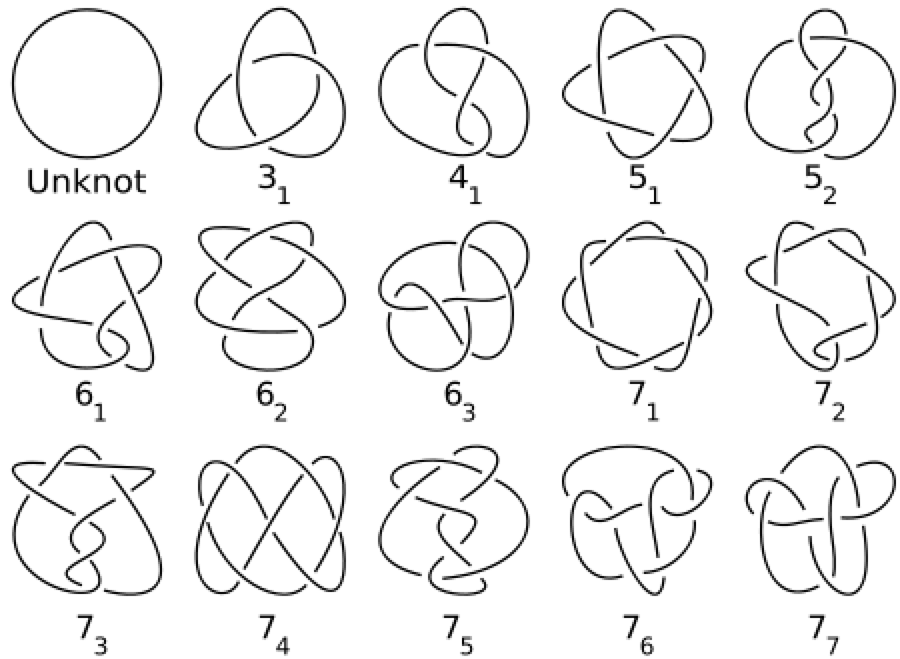
Knot semigroups:
a new algebraic approach to knots and braids

Alexei Vernitski
University of Essex

On the picture: the logo of a shopping centre in Colchester presents what in this talk we shall treat as the generators of the semigroup of the trefoil knot

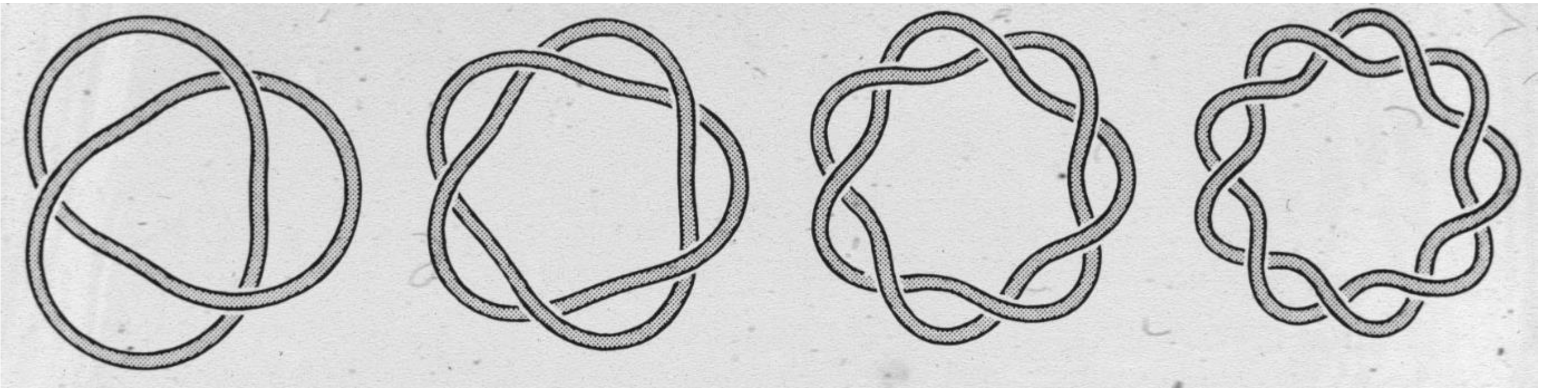
Small knots



(The picture is from the Wikipedia)

Torus knots $T(2,n)$

- The knots shown below are $T(2,3), T(2,5), T(2,7), T(2,9)$

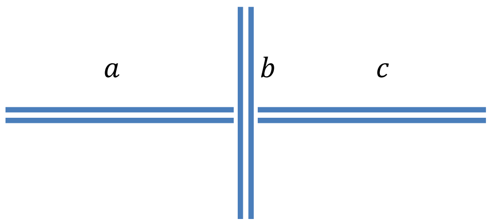


(The picture is from http://www.mit.edu/~kardar/research/seminars/knots/stasiak/stasiak.html)

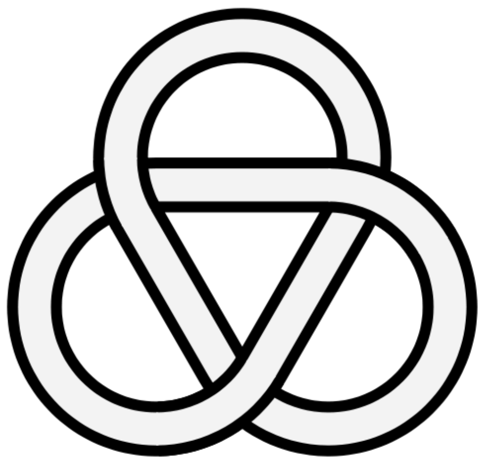
Arc labelling

- An arc is a part of the knot from one undercrossing to another undecrossing
- As one useful construction, we shall need to label all arcs (by pairwise distinct labels) in a way satisfying one special condition:

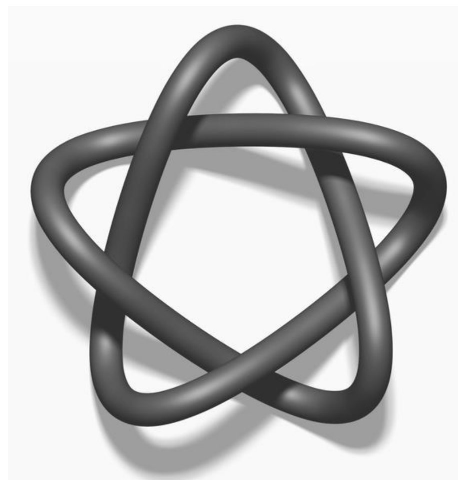
At each crossing, $a - b = b - c \pmod{n}$, where n is a fixed number



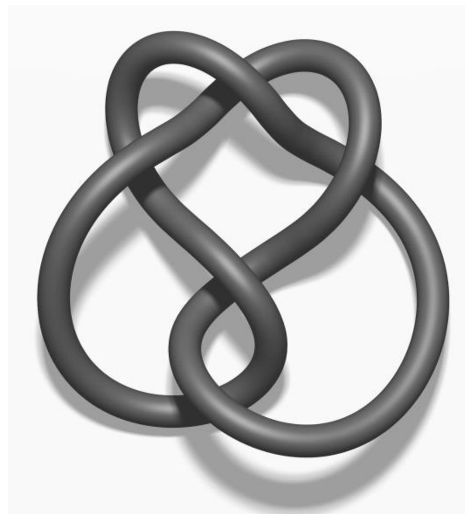
Knot 1



Knot 2

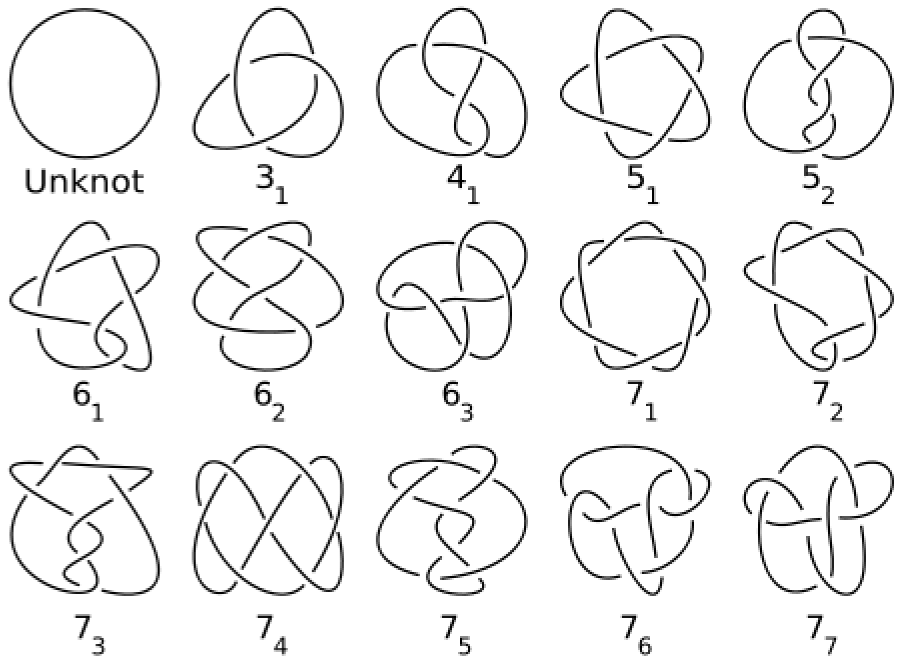


Knot 3



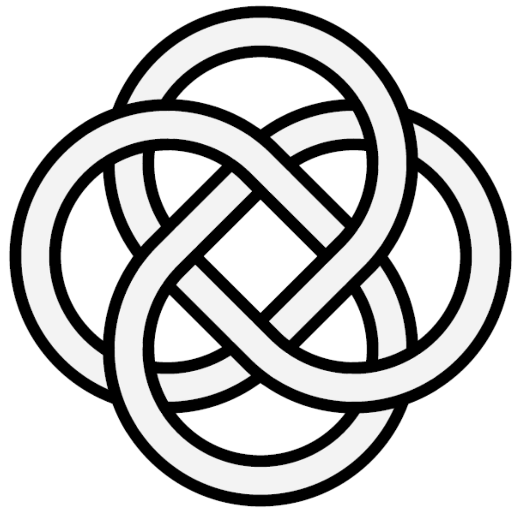
Does it not make sense to think of the number n, which we may call the *modulus* of the knot, as a special characteristic of the knot?

Small knots



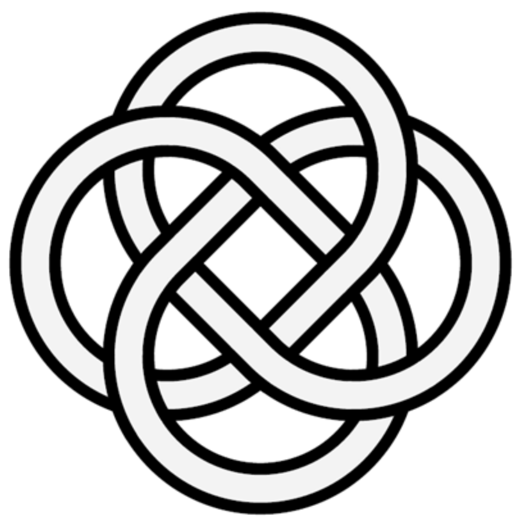
All small knots have such labelling

What about larger knots?



What is this arc labelling like? – 1

- *Fox colouring* is, basically, the same construction, except that it does not assume that labels are pairwise distinct



What is this arc labelling like? – 2

- Quandles; also known as wracks (or racks) and distributive groupoids
- Axioms are:
  - $x * x = x$
  - $\forall y, z \exists x : x * y = z$
  - $(x * y) * z = (x * z) * (y * z)$

Semigroup

- A semigroup is a system of rules defining which words are equal to each other
- A semigroup is defined by its *generators* (letters) and *relations* (equalities)
- For example:
  - $\langle a, b | ab = ba \rangle$
  - $\langle a, b, c | ab = bc = ca, ba = cb = ac \rangle$

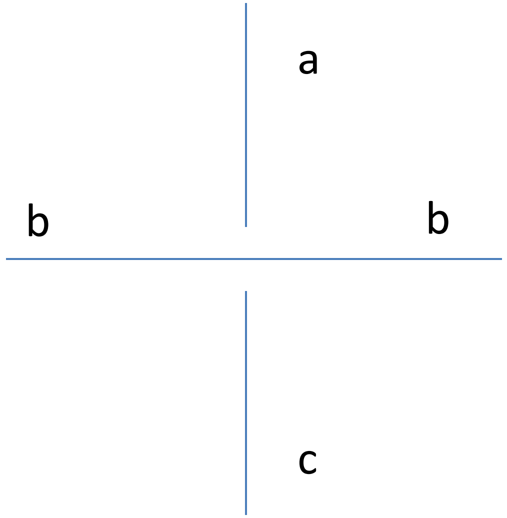
Working with semigroups

- $\langle a, b | ab = ba \rangle$
  - Prove that $abb = bba$
- $\langle a, b, c | ab = bc = ca, ba = cb = ac \rangle$
  - Prove that $abc = cba$

- Semigroups with relations of the form $xy = yx$ are overused
- Semigroups with relations of the form $xy = yz$ are underused
- Only group theorists use such relations a lot, often writing them as $x^y = z$.

Knot semigroup

- Arcs are considered as generators
- Each relation is an equality of two products found at a crossing: read the letters in the opposite angles, clockwise in one of them and anticlockwise in the other

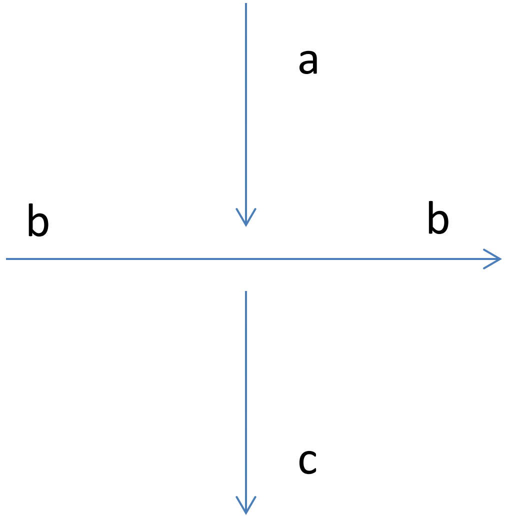


$$ab = bc$$

$$ba = cb$$

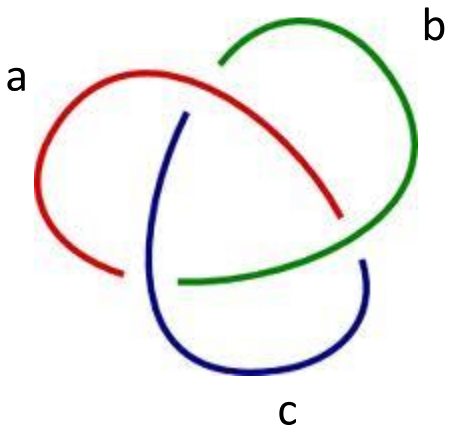
For comparison – the standard knot group (Wirtinger presentation)

- Arcs (treated as directed arcs with a consistent orientation throughout) are considered as generators
- Each relation is 'read around' a crossing: move anti-clockwise and read out the letters on arcs coming from the right (or coming from the left, inverted)



$$abc^{-1}b^{-1} = 1$$

The semigroup of the trefoil knot

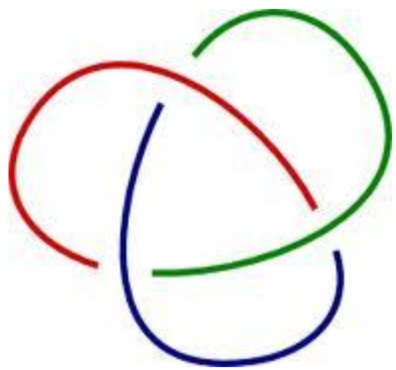


- Relations:
  - $ab = ca, ba = ac$
  - $ba = cb, ab = bc$
  - $ca = bc, ac = cb$

Cancellations: a useful property

- Examples with positive integers:
  - The equation $x + y = z$ cannot be solved for x
  - But the equation $x + y = z + y$ can be solved for x
- In knot semigroups,
  - if $uv = wv$ then $u = w$
  - if $vu = vw$ then $u = w$
- For example, in the semigroup of the trefoil knot $aa = bb = cc$, but only if you allow using cancellation

The semigroup of the trefoil knot

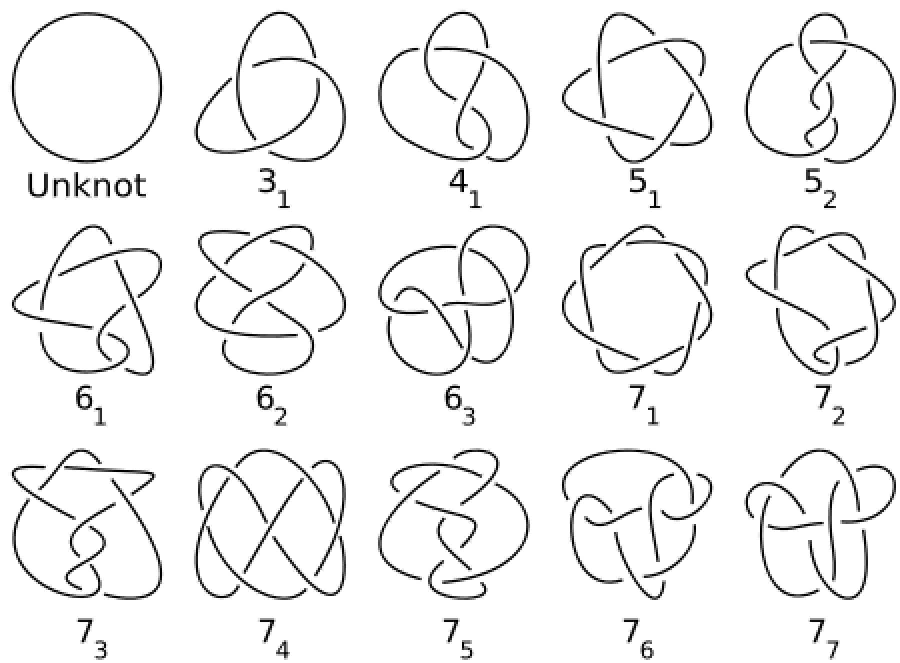


- Words of each given length split in exactly 3 classes, with words within each class being equal to each other.

Connection with Fox colouring

- Label the three arcs $0, 1, 2$, producing a pairwise distinct Fox colouring
  - Say, $a = 0, b = 1, c = 2$
- Two words xy and zt are equal in the semigroup of the trefoil knot if and only if $x - y = z - t \pmod 3$, that is:
  - $aa = bb = cc$
  - $ab = bc = ca$
  - $ba = cb = ac$

Small knots



All small knots have this property:
$xy = zt$ if and only if $x - y = z - t \pmod n$

Connection with Fox colouring

- The number n from the previous slide is equal to the number of classes of equal words of each (sufficiently large) length

Torus knots

- For torus knots $T(2, n)$ which wind only twice around the central void of the torus, the knot semigroup is especially simple.
- Each such knot has a pairwise distinct Fox colouring modulo n, and for each length of words the number of classes of equal words is n.

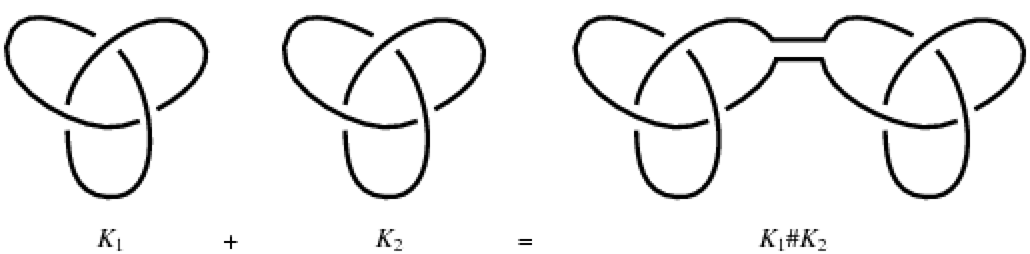
Useful software: Python

- Most equalities were found by code I wrote in Python
- Python is an extremely convenient language for word manipulation – a typical fragment of code is below

```
# remove common prefixes
while u[0] == v[0]:
    u = u[1:]
    v = v[1:]
# remove common suffixes
while u[-1] == v[-1]:
    u = u[:-1]
    v = v[:-1]
```

Future plans

- A number of semigroup-theoretical questions, for example:
  - Are knot semigroups group-embeddable?
  - What is the semigroup of the sum of knots? (in the original ‘semigroup of knots’)



The diagram is from http://mathworld.wolfram.com/KnotSum.html

Future plans

- A number of geometry-inspired questions, for example:
  - How do knot semigroups of braids characterise braids?
  - What is the semigroup of $T(3,4)$ ('the' nasty knot with 8 crossings)?