The diameter of endomorphism monoids II: Oligomorphicity

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Right congruences

- Global aim: Study finitary conditions of semigroups that naturally arise from one sided congruences.

Definition

A **right congruence** on a semigroup S is an equivalence relation ρ such that for every $a, b, c \in S$,

$$a \,\rho\, b \Rightarrow ac \,\rho\, bc.$$

- If $U \subseteq S \times S$, then the **right congruence generated by** U, denoted $\langle U \rangle$, is the smallest right congruence containing U.

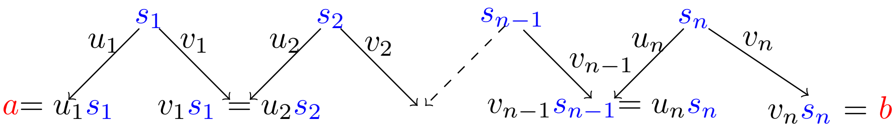
Generating right congruences

Lemma (Kilp, Knauer, Mikhalev, 2000)

Let $U \subseteq S \times S$*. Then* $a \,\langle U \rangle\, b$ *if and only if either* $a = b$ *or there exists a* U*-path from* a *to* b*, that is,*

$$a = u_1 s_1, v_1 s_1 = u_2 s_2, \ldots, v_n s_n = b$$

where $(u_i, v_i) \in U \cup U^{-1}$ *and* $s_i \in S^1$*.*



Diameter

Corollary

Let S be a semigroup. Then ω_r^S is f.g. if and only if there exists a finite subset U of $S \times S$ such that for any $a, b \in S$, we have $a = b$ or there exists a U-path from a to b, that is,

$$a = u_1 s_1, v_1 s_1 = u_2 s_2, \ldots, v_n s_n = b$$

where $(u_i, v_i) \in U$ and $s_i \in S^1$.

Definition

Let S be a semigroup in which ω_r^S is f.g.

- If $\omega_r^S = \langle U \rangle$, then define $D_r(S; U) =$sup{length of the smallest U-path from a to b : $a, b \in S$}.
- The **right diameter of** S is then $D_r(S) =$min$\{D_r(S; U) : \omega_r^S = \langle U \rangle, |U| < \infty\}$.
- If $D_r(S)$ is finite, then S is called **right pseudo-finite**.

Pseudo-finite semigroups

- (Left) pseudo-finite semigroups were first studied by Dales and White in 2017 with regards to Banach algebras.
- Boring property for groups: pseudo-finite groups are finite.
- Kobyashi (2007): ω_r^M is f.g. if and only if M is of type right-FP1.
- I first joined the project for "Semigroups with finitely generated universal left congruence" (2019, Dandan, G, Q-G, Zenab).
- Clear picture for key classes including inverse semigroups, completely regular, and Rees matrix.
- Far more complex then first thought: there exists pseudo-finite regular semigroups without a completely simple minimal ideal "On minimal ideals in pseudo-finite semigroups" (2022, G,M, Q-G, R).
- The diameters of transformation semigroups calculated in "On the diameter of semigroups of transformations and partitions" (2023, E,G,M,Q-G,R).
- The exact diameters of the endomorphism monoid of a chain calculated in "Diameters of endomorphism monoids of chains" (2024, E,G, M,Q-G). Extended the diameter 1 case of Gallagher and N (2005).

A curious occurrence

Each semigroup we considered had left and right diameter at most 4 (if pseudo-finite).

Semigroup S	ω_S^r f.g.?	$D_r(S)$	ω_S^l f.g.?	$D_l(S)$
$\mathcal{B}_X$	Yes	1	Yes	1
$\mathcal{PT}_X$	Yes	1	Yes	1
$\mathcal{I}_X$	Yes	2	Yes	2
$\mathcal{T}_X$	Yes	1	Yes	1
$\mathcal{F}_X$	Yes	1	No	n.a.
$\mathcal{I}nj_X$	Yes	4	No	n.a.
$\mathcal{BL}_{X,q}, q < \lvert X\rvert$	No	n.a.	No	n.a.
$\mathcal{BL}_X$	Yes	3	No	n.a.
$\mathcal{S}urj_X$	No	n.a.	Yes	4
$\mathcal{DBL}_{X,q}, q < \lvert X\rvert$	No	n.a.	No	n.a.
$\mathcal{DBL}_X$	No	n.a.	Yes	2
$\mathcal{T}_X \setminus \mathcal{I}nj_X$	No	n.a.	Yes	2
$\mathcal{T}_X \setminus \mathcal{S}urj_X$	Yes	2	No	n.a.
$\mathcal{P}_X$	Yes	1	Yes	1
$\mathcal{PB}_X$	Yes	1	Yes	1
$\mathrm{End}(C)$ (C chain)	Yes	2 or 3	Yes	2

A curious occurrence

Problem

For each $n \in \mathbb{N}$ does there exist a semigroup of right diameter n?

Answer: Almost definitely.

Problem

For each $n \in \mathbb{N}$ construct a "nice" semigroup of right diameter n.

Problem (Meta)

Which "characteristics" of a semigroup determines its right diameter?

First order structures

- We restrict to transformation monoids which have an inherited global structure: Endomorphism monoids.

Definition

A **(first order) structure** $\mathbb{M} = (M; \mathfrak{K})$ is a set M together with a collection $\mathfrak{K}$ of **basic** relations and functions defined on M.

- A semigroup is considered as a set together with a binary (associative) operation.
- Both partially ordered sets (posets) and graphs can be considered as sets together with a single binary relation.
- A semilattice can also be considered as the structure $(Y; \wedge, \leq)$ where $a \leq b$ if and only if $a \wedge b = a$.

Endomorphisms

Definition

Let $\mathbb{M} = (M; \mathfrak{K})$ be a structure. Then a map $\theta \colon M \to M$ is an **endomorphism** of $\mathbb{M}$ if it preserves each function and relation from $\mathfrak{K}$, that is, for each function $f \in \mathfrak{K}$, relation $R \in \mathfrak{K}$, and $a_1, \ldots, a_n \in M$,

$$((a_1, \ldots, a_n)f)\theta = (a_1\theta, \ldots, a_n\theta)f,$$
$$(a_1, \ldots, a_n) \in R \Rightarrow (a_1\theta, \ldots, a_n\theta) \in R.$$

The set of all endomorphisms of $\mathbb{M}$ is denoted $\mathrm{End}(\mathbb{M})$, and forms a submonoid of $\mathcal{T}_M$.

E.g. If $\mathcal{Y} = (Y; \wedge, \leq)$ is a semilattice then $\theta \in \mathrm{End}(\mathcal{Y})$ if

$$(x \wedge y)\theta = x\theta \wedge y\theta \text{ and } x \leq y \Rightarrow x\theta \leq y\theta.$$

Note: $\mathrm{End}(Y; \wedge) = \mathrm{End}(Y; \wedge, \leq) \subseteq \mathrm{End}(Y; \leq)$.

Our plan of attack

- The method: Properties of $\mathrm{End}(\mathbb{M})$ often depend solely on those of the underlying structure $\mathbb{M}$, which is easier to work with.

Problem

Construct a structure $\mathbb{M}_n$ *such that* $\mathrm{End}(\mathbb{M}_n)$ *has right diameter* n.

- We study relational structures in which it is easier to pass between elements of the structure and endomorphisms: Reflexive structures.

Reflexive structures

- Given an n-ary relation R of a set M, we define an R-**loop** to be an element $x \in M$ with $(x, x, \ldots, x) \in R$.
- We call R **reflexive** if each $x \in M$ is an R-loop.
- A relational structure $\mathbb{M}$ is called **reflexive** if each of its basic relations are reflexive.

Lemma

If $\mathbb{M}$ is reflexive then the constant map $c_x \colon M \to M$ $(a \mapsto x)$ is an endomorphism of $\mathbb{M}$ for each $x \in A$. Moreover, $\mathcal{C}_M = \{c_x : x \in M\}$ is the minimum ideal of $\mathrm{End}(\mathbb{M})$.

Example

Posets $(P; \leq)$, chains, prosets, looped graphs, and bands(!) are all reflexive.

- Right pseudo-finite monoids with a completely simple minimal ideal were classified in "On minimal ideals in pseudo-finite semigroups". Exact diameter not considered.

The minimal right zero ideal

- Let $S = \mathrm{End}(\mathbb{M})$ for some reflexive structure $\mathbb{M}$ and fix some $x \in M$.
- Recall $\mathcal{C} = \mathcal{C}_M$ is a (right zero) minimum ideal of S.
- In particular, if $\theta = \alpha\delta$ in S then

$$c_x\theta = c_{x\theta} = c_{x\alpha}\delta = c_x\alpha\delta.$$

- Hence $D_r^S(\mathcal{C}) \leq D_r(S)$, where $D_r^S(\mathcal{C})$ denotes the right diameter of $\mathcal{C}$ corresponding to the right congruence $\omega_r^S|_{\mathcal{C}\times\mathcal{C}} \cup \Delta_S$ of S.
- We call $D_r^S(\mathcal{C})$ the **constant right diameter** of $\mathbb{M}$.
- Paths are therefore of the form

$$c_a = c_{u_1}\delta_1, c_{v_1}\delta_1 = c_{u_2}\delta_2, \ldots, c_{v_n}\delta_n = c_b$$

for $\delta_i \in S$ and $(c_{u_i}, c_{v_i}) \in U$.

Constant right diameter

Recall: Paths witnessing $D_r^S(\mathcal{C};U)=n$ are of the form

$$c_a = c_{u_1}\delta_1, c_{v_1}\delta_1 = c_{u_2}\delta_2, \ldots, c_{v_n}\delta_n = c_b$$

for $\delta_i \in S$ and $(c_{u_i}, c_{v_i}) \in U$.

Lemma (EGMQ-GR)

Let $S = \mathrm{End}(\mathbb{M})$ for some reflexive structure $\mathbb{M}$ and let $\mathcal{C} = \mathcal{C}_A$. Then

$$D_r^S(\mathcal{C}) \leq D_r(S) \leq D_r^S(\mathcal{C}) + 2.$$

Proof.

- $D_r^S(\mathcal{C}) \leq D_r(S)$ is clear.
- Fix $x \in M$ and let $D_r^S(\mathcal{C};U) = n$.
- For any $\theta, \psi \in S$ we get a $U \cup (U,1) \cup (1,U)$-path

$$\theta = 1\theta, c_x\theta = c_{x\theta} \to_n c_{x\psi} = c_x\psi, 1\psi = \psi$$

  of length $n+2$.

Swapping to orbits

- Let $S = \mathrm{End}(\mathbb{M})$ for some reflexive $\mathbb{M}$ and let $\mathcal{C} = \mathcal{C}_M$.
- Recall: paths witnessing $D_r^S(\mathcal{C}; U)$ are chains of equalities $c_u\delta = c_v\delta'$ where $(c_u, c_v) \in U$ and $\delta, \delta' \in S$.
- But $c_u\delta = c_v\delta'$ if and only if $u\delta = v\delta'$.
- Hence any U-path from c_x to c_y is equivalent to a path (in $\mathbb{M}$)

$$x = u_1\delta_1, v_1\delta_1 = u_2\delta_2, \ldots, v_n\delta_n = y,$$

where $(u_i, v_i) \in \{(u, v) \in M^2 \colon (c_u, c_v) \in U \cup U^{-1}\}$.

Lemma

A reflexive structure $\mathbb{M}$ has constant right diameter 1 if and only if there exists a finite collection $(u_1, v_1), \ldots, (u_n, v_n) \in M \times M$ such that for each $x, y \in M$ there exists $1 \leq i \leq n$ and $\delta \in \mathrm{End}(\mathbb{M})$ with $(x, y) = (u_i, v_i)\delta$. In which case $1 \leq D_r(\mathrm{End}(\mathbb{M})) \leq 3$.

Orbits and oligomorphicity

- Let $S \leq \mathcal{T}_X$ be a transformation monoid.
- S acts on the right of X^n by $(x_1, \ldots, x_n)\theta = (x_1\theta, \ldots, x_n\theta)$.
- We call $(x_1, \ldots, x_n)S$ an n-**orbit**.
- So $\mathbb{M}$ has constant-diameter 1 if and only if $M \times M$ can be written as a finite union of 2-orbits (of the action of $\mathrm{End}(\mathbb{M})$ on $M \times M$).

Definition

S is called n-**oligomorphic** if it has only finitely many n-orbits. If S is n-oligomorphic for each n then S is **oligomorphic**.

- Studied extensively by P. Cameron, M. Pech, J. Nešetřil, D. Mašulović etc.
- Oligomorphic groups are central in a number of model theoretic conceps e.g. ω-categoricity, quantifier elimination, and homogeneity.
- Oligomorphic transformation monoids give an endomorphism-dual to these concepts.

Corollary (EGMQ-GR)

Let $\mathbb{M}$ be a 2-oligomorphic reflexive structure. Then $D_r^S(\mathcal{C}_M) = 1$, so that $D_r(\mathrm{End}(\mathbb{M})) \leq 3$.

The converse

2-oligomorphicity is a stricter condition than having constant diameter 1.

Fact

Let $\mathbb{G} = (V; E)$ be a looped graph (so $(x, x) \in E$ for each $x \in V$). Then

$$(x, y)\,\mathrm{End}(\mathbb{G}) = \{(a, b) \in V^2 \colon d_E(a, b) \leq d_E(x, y)\}.$$

Corollary

A looped graph has constant diameter 1 if and only if it is either unconnected, or connected of bounded (graph) diameter.

Since the 2-orbits correspond to the distances of elements in the graph:

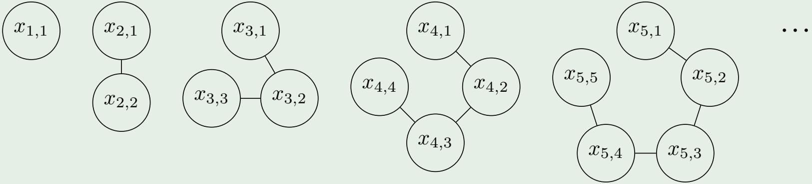
Corollary

A looped graph is 2-oligomorphic if and only if there exists $n \in \mathbb{N}$ such that each connected component has (graph) diameter at most n.

The converse

Example

The looped graph $\mathbb{G} = L_1 \cup L_2 \cup L_3 \cup \cdots$ has constant diameter 1 but isn't 2-oligomorphic, where L_n is the line graph of length n.



- Distinct 2-orbits: $(x_{1,1}, x_{2,1}), (x_{1,1}, x_{1,1}), (x_{2,1}, x_{2,2}), (x_{3,1}, x_{3,3}), \ldots$.
- $(x_{1,1}, x_{2,1})\,\mathrm{End}(\mathbb{G}) = G \times G$.

2-oligomorphicity and posets

Theorem (Mašulović, 2007)

The endomorphism of a tree (and hence a chain) is oligomorphic. Not all posets have oligomorphic endomorphism monoids.

Lemma (EGMQ-GR)

The endomorphism monoid of a poset is 2-oligomorphic, with at most eight 2-orbits.

Corollary (EGMQ-GR)

If $\mathbb{P}$ is a poset then $D_r(\mathrm{End}(\mathbb{P})) \leq 3$.

Higher constant right diameters

- Recall: Any U-path from c_x to c_y is equivalent to a $\{(u,v) \in M^2 \colon (c_u, c_v) \in U\}$-path

$$x = u_1\delta_1, v_1\delta_1 = u_2\delta_2, \ldots, v_n\delta_n = y.$$

- A transformation semigroup $S \leq \mathcal{T}_X$ is called (properly) k-**weakly** 2-**oligomorphic** if ($k \in \mathbb{N}$ is minimal such that) there exists a finite set $U \subseteq X \times X$ such that for each $(x, y) \in X \times X$, there is a path $x = z_1, \ldots, z_n = y$ with

$$(z_i, z_{i+1}) \in US = \bigcup_{(u,v) \in U} (u, v)S.$$

Corollary

Let $\mathbb{M}$ be a reflexive structure. Then $D_r^S(\mathcal{C}_M) = n$ if and only if $\mathbb{M}$ is properly n-weakly 2-oligomorphic. In which case $n \leq D_r(\mathrm{End}(\mathbb{M})) \leq n + 2.$

The (big!) missing piece

Problem

For each $n \geq 2$ construct a structure $\mathbb{M}_n$ which is properly n-weakly 2-oligomorphic. Manipulate the construction so that $D_r(\mathbb{M}_n) = n$.

- No examples found yet of properly 2-weakly 2-oligomorphic structures!

Lemma (EGMQ-GR)

Let $\mathbb{G}$ be a looped graph and $S = \mathrm{End}(\mathbb{G})$. Then ω_r^S is f.g., and t.f.a.e:

(1) *S is right pseudo-finite;*

(2) *$\mathbb{G}$ is either unconnected, or connected of bounded diameter.*

(2) *Let $\mathbb{G}$ is 1-weakly 2-oligomorphic.*

In which case $D_r(S) \leq 3$.

Problem

Mirroring the achievements of D. Mašulović and M. Pech, find links between weak oligomorphicity and forms of ω-categoricity, homogeneity, and quantifier elimination.

Further usage and ideas

- Similar concept of weak 2-oligomorphic groups (with respect to acting on some set) is a key missing piece to solving the diameter of endomorphism monoids of independence algebras.
- We can also consider structures which are "close" to being reflexive, i.e. have infinitely many loops. The outlined results easily generalize. Key: Includes all graphs of interest.

Thank you!