Simplicity of contracted inverse semigroup algebras

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Inverse semigroups

Definition

A semigroup S is called an **inverse semigroup** if for any $s \in S$, there exists a unique element $s^{-1} \in S$ for which

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The archetypal example

The set of partial one-to-one maps on a set A under composition and inverse: the symmetric inverse semigroup $\mathcal{I}_A$.

The polycylic monoid

Example

Fix a set $|X| > 1$ (alphabet). The **polycyclic monoid** $P(X)$ on X is

- an inverse semigroup with a zero 0 and an identity 1 generated by X,
- defined by relations

$$x^{-1}y = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$.

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Elements: $\alpha\beta^{-1}$ with $\alpha, \beta \in X^*$, and 0
Idempotents: $\alpha\alpha^{-1}$ with $\alpha \in X^*$, and 0.

Semigroup algebras

Let S be a semigroup, K a field.

The **semigroup algebra** KS consists of finite linear combinations of elements of S over K. It is

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Question: Suppose S in an inverse semigroup. When is the ring KS simple?

A simple answer

Let S be a nontrivial inverse semigroup, K a field.

Then

$$KS \to K,\ \sum_{s \in S} a_s s \mapsto \sum_{s \in S} a_s$$

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$$\Longrightarrow KS \text{ is not simple.}$$

The contracted inverse semigroup algebra

Let S be an inverse semigroup **with a zero** z, K a field.

Let $K_0S = KS/(z)$ – this effectively identifies z with 0. We call it the **contracted inverse semigroup algebra**.

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This can be simple, e.g. if S is the Brandt semigroup B_n, then $K_0S \cong M_n(K)$.

Notice: a congruence $\equiv$ on S induces a surjective homomorphism $K_0S \to K_0[S/\equiv]$, so

$$K_0S \text{ is simple} \Longrightarrow S \text{ is congruence-free.}$$

The contracted inverse semigroup algebra

But

$$K_0S \text{ is simple } \not\Leftarrow S \text{ is congruence-free.}$$

$P(x, y)$ is congruence-free, but $K_0[P(x, y)]$ is not:

$$I = (xx^{-1} + yy^{-1} - 1)$$

is a proper ideal, in fact $K_0[P(x, y)]/I$ is the Leavitt algebra $L_K(1, 2)$.

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Problem (Munn, 1978)

Characterize those congruence-free inverse semigroups with zero which have a simple contracted algebra.

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- 0-simple: it has no proper, nonzero ideals,
- fundamental: it has no nontrivial idempotent-separating congruences,
- and $E(S)$ is 0-disjunctive: for all idempotents $0 \neq f < e$, there exists $0 \neq f' < e$ such that $ff' = 0$.

Tight inverse semigroups

Let S be an inverse monoid with zero 0, E its semilattice of idempotents, $e \in E$.

$F \subseteq (e)^{\downarrow}$ **covers** e if for all $h \in E$

$$hf = 0 \text{ for all } f \in F \Longrightarrow he = 0.$$

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Note: E is 0-disjunctive $\Longleftrightarrow$ if F is a 1-element cover then $e \in F$.

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Nontrivial finite covers give rise to an ideal of K_0S called the **tight ideal**.

$$K_0S \text{ is simple} \implies S \text{ is tight}$$

Previous results

S is called **Hausdorff** if for each $s, t \in S$, the set $(s)^{\downarrow} \cap (t)^{\downarrow}$ has finitely many maximal elements.

Remark

E^*-unitary $\Longrightarrow$ Hausdorff

Theorem (Steinberg, 2014)

A Hausdorff inverse semigroup S with a zero has a simple contracted algebra over any field K
$\Longleftrightarrow$ *S is congruence-free and tight,*
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$\Longleftrightarrow$ *S is congruence-free and tight,*
$\Longleftrightarrow$ *S is 0-simple, fundamental and tight,*

In the general case, congruence-free and tight are necessary conditions, but it was not known if they were sufficient.

The full characterization

Let

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Theorem (Steinberg, Sz.)

1. *$K_0 S$ is simple*
   $\iff$ *S is congruence free and $I = \{0\}$,*
   $\iff$ *S is 0-simple, fundamental and $I = \{0\}$.*

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1. *$K_0 S$ is simple*
   $\Longleftrightarrow$ S is congruence free and $I = \{0\}$,
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Remark: Simplicity depends on the field K.

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$K_0S \cong K\mathcal{G}(S)$, the Steinberg algebra of the ample groupoid $\mathcal{G}(S)$.

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Theorem (Clark, Exel, Pardo, Sims and Starling (2018))

If $\mathcal{G}$ is a second-countable ample groupoid with $\mathcal{G}^{(0)}$ Hausdorff, then $K\mathcal{G}$ is simple $\iff$ $\mathcal{G}$ is minimal, effective, and $K\mathcal{G}$ has no nonzero singular functions.

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They ask: is there a minimal, effective $\mathcal{G}$ where $\mathbb{C}\mathcal{G}$ has nonzero singular functions?

A class of congruence-free inverse semigroups

Fix an alphabet X, and consider the polycyclic monoid $P(X)$.

Recall: $P(X)$ is congruence free, and tight whenever X is infinite.
We build congruence-free [tight] inverse semigroups from polycyclic monoids and a groups.

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$P(X)$ can be represented by partial one-to-one (left) maps on X^*:

$$\alpha\beta^{-1}\colon \beta X^* \to \alpha X^*$$
$$\beta w \mapsto \alpha w$$

So $P(X) \leq \mathcal{I}_{X^*}$.

Self-similar groups

Let $G \leq S_{X^*}$ such that $g(\cdot)$ is length preserving.
We call the G a **self-similar group** if for every $g \in G$, $u \in X^*$ there exists $g|_u \in G$ such that for all $w \in X^*$

$$g(uw) = g(u)g|_u(w).$$

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An easy example: $G = C_2 = \{1, a\}$, $X = \{x, y\}$, a acts by switching the first letter.

Inverse semigroups from self-similar actions

Let $G \leq S_{X^*}$ a self-similar group, and let

$$S = \langle G, P_X \rangle \leq \mathcal{I}_{X^*}.$$

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Fact: S is always congruence-free, and tight whenever X is infinite.

Nonzero elements of S are of the form $\alpha g \beta^{-1}$, where $\alpha, \beta \in X^* (\subseteq P_X)$, $g \in G$.

A congruence-free, tight inverse semigroup S with $I \neq \{0\}$

Let $A = \{x, y\} \dot{\bigcup} Z$ with Z infinite, $G = C_2 = \{1, a\}$, and consider the self-similar action

$$a(xw) = yw, a(yw) = xw, a(zw) = zw$$

for all $z \in Z$, $w \in X^*$.

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Let $S = \langle G, P_A \rangle$.

Recall:

$$I = \{A \in K_0 S : \forall\, e \in E \backslash \{0\}\ \exists\, f \leq e, f \neq 0 \text{ such that } Af = 0\}.$$

Claim:

$$A = (1 - xx^{-1} - yy^{-1}) - (a - axx^{-1} - ayy^{-1}) \in I.$$

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$Ax = Ay = Az = 0 \Longrightarrow$ for all $f \in E \setminus \{1\}$ we have $Af = 0$, so certainly for all $e \in E \setminus \{0\}$ there exists $f \leq e, f \neq 0$ such that $Af = 0$

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S is congruence-free an tight, but $K_0 S$ is not simple (for any K).

$\mathcal{G}(S)$ is minimal, effective, but $K\mathcal{G}(S)$ has a nonzero ideal of singular functions for any K.

Thanks!