -)$ to $\alpha(h, -) \circ \alpha(h', -)$ for each pair of elements $h, h' \in H$.

Relation to the usual characterisation: the data

A normal oplax functor $L\colon \mathcal{B}H \to \mathbf{Mon}$ consists of

- a monoid N,
- a map $\alpha\colon H \times N \to N$ with $\alpha(h, 1) = 1$ and $\alpha(h, nn') = \alpha(h, n)\alpha(h, n')$,
- a requirement that $\alpha(1, n) = n$,
- a map $\chi\colon H \times H \to N$ such that $\chi(h, h')\alpha(hh', n) = \alpha(h, \alpha(h', n))\chi(h, h')$.

These must then satisfy the coherence conditions. We omit the details, but these lead to the additional requirements:

- $\chi(1, h) = 1 = \chi(h, 1)$,
- $\chi(x, y)\chi(xy, z) = \alpha(x, \chi(y, z))\chi(x, yz)$.

We have recovered the usual data for specifying Schreier extensions.

Relation to the usual characterisation: constructing extensions

Given such a pair (α, χ), we can now apply the Grothendieck construction to construct the associated extension.

For a normal oplax functor $L \colon \mathcal{B}H \to \mathbf{Mon}$, the category $\int L$ will have a single object and a morphism (h, n) for each $h \in H$ and $n \in \mathrm{Hom}(L(h)(*), *) = N$.

Multiplication is given by $(h, n) \cdot (h', n') = (hh', n \cdot L(h)(n') \cdot (\gamma^L_{h,h'})_*)$.

In terms terms of our data,

$$(h, n) \cdot (h', n') = (hh', n\alpha(h, n')\chi(h, h')).$$

Then the cokernel sends (h, n) to h and the kernel sends n to $(1, n)$.

This accords with the usual construction.

Further applications

Variants of this approach can be used to give a number of further characterisations related to extensions of monoids, including

- weakly Schreier extensions,
- (weakly) Schreier *split* extensions,
- morphisms of extensions or split extensions.

It seems likely that this approach could also be used to classify other classes of monoid extensions.