Axiomatisability of free, projective and flat S-acts

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For Mati and Ulrich on the occasion of their 65th birthdays

ABSTRACT. We survey the known results characterising those monoids S such that the classes of free, projective or (weakly, strongly) flat S-acts are first order axiomatisable. The conditions on the monoid S that arise are finitary, and intricately related. We examine their inter-connections and illustrate the independence of certain pairs.

1. Introduction

The so-called *homological classification of monoids*, that is, characterising monoids by properties of their acts, is now almost 40 years old. Initiated by Skornjakov [**17, 18**] in the late 1960's, the project grew apace in subsequent decades, as a glance at the comprehensive monograph *Monoids, Acts and Categories* by Kilp, Knauer and Mikhalev [**14**] makes evident. Of the several distinct strands of study that emerged, perhaps the most notable is the consideration of acts that are free or that satisfy a weaker property such as projectivity or flatness. The structure of projective acts was determined in 1972 by Knauer [**15**]. There are several candidates for the notion of flatness for acts, all of which coincide in the analogous situation of modules over a ring. In a crucial paper [**19**] of 1970, Strenström considered acts that are directed colimits of finitely generated free acts; such acts are now called *strongly flat*. The notion of tensor product of acts was also introduced in [**19**] and elucidated by Kilp in [**11**]. Stenström shows that for a monoid S and a left S-act B, the functor $-\otimes B$ preserves pullbacks and equalisers if and only if B is strongly flat. Rather later, Bulman-Fleming [**1**] improved upon this result by demonstrating that if $-\otimes B$ preserves pullbacks, then perforce it preserves equalisers. Weakening Stenström's concept, Kilp defined an act to be *flat* if $-\otimes B$ preserves monomorphisms [**11**] and *weakly flat* if $-\otimes B$ preserves embeddings of right ideals into S [**12**].

We thus have five classes of acts: for a monoid S we will denote the classes of free, projective, strongly flat, flat and weakly flat *left* S-acts by $\mathcal{F}r, \mathcal{P}, \mathcal{SF}, \mathcal{F}$ and $\mathcal{WF}$, respectively. It is known that

$$\mathcal{F}r \subseteq \mathcal{P} \subseteq \mathcal{SF} \subseteq \mathcal{F} \subseteq \mathcal{WF}$$

where in general all of the inclusions are strict [**14**]. In this article we consider the question: for which monoids S are these classes axiomatisable?

Any class of algebras $\mathcal{A}$ of a given fixed type has an associated first order language L. A subclass $\mathcal{B}$ of $\mathcal{A}$ is *axiomatisable* (or *elementary*) if there is a set of sentences Π

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of L such that for any member C of $\mathcal{A}$, C lies in $\mathcal{B}$ if and only if all sentences of Π are true in C, that is, C is a *model* of Π. We say in this case that Π *axiomatises* $\mathcal{B}$. It is perhaps helpful here to demonstrate with an example pertinent to this paper.

Throughout this article, S will denote a monoid with set of idempotents $E = E(S)$. We recall that a left S-act A is a set A together with a map $S \times A \to A$, denoted by $(s, a) \mapsto sa$, such that for any $a \in A$ and $s, t \in S$,

$$1a = a \text{ and } (st)a = s(ta);$$

right S-acts are defined dually. It is convenient to allow the empty set to be a left and a right S-act. Left (right) ideals of S are, of course, examples of left (right) S-acts; again, we allow the empty set to be regarded as a left (right) ideal. We denote by S-*Act* the class of all left S-acts. The first order language L_S associated with S-*Act* has no constant or relational symbols, other than $=$, and merely a unary function symbol λ_s for each $s \in S$. An S-act A becomes an L_S-*structure* (or an *interpretation* of L_S), if we interpret each λ_s by the function $x \mapsto sx$. The class S-*Act* is axiomatised (amongst all L_S-structures) by the set of sentences

$$\Pi = \big\{(\forall x)(\lambda_1(x) = x)\big\} \cup \big\{\mu_{s,t} : s, t \in S\big\}$$

where $\mu_{s,t}$ is the sentence

$$(\forall x)\big(\lambda_{st}(x) = \lambda_s(\lambda_t(x))\big).$$

Some special classes of left S-acts are axiomatisable for *any* monoid S. An S-act A is *torsion free* if for any $a, b \in A$ and $s \in T$, where T is the set of left cancellable elements of S, from the equality $sa = sb$ we can deduce that $a = b$. Clearly the class $\mathcal{TF}r$ of torsion free left S-acts is axiomatised by $\Pi \cup \Sigma_{\mathcal{TF}r}$ where

$$\Sigma_{\mathcal{TF}r} = \big\{(\forall x)(\forall y)\big(\lambda_s(x) = \lambda_s(y) \to x = y\big) : s \in T\big\}.$$

Other natural classes of left S-acts are axiomatisable for some monoids and not for others: the classes $\mathcal{F}r, \mathcal{P}, \mathcal{SF}, \mathcal{F}$ and $\mathcal{WF}$ are all of this kind.

In a series of articles [**7, 20, 2, 8**], characterisations are given of those monoids S such that $\mathcal{F}r, \mathcal{P}, \mathcal{SF}, \mathcal{F}$ or $\mathcal{WF}$ is axiomatisable. The latest of these results is in the case for $\mathcal{F}r$ and appears in [**8**], which also contains a survey of the earlier work, including full proofs, together with some further model-theoretic material. Not surprisingly, the conditions on the monoid S that arise are finitary. The aim of the current article is to investigate the relation between pairs of these conditions. It is known that if $\mathcal{F}r$ is axiomatisable, then so is $\mathcal{P}$, and if $\mathcal{P}$ is axiomatisable, then so is $\mathcal{SF}$. We give new examples of monoids such that $\mathcal{SF}$ is axiomatisable, but $\mathcal{P}$ is not, and such that $\mathcal{P}$ is axiomatisable, but $\mathcal{F}$ is not.

The structure of this paper is as follows. For completeness we visit briefly in Section 2 the question of the conditions on S such that $\mathcal{F}$ or $\mathcal{WF}$ is axiomatisable. We do not pursue these cases further here. In Section 3 we consider strongly flat acts, and recall the characterisation from [**7**] of those monoids S such that $\mathcal{SF}$ is axiomatisable; such monoids must satisfy conditions we refer to as (FGR) and (FGr). In Section 4 we look at projective acts and give the result from [**20**] determining those S such that $\mathcal{P}$ is axiomatisable. Such monoids S must in particular be left perfect and hence local. Section 5 looks at the remaining case of free acts; we present the result from [**8**] characterising the monoids such that $\mathcal{F}r$ is axiomatisable. Here we find a new and rather curious condition for a monoid S which we label (F), which is implied by the property that the group of units H_1 of S has finite right index.

The new material of this article comprises examples, for the most part. Some are scattered through the text; we present two rather longer ones in our final section. We investigate the connections between (FGR) and (FGr), left perfection, localness, Condition (F) and H_1 having finite right index, summarising our findings in a table.

For further details concerning the theory of S-acts, we refer the reader to 'The Book' [**14**]. The reader interested in the techiques used to arrive at the results presented here should consult [**4**]. Our main tool is that of an *ultraproduct*, and the celebrated theorem of Łos that tells us in particular that if $\mathcal{A}$ is an axiomatisable class of S-acts, then $\mathcal{A}$ is closed under the formation of ultraproducts; however, we make no explicit use of these ideas in this paper. We follow standard semigroup convention in denoting the four Green's relations on a monoid S of which we make use by $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ and for an element $a \in S$, the corresponding equivalence classes containing a by L_a, R_a, H_a and D_a.

2. Flat and weakly flat acts

To define classes of flat left S-acts we need the notion of *tensor product*. If A is a right S-act and B a left S-act then the tensor product of A and B, written $A \otimes B$, is the set $A \times B$ factored by the equivalence generated by

$$\big\{\big((as, b), (a, sb)\big) : a \in A, b \in B, s \in S\big\}.$$

For $a \in A$ and $b \in B$ we write $a \otimes b$ for the equivalence class of (a, b).

A function θ from a left S-act A to a left S-act B is an S-*morphism* if $(sa)\theta = s(a\theta)$ for all $s \in S$ and $a \in A$. The class of all left S-acts, S-*Act*, together with all S-morphisms, forms a category **S–Act**; S-morphisms between right S-acts, *Act*-S and the category **Act–S** of right S-acts and S-morphisms, are defined dually. As usual we denote the category of sets and functions as **Set**. For a left S-act B, the map $-\otimes B$ takes a right S-act A to a set $A \otimes B$. We can lift $-\otimes B$ to a functor from **Act–S** to **Set** by defining its value at θ to be $\theta \otimes I_B$, where $\theta : A \to A'$ is an S-morphism between right S-acts A and A', and

$$(a \otimes b)(\theta \otimes I_B) = a\theta \otimes b.$$

It is from this functor that the various notion of *flatness* are derived.

A left S-act B is *weakly flat* if the functor $-\otimes B$ preserves embeddings of right ideals of S into S, *flat* if it preserves arbitrary embeddings of right S-acts, and *strongly flat* if it preserves pullbacks (equivalently, equalisers and pullbacks [**1**]). Clearly flat left acts are weakly flat and since embeddings are equalisers in **Act–S**, strongly flat left S-acts are flat. It is unnecessary here to recap the categorical notions of pullback and equaliser, since strongly flat acts may be characterised via interpolation conditions, explained in the next section. We give the reader a warning that terminology has changed over the years; in particular, left S-acts B such that $-\otimes B$ preserves equalisers and pullbacks are called weakly flat in [**6**] and [**19**], flat in [**7**], and pullback flat in [**14**].

To describe the conditions on a monoid S such that $\mathcal{F}$ or $\mathcal{WF}$ are axiomatisable, we need to look a little more carefully at equalities of the form $a \otimes b = a' \otimes b'$.

LEMMA 2.1. [**14**] *Let A be a right S-act and B a left S-act. Then for $a, a' \in A$ and $b, b' \in B$, $a \otimes b = a' \otimes b'$ if and only if there exist $s_1, t_1, s_2, t_2, \ldots, s_m, t_m \in S$,*

$a_2,\ldots,a_m \in A$ and $b_1,\ldots,b_m \in B$ such that

$$\begin{array}{rclrcl} & & & b & = & s_1b_1 \\ as_1 & = & a_2t_1 & t_1b_1 & = & s_2b_2 \\ a_2s_2 & = & a_3t_2 & t_2b_2 & = & s_3b_3 \\ & \vdots & & & \vdots & \\ a_ms_m & = & a't_m & t_mb_m & = & b'. \end{array}$$

The sequence presented in Lemma 2.1 will be called a *tossing* (or scheme) $\mathcal{T}$ of length m over A and B connecting (a,b) to (a',b'). The *skeleton* $\mathcal{S}=\mathcal{S}(\mathcal{T})$ of $\mathcal{T}$, is the sequence

$$\mathcal{S}=(s_1,t_1,\ldots,s_m,t_m)\in S^{2m}.$$

The set of all skeletons is denoted by $\mathbb{S}$. By considering trivial acts it is easy to see that $\mathbb{S}$ consists of all even length sequences of elements of S.

We know therefore that if $a,a'\in A$ and $b,b'\in B$, where A is a right S-act and B a left S-act, then $a\otimes b=a'\otimes b'$ in $A\otimes B$ if and only if there exists a tossing $\mathcal{T}$ from (a,b) to (a',b') over A and B, with skeleton $\mathcal{S}$, say. If the equality $a\otimes b=a'\otimes b'$ holds also in $(aS\cup a'S)\otimes B$ (certainly in the case B is flat) and is determined by some tossing $\mathcal{T}'$ from (a,b) to (a',b') over $aS\cup a'S$ and B with skeleton $\mathcal{S}'=\mathcal{S}(\mathcal{T}')$ then we say that $\mathcal{T}'$ is a *replacement tossing* for $\mathcal{T}$, $\mathcal{S}'$ is a *replacement skeleton for* $\mathcal{S}$ and (in case $A=S$) triples $(a,\mathcal{S}',a')$ will be called *replacement triples* for $(a,\mathcal{S},a')$.

THEOREM 2.2. [**2**] *The following conditions are equivalent for a monoid S:*
(1) the class $\mathcal{WF}$ is axiomatisable;
(2) the class $\mathcal{WF}$ is closed under formation of ultraproducts;
(3) for every skeleton $\mathcal{S}$ over S and $a,a'\in S$ there exist finitely many skeletons $\mathcal{S}_1,\ldots,\mathcal{S}_{\alpha(a,\mathcal{S},a')}$ over S, such that for any weakly flat left S-act B, if elements $(a,b),(a',b')$ of $S\times B$ are connected by a tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T})=\mathcal{S}$, then (a,b) and (a',b') are connected by a tossing $\mathcal{T}'$ over $aS\cup a'S$ and B such that $\mathcal{S}(\mathcal{T}')=\mathcal{S}_k$, for some $k\in\{1\,\ldots,\alpha(a,\mathcal{S},a')\}$.

THEOREM 2.3. [**2**] *The following conditions are equivalent for a monoid S:*
(1) the class $\mathcal{F}$ is axiomatisable;
(2) the class $\mathcal{F}$ is closed under formation of ultraproducts;
(3) for every skeleton $\mathcal{S}$ over S there exist finitely many replacement skeletons $\mathcal{S}_1,...,\mathcal{S}_{\beta(\mathcal{S})}$ over S such that, for any right S-act A and any flat left S-act B, if $(a,b),(a',b')\in A\times B$ are connected by a tossing $\mathcal{T}$ over A and B with $\mathcal{S}(\mathcal{T})=\mathcal{S}$, then (a,b) and (a',b') are connected by a tossing $\mathcal{T}'$ over $aS\cup a'S$ and B such that $\mathcal{S}(\mathcal{T}')=\mathcal{S}_k$, for some $k\in\{1,...,\beta(\mathcal{S})\}$.

Theorems 2.2 and 2.3 are, sadly, rather hard to use directly. Clearly they imply that for a monoid S, if $\mathcal{F}$ is an axiomatisable class, then so is $\mathcal{WF}$. It is also worth pointing out that if *all* left S-acts are weakly flat (flat), then $\mathcal{WF}$ ($\mathcal{F}$) is axiomatisable. We argue via indirect means in [**2**] that for an infinite null monoid, $\mathcal{WF}$ is not axiomatisable (hence neither is $\mathcal{F}$), whereas for the monoid of natural numbers under multiplication, $\mathcal{F}$ (and hence $\mathcal{WF}$) is axiomatisable, without all S-acts being flat. To date we do not have an example of a monoid S such that $\mathcal{WF}$ is axiomatisable, but $\mathcal{F}$ is not. Perhaps this is not entirely surprising, since it is in general difficult to determine when a weakly flat S-act is flat, as demonstrated in [**3**]. The question of for which monoids S we have that $\mathcal{F}=\mathcal{WF}$ remains open.

3. Strongly flat acts

We recall from Section 2 that a left S-act B is *strongly flat* if the functor $-\otimes B$ from **Act–S** to **Set** preserves pullbacks. Strongly flat acts are, however, made rather more manageable than flat and weakly flat acts, thanks to the following result of Stenström [**19**]. The separating of Stenström's condition into two separate conditions labelled (P) and (E) first appears in [**16**].

THEOREM 3.1. [**19**] *A left S-act B is strongly flat if and only if B satisfies Conditions (P) and (E):*

(P): if $x, y \in B$ and $s, t \in S$ with $sx = ty$, then there is an element $z \in B$ and elements $s', t' \in S$ such that $x = s'z, y = t'z$ and $ss' = tt'$;

(E): if $x \in B$ and $s, t \in S$ with $sx = tx$, then there is an element $z \in B$ and $s' \in S$ with $x = s'z$ and $ss' = ts'$.

We recall from [**7**] that for any $s, t \in S$, $R(s, t)$ and $r(s, t)$ are defined by

$$R(s, t) = \{(u, v) \in S \times S \mid su = tv\}$$

and

$$r(s, t) = \{u \in S \mid su = tu\}$$

so that $R(s, t)$ is a right S-subact of $S \times S$, and $r(s, t)$ is a right ideal of S. We say that S satisfies *Condition (FGR)* (respectively *Condition (FGr)*), if $R(s, t)$ (respectively $r(s, t)$) is finitely generated for all $s, t \in S$. We remark that if every right ideal of S is finitely generated, that is, S is *right noetherian*, then certainly S satisfies (FGr); we see in Example 6.1 that a monoid can satisfy (FGr) without being right noetherian.

Stenström's theorem, together with a straightforward use of ultraproducts, enables us to prove the next result.

THEOREM 3.2. [**7**] *The following conditions are equivalent for a monoid S:*
(1) $\mathcal{SF}$ is axiomatisable;
(2) $\mathcal{SF}$ is closed under ultraproducts;
(3) every ultrapower of S as a left S-act is strongly flat;
(4) S satisfies (FGr) and (FGR).

It is implicit in the proof of [**7**, Theorem 3.1] and made explicit in [**8**] that the class of left S-acts satisfying Condition (E) is axiomatisable if and only if S satisfies (FGr), and the class of left S-acts satisfying Condition (P) is axiomatisable if and only if S satisfies (FGR).

As remarked in [**7**], any finite monoid, any group, and any inverse ω-chain are such that $\mathcal{SF}$ is axiomatisable.

We will return to the next examples in later sections. The first and third were arrived at via discussions with the author's student, Lubna Shaheen.

EXAMPLE 3.3. *Let G be a group with identity ϵ, let 1 be an adjoined identity and put $S_1 = G^1$. For the monoid S_1 the class $\mathcal{SF}$ is axiomatisable.*

PROOF. By the remarks above, it is clear that (FGr) holds. To see that (FGR) holds, we first observe that $R(1, 1) = (1, 1)S_1$ and for any $a \in G$,

$$R(a, a) = (1, 1)S_1 \cup (1, \epsilon)S_1 \cup (\epsilon, 1)S_1$$

and so is finitely generated.

Suppose now that $a \in G$. We claim that with $R = R(1,a)$ we have that $R = (a,1)S_1$. Clearly $(a,1) \in R$ so that $(a,1)S_1 \subseteq R$. On the other hand, if $(u,v) \in R$ then as $u = 1u = av$ we have that $(u,v) = (a,1)v \in (a,1)S_1$, so that $R = (a,1)S_1$ is monogenic. Dually, $R(a,1) = (1,a)S_1$.

Finally, if $a,b \in G$ with $a \neq b$, put $R = R(a,b)$ and notice that $(1, b^{-1}a), (a^{-1}b, 1) \in R$. On the other hand we know that $(1,1) \notin R$ and if $(1,x) \in R$ with $x \in G$, then from $a1 = bx$ we have that $x = b^{-1}a$ and dually, if $(y,1) \in R$ then $y = a^{-1}b$. Finally, if $h,k \in G$ and $(h,k) \in R$, then from $ah = bk$ we obtain that $h = a^{-1}bk$ and so $(h,k) = (a^{-1}b,1)k$. We have argued that $R = (1,b^{-1}a)S_1 \cup (a^{-1}b,1)S_1$ and hence is finitely generated as required.

The result now follows from Theorem 3.2. □

Example 3.4. *Let G and H be isomorphic groups with isomorphism from G to H given by $g \mapsto g'$. Let S_2 be the semilattice $\{1,0\}$ of groups $G_1 = G, G_0 = H$ and connecting morphism $'$. For the monoid S_2 the class $\mathcal{SF}$ is axiomatisable.*

Proof. We write the identity of G, hence of S_2, as 1, and the identity of H as ϵ. As in Example 3.3, certainly (FGr) holds since S_2 is right noetherian. It remains to show that (FGR) holds, whence from Theorem 3.2, $\mathcal{SF}$ is axiomatisable.

For any $b \in G$, and *any* $z \in S_2$ we claim that $R(z,b) = (1, b^{-1}z)S_2$. Certainly $(1, b^{-1}z) \in R(z,b)$. Conversely, if $zu = bv$ for $u,v \in S_2$, then $v = b^{-1}zu$ so that $(u,v) = (1,b^{-1}z)u$ lies in $(1,b^{-1}z)S_2$. Dually, if $a \in G$ then $R(a,z) = (a^{-1}z,1)S_2$ for any $z \in S_2$.

Suppose now that $a,b \in G$; we claim that

$$R(a',b') = (1,b^{-1}a)S_2 \cup (\epsilon, b^{-1}a)S_2 \cup (a^{-1}b,\epsilon)S_2.$$

It is easy to check that $(1,b^{-1}a), (\epsilon, b^{-1}a)$ and $(a^{-1}b, \epsilon)$ lie in $R(a',b')$. If $u,v \in G$ and $a'u = b'v, a'u' = b'v, a'u = b'v'$ or $a'u' = b'v'$, we obtain that $a'u' = b'v'$, and so $au = bv$ as $'$ is an isomorphism. Hence $(u,v) = (1,b^{-1}a)u$, $(u',v) = (\epsilon, b^{-1}a)u$, $(u,v') = (a^{-1}b,\epsilon)v$ and $(u',v') = (1,b^{-1}a)u'$. Hence $R(a',b')$ has three generators as claimed. □

Example 3.5. *Let S_3 be a semilattice $\{1,0\}$ of groups G_1, G_0 with trivial connecting homomorphism. If S_3 satisfies (FGR), then G_1 is finite.*

Proof. Let $a \in G_0$, so that for any $u,v \in G_1$, $(u,v) \in R(a,a)$. If $R(a,a)$ is finitely generated we must therefore have a finite sublist $(c_1,d_1),\ldots,(c_k,d_k)$ of generators of $R(a,a)$ such that

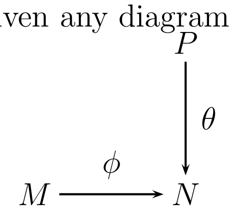
$$G_1 \times G_1 = (c_1,d_1)G_1 \cup \ldots \cup (c_k,d_k)G_1.$$

For any $g \in G_1$ we have that $(g,1) = (c_i,d_i)h$ for some $i \in \{1,\ldots,k\}$ and $h \in G_1$. It follows that $h = d_i^{-1}$ and $g = c_i d_i^{-1}$. Hence G_1 is finite.

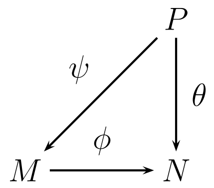
□

4. Projective acts

A left S-act P is *projective* if given any diagram of left S-acts and S-morphisms



where $\phi : M \to N$ is onto, there exists an S-morphism $\psi : P \to M$ such that the diagram



is commutative.

It is easy to see from the following structure theorem for projective acts, that $\mathcal{P} \subseteq \mathcal{SF}$ for any monoid S

THEOREM 4.1. [**15, 5**] *A left S-act P is projective if and only if $P \cong \coprod_{i\in I} Se_i$, where $e_i \in E$ for all $i \in I \neq \emptyset$.*

The notion of left perfection appears in the characterisation of those monoids S such that $\mathcal{P}$ is axiomatisable. We therefore pause to consider this concept.

A left S-act B is called a *cover* of a left S-act A if there exists an S-epimorphism $\theta : B \to A$ such that the restriction of θ to any proper S-subact of B is not an epimorphism to A. If B is in addition projective, then B is a *projective cover* for A. We remark here that the epimorphisms in **S–Act** are precisely the surjective S-morphisms [**14**]. A monoid S is *left perfect* if every left S-act has a projective cover.

Left perfect monoids have a number of equivalent descriptions, involving conditions which for convenience we present in the following list, together with related conditions given here for future reference. We recall that a submonoid T of S is *left unitary* if $t, ts \in T$ implies that $s \in T$, and *right collapsible* if for any $s, t \in T$ there exists $r \in T$ with $sr = tr$.

(A) Every left S-act satisfies the ascending chain condition for cyclic S-subacts.

(D) Every left unitary submonoid of S has a minimal left ideal generated by an idempotent.

$(M_R)/(M_L)$ The monoid S satisfies the descending chain condition for principal right/left ideals.

$(M^R)/(M^L)$ The monoid S satisfies the ascending chain condition for principal right/left ideals.

(K) Every right collapsible submonoid contains a right zero.

Clearly (A) implies (M^L); the converse is not true. It is shown in [**10**] that a monoid may satisfy (M^L) and (M_R) but not (A).

THEOREM 4.2. [**6, 10, 13**] *The following conditions are equivalent for a monoid S:*

(1) S is left perfect;
(2) S satisfies Conditions (A) and (D);
(3) S satisfies Conditions (A) and (M_R);
(4) S satisfies Conditions (A) and (K);
(5) $\mathcal{SF} = \mathcal{P}$.

As examples of left perfect monoids we give groups, or the monoids S_1, S_2 and S_3 appearing in Examples 3.3, 3.4 and 3.5. Indeed if S is any semilattice $\{1, 0\}$ of groups G_1, G_0, then S is left perfect. For clearly S has M_R: to see that S has Condition (A) we make use of the following alternative characterisation of that condition.

Proposition 4.3. [**10**] *A monoid S satisfies Condition (A) if and only if for any elements $a_1, a_2, \ldots$ of S, there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geq n$, there exists $j_i \in \mathbb{N}, j_i \geq i+1$, such that*

$$Sa_ia_{i+1}\ldots a_{j_i} = Sa_{i+1}\ldots a_{j_i}.$$

It is clear from Proposition 4.3 that any group has (A). If S is a semilattice $\{1,0\}$ of groups G_1 and G_0, then consider any sequence $a_1, a_2, \ldots$ of elements of S. If only finitely many of these elements lie in G_0, then let n be chosen such that $a_i \in G_1$ for all $i \geq n$. Clearly $Sa_ia_{i+1} = Sa_{i+1} = S$ for all $i \geq n$. If the sequence contains infinitely many elements from G_0, then for any $i \in \mathbb{N}$ we can choose a $j_i \geq i+1$ such that $a_{j_i} \in G_0$; then $Sa_ia_{i+1}\ldots a_{j_i} = G_0 = Sa_{i+1}\ldots a_{j_i}$. Thus S satisfies (A) and hence is left perfect.

We have already observed that if S is a finite monoid, then conditions (FGR), (FGr) hold, and clearly so does (M_R). Moreover, if A is a cyclic S-act we must have $|A| \leq |S|$, from which it follows that S has condition (A) and hence is left perfect. Consequently, $\mathcal{SF} = \mathcal{P}$ is axiomatisable.

We remind the reader that a monoid S is *local* if $S \setminus H_1$ is an ideal. Left perfect monoids are local, as are the monoids satisfying conditions that appear in the next chapter.

Lemma 4.4. *The following conditions are equivalent for a monoid S:*
(i) S is local;
(ii) $H_1 = D_1$;
(iii) if $e\,\mathcal{D}\,1$ for any $e \in E$, then $e = 1$.

Proof. Let $A = S \setminus H_1$.

(i) $\Rightarrow$ *(ii)* If $a\,\mathcal{D}\,1$ for some $a \in S$, then there exists $b \in S$ with $a\,\mathcal{R}\,b\,\mathcal{L}1$ and hence $c \in S$ with $cb = 1$. From the latter equation and the fact that S is local we deduce that $b \in H_1$. Now $a\,\mathcal{R}\,1$ and so $ad = 1$ for some $d \in S$, so that locality again gives that $a \in H_1$.

(ii) $\Rightarrow$ *(iii)* This is clear, as any $\mathcal{H}$-class contains at most one idempotent.

(iii) $\Rightarrow$ *(i)* Let $a \in A$ and suppose that $ab \notin A$, so that $ab\,\mathcal{H}\,1$. Then $abc = 1$ for some $c \in S$. Putting $d = bc$ we have that $da \in E$ and $1, a, d$ and da are related as in the egg-box picture below

1	a
d	da

By our assumption, $da = 1$, so that $a\,\mathcal{H}\,1$, a contradiction. Hence $ab \in A$ and A is a right ideal. The dual argument completes the proof.

□

The next result is folklore.

Lemma 4.5. *Let S be a monoid satisfying (M_R). Then S is local.*

Proof. Let $e \in E$ with $e\,\mathcal{D}\,1$. By [**9**, Proposition 2.3.5], there is an element $a \in S$ with inverse a' such that $1 = a'a$ and $e = aa'$, so that $1, e, a, a'$ are related as in the egg-box picture below

1	a'
a	e

From $a\,\mathcal{L}\,1$ we obtain that $a^n\,\mathcal{L}\,1$ for all $n \in \mathbb{N}$. Clearly

$$aS \supseteq a^2S \supseteq \dots .$$

Since S has (M_R), $a^nS = a^{n+1}S$ for some n. It follows that $a^n = a^{n+1}t$ for some $t \in S$, whence $1 = at$ since $a^n\,\mathcal{L}\,1$. This tells us that $1\,\mathcal{R}\,a\,\mathcal{R}\,e$, hence $1 = e$. From Lemma 4.4 we deduce that S is local. □

For an example of a monoid that is local but which does not satisfy (M_R), we cite an inverse ω-chain, and also we refer the reader to our final section.

Our interest in left perfect monoids lies in the next result, due to Stepanova.

Theorem 4.6. [**20**], *cf.* [**7, 2**] *The following conditions are equivalent for a monoid S:*

(1) every ultrapower of the left S-act S is projective;

(2) $\mathcal{SF}$ is axiomatisable and S is left perfect.

(3) $\mathcal{P}$ is axiomatisable.

It follows that for any finite monoid, any group, and for the semilattices of groups S_1 or S_2, $\mathcal{P}$ is axiomatisable. On the other hand, if S_3 is a semilattice of groups as in Example 3.5 with G_1 infinite, then as shown in that example, $\mathcal{SF}$ is not axiomatisable, but as argued above, S is left perfect. As remarked in [**7**], an inverse ω-chain is an example of a monoid such that $\mathcal{SF}$ is axiomatisable, but clearly $\mathcal{P}$ is not, as the monoid fails to satisfy (M_R) and hence is not left perfect. We give a *non-local* example of a monoid such that $\mathcal{SF}$ but not $\mathcal{P}$ is axiomatisable in Section 6.

5. Free acts

We remind the reader that a left S-act F is *free* (on a subset X, in **S–Act**) if and only if for any left S-act A and function $\theta : X \to A$, there is a unique S-morphism $\overline{\theta} : F \to A$ such that $\iota\overline{\theta} = \theta$, where $\iota : X \to F$ is the inclusion function.

For any set X and element $x \in S$, the left S-act Sx consists of all formal symbols sx, where $s \in S$ and $x \in X$, with action given by $t(sx) = (ts)x$, for any $s, t \in S$ and $x \in X$. Clearly Sx is isomorphic to S, where S is regarded as a left ideal.

Theorem 5.1. [**14**] *A left S-act F is free on X if and only if $F \cong \coprod_{x\in X} Sx$.*

Notice from the above that a free left S-act is isomorphic to a coproduct of copies of the left S-act $S = S1$, so that free left S-acts are clearly projective.

We now present a rather curious condition, that is crucial in determining those monoids S such that $\mathcal{F}r$ is axiomatisable. To state and use our condition, it is convenient to make use of the preorders $\leq_{\mathcal{R}}$ ($\leq_{\mathcal{L}}$) on a monoid S, defined by the rule that $a \leq_{\mathcal{R}} b$ ($a \leq_{\mathcal{L}} b$) if and only if $a = bs$ ($a = sb$) for some $s \in S$. Clearly, the equivalence relations associated with $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ are Green's relations $\mathcal{R}$ and $\mathcal{L}$, respectively. Let $e \in E$ and $a \in S$. We say that $a = xy$ is an *e-good factorisation of a through x* if $y \not\leq_{\mathcal{R}} w$ for any w with $e = xw$ and $w\,\mathcal{L}\,e$.

A monoid S *satisfies condition (F)* if, for any $e \in E$ with $e \neq 1$, there is a finite set $F_e \subseteq S$ such that every $a \in S$ has an e-good factorisation through some $u \in F_e$. We remark that condition (F) is only imposing restrictions on those elements a such that $a \leq_{\mathcal{R}} e$, that is, $a \in eS$. For any $a \in S$ may be written as $a = 1\,a$; if $a \not\leq_{\mathcal{R}} e$, then this factorisation is easily seen to be e-good, since if $e = 1\,s$ then we must have $s = e$, and by assumption, $a \not\leq_{\mathcal{R}} e$.

If S is unipotent (that is, $E = \{1\}$), then S satisfies (F) vacuously. Any finite monoid satisfies (F), for let $e \in E$ with $e \neq 1$ and put $F_e = S$. Then any $a \in S$ can be written as $a = a\,1$ and as by Lemma 4.5 S is local, $1 \not\leq_{\mathcal{R}} v$ for any $v \in L_e$. Hence S satisfies (F).

Lemma 5.2. *Let S be a monoid satisfying condition (F). Then S is local.*

Proof. Suppose that $e \in E$ and $e\,\mathcal{D}\,1$. By [**9**, Proposition 2.3.5] there is an element $a \in S$ with inverse a' such that $1 = aa'$ and $a'a = e$. If $e \neq 1$ then by assumption that S satisfies (F), a' has an e-good factorisation through u, say. Suppose that $a' = uv$ is such a factorisation. Then $e = a'a = uva$ and as $e \leq_{\mathcal{L}} va \leq_{\mathcal{L}} a\,\mathcal{L}\,e$ we have that $e\,\mathcal{L}\,va$. From $1\,\mathcal{R}\,a$ we see that $v = v\,1\,\mathcal{R}\,va$ (and so in particular $v \leq_{\mathcal{R}} va$), contradicting the assumption that $a' = uv$ is an e-good factorisation through u. We conclude that $e = 1$ so that by Lemma 4.4, S is local. □

At the end of this section we demonstrate that a monoid can be local without satisfying (F).

The question of axiomatisability of $\mathcal{F}r$ was solved in some special cases by Stepanova in [**20**], and most recently by the author as below.

Theorem 5.3. [**8**] *The following conditions are equivalent for a monoid S:*
(1) every ultrapower of the left S-act S is free;
(2) $\mathcal{P}$ is axiomatisable and S satisfies (F);
(3) $\mathcal{F}r$ is axiomatisable.

We say that the group of units H_1 of a monoid S has *finite right index* if there exist $u_1, \dots, u_n \in S$ such that $S = u_1H_1 \cup \dots u_nH_1$. It is immediate that if H_1 has finite right index, then S has only finitely many $\mathcal{R}$-classes $R_{u_1}, \dots, R_{u_n}$ and hence only finitely many right ideals. The following corollary is now immediate from Lemma 4.5.

Corollary 5.4. *Let S be a monoid in which H_1 has finite right index. Then S satisfies Condition (M_R) and is local.*

Lemma 5.5. *Let S be a monoid in which H_1 has finite right index. Then S satisfies condition (F).*

Proof. Since H_1 has finite right index in S we may write S as $S = u_1H_1 \cup \dots u_nH_1$ for some $u_1, \dots, u_n \in S$. Let $e \in E$ with $e \neq 1$ and put $F_e = \{u_1, \dots, u_n\}$. Let $a \in S$. By assumption may write a as $a = u_ig$ for some $g \in H_1$. If $e = u_iw$ where $w\,\mathcal{L}\,e$, and if $g \leq_{\mathcal{R}} w$, then since $g \in H_1$ we deduce that $e\,\mathcal{L}\,w\,\mathcal{R}\,1$, contradicting the fact that S is local. Hence the factorisation $a = u_ig$ is e-good. □

The converses to both Corollary 5.4 and Lemma 5.5 fail, as we see below and in the next section.

For some restricted classes of monoids, we can simplify the condition given in Theorem 5.3.

Proposition 5.6. [**8**] *Let S be a monoid such that*

$$S \setminus R_1 = s_1S \cup \dots \cup s_mS$$

for some $s_1, \dots, s_m \in S$. Then $\mathcal{F}r$ is an axiomatisable class if and only if $\mathcal{P}$ is axiomatisable and H_1 has finite right index in S.

In the case where S is inverse, we can simplify further.

Corollary 5.7. [**20**] *Let S be an inverse monoid. Then $\mathcal{F}r$ is an axiomatisable class if and only if $\mathcal{P}$ is axiomatisable and H_1 has finite right index in S.*

Let us end this section by considering some examples. By earlier remarks we have that $\mathcal{F}r$ is axiomatisable for S finite or S a group. For the monoids S_1 and S_2 of Examples 3.3 and 3.4, we have remarked that $\mathcal{P}$ is axiomatisable. For S_1, $H_1 = \{1\}$, so that the index of H_1 is $|S|$, hence by Corollary 5.7, $\mathcal{F}r$ is axiomatisable if and only if G is finite. Thus if G is infinite, we have an example of a monoid which is left perfect but such that the index of H_1 is infinite and (F) fails, and which is such that $\mathcal{P}$ is axiomatisable but $\mathcal{F}r$ is not. For the monoid S_2 we have that $H_1 = G_0$ and $1G_0 = G_0$, $\epsilon G_0 = G_1$, so that H_1 has finite right index and $\mathcal{F}r$ is axiomatisable.

6. Examples

The aim of this section is to present two monoids demonstrating the independence of some of the conditions we have encountered in this paper. Our examples are the bicyclic monoid B and the monoid D^1, where $D = \mathbb{Z} \times \mathbb{Z}$ with multiplication given by the rule

$$(a, b)(c, d) = (a - b + \max\{b, c\}, d - c + \max\{b, c\}).$$

The semigroup D first appears in the papers of Warne in the 1960's, dubbed in [**21**] the *extended bicyclic semigroup*. It is easy to see that

$$E(D) = \{(a, a) \mid a \in \mathbb{Z}\}$$

forming a chain

$$\ldots < (2, 2) < (1, 1) < (0, 0) < (-1, -1) < (-2, -2) < \ldots$$

Moreover, D is regular, hence inverse, with

$$(a, b)' = (b, a)$$

for any $(a, b) \in D$. It then follows that for any $(a, b), (c, d) \in D$,

$$(a, b)D \subseteq (c, d)D \text{ if and only if } a \geq c$$

and dually,

$$D(a, b) \subseteq D(c, d) \text{ if and only if } b \geq d.$$

As Warne remarks in [**21**], D is bisimple. It is perhaps useful to point out that for any $(a, b), (c, d) \in D$, if $(a, b)(c, d) = (x, y)$, then $x \geq a$ and $y \geq d$.

The semigroup D contains $B = \mathbb{N}^0 \times \mathbb{N}^0$ as an inverse subsemigroup. Certainly B is a monoid (with identity $(0, 0)$), but D is not. We therefore adjoin an identity and consider D^1; to avoid confusion with the integer 1, we denote the adjoined identity of D^1 by ϵ. Note that for any $(a, b), (c, d) \in D$, $(a, b) \leq_{\mathcal{R}} (c, d)$ *in* D^1 if and only if $(a, b) \leq_{\mathcal{R}} (c, d)$ *in* D, with the dual comment valid for principal left ideals.

The monoid B is an old friend to semigroup theorists. It is well known to be bicyclic, and to have a monoid presentation given by

$$B = \langle a, b : ab = 1 \rangle.$$

By contrast, D appears only sporadically in the literature. We see by a comment above that D cannot be finitely generated, either as a semigroup or as an inverse semigroup. For if $T = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ is a finite subset of D, then choosing $z \in \mathbb{Z}$ strictly smaller than any a_i or b_i, it is impossible to write (z, z) as a product of elements of T and their inverses.

The following example shows us that D^1 has condition (F), but is not left perfect, since both condition (A) and (M_R) fail. Moreover, since (M_R) fails, clearly the group of units $\{\epsilon\}$ does *not* have finite right index in D^1. Finally, D^1 satisfies one but not both conditions required for $\mathcal{SF}$ to be axiomatisable.

EXAMPLE 6.1. *The extended bicyclic monoid D^1 satisfies (F) and (FGr). On the other hand, D^1 does not satisfy (FGR) and satisfies none of the chain conditions (M_R),(M_L),(M^R) and (M^L).*

PROOF. We show first that D^1 satisfies (F). Let (a,a) be a non-identity idempotent of D^1 and put

$$F_{(a,a)} = \{1, (a+1,a+1), (a,a+1)\}.$$

As commented in Section 5, any element that is not $\leq_{\mathcal{R}}$-related to (a,a) has an (a,a)-good factorisation through 1. Suppose now that $(b,c) \leq_{\mathcal{R}} (a,a)$ in D^1, so that $b \geq a$. If $b > a$, then

$$(b,c) = (a+1,a+1)(b,c)$$

and as $(a,a) \notin (a+1,a+1)D^1$, this factorisation of (b,c) is (a,a)-good. Finally, consider $(a,d) \in D$. We have

$$(a,d) = (a,a+1)(a,d-1)$$

and this factorisation is (a,a)-good. For, if

$$(a,a) = (a,a+1)(p,q)$$

with $(p,q)\,\mathcal{L}\,(a,a)$ then $q = a$ and we are forced to have $p = a+1$. But then

$$(a,d-1) \not\leq_{\mathcal{R}} (p,q).$$

Hence every element of D^1 has an (a,a)-good factorisation through an element of $F_{(a,a)}$, so that D^1 satisfies (F).

Since we have the strict doubly infinite chain of principal right ideals

$$\ldots \subset (2,2)D^1 \subset (1,1)D^1 \subset (0,0)D^1 \subset (-1,-1)D^1 \subset \ldots$$

it is clear that both (M^R) and (M_R) fail; dually, (M^L) and (M_L) fail.

To see that (FGr) holds, we remark first that $r(\epsilon,\epsilon) = D^1$ so is principal.

Suppose now that $r\big(\epsilon,(a,b)\big) \neq \emptyset$. Clearly $\epsilon \notin r\big(\epsilon,(a,b)\big)$, so that we must have $(s,t) \in D$ with

$$(s,t) = (a,b)(s,t) = (a-b+\max\{b,s\}, t-s+\max\{b,s\}).$$

Consequently, $a = b$. It is then easy to see that $r\big(\epsilon,(a,b)\big) = (a,a)D^1$. The dual argument gives that $r\big((a,b),\epsilon\big)$ is either empty or principal, for any $(a,b) \in D$.

Finally, suppose that $r = r\big((a,b),(c,d)\big) \neq \emptyset$ for some $(a,b),(c,d) \in D$. If $\epsilon \in r$, then $(a,b) = (c,d)$, so that $r = D^1$. Otherwise, we must have that $(a,b)(s,t) = (c,d)(s,t)$ for some $(s,t) \in D$. Then

$$a-b+\max\{b,s\} = c-d+\max\{d,s\}$$

and

$$t-s+\max\{b,s\} = t-s+\max\{d,s\},$$

so that $\max\{b,s\} = \max\{d,s\}$ and $a-b = c-d$. Further, if there exists $(s,t) \in r$ with $s < b$ and $s < d$, then $(a,b) = (c,d)$, a contradiction. Let $u = \max\{b,d\}$. Then

$(u,u) \in r$ and if $(s,t) \in r$ then $s \geq u$ so that $(s,t) \in (u,u)D^1$ and $r = (u,u)D^1$ is finitely generated as required.

On the other hand, D^1 does not satisfy (FGR). To see this, consider $R = R\big((0,0),(0,1)\big)$; certainly $(\epsilon,\epsilon) \notin R$. Notice that for any $n \in \mathbb{N}$,

$$(0,0)(-n,-n) = (0,0) = (0,1)(-n,-n-1)$$

so that $\big((-n,-n),(-n,-n-1)\big) \in R$. If R were finitely generated, say $R = UD^1$ for some finite $U \subseteq R$, then we could choose $n \in \mathbb{N}$ with $-n < t$ for any $t \in \mathbb{Z}$ such that $\big((\alpha,(t,s)\big)$ or $\big((t,s),\alpha\big)$ lies in U, for any $\alpha \in D^1$ and any $s \in \mathbb{Z}$. Since U generates R,

$$\big((-n,-n),(-n,-n-1)\big) = (\alpha,\beta)\gamma$$

for some $(\alpha,\beta) \in U$ and $\gamma \in D^1$. Since we cannot have $\alpha = \beta = \epsilon$, we contradict the choice of n. We deduce that R cannot be finitely generated. □

We commented in Section 4 that an inverse ω-chain C is such that $\mathcal{SF}$ is axiomatisable, but since C is not left perfect, $\mathcal{P}$ is not. The bicyclic monoid provides us with a non-local example having the same properties.

Example 6.2. *The bicyclic monoid B has (FGR) and (FGr), is not local and is not left perfect.*

Proof. Since every left ideal of B is principal, certainly (FGr) holds. On the other hand B does not have (M_R), and so cannot be left perfect. Moreover, being bisimple and possessing an inverse ω-chain of idempotents, B is not local.

It remains to prove that (FGR) holds for B. To this end, let $(a,b),(c,d) \in B$ and let $R = R\big((a,b),(c,d)\big)$.

We consider first the case where $a \neq c$; without loss of generality we assume that $a > c$. Observe that

$$(a,b)(0,0) = (a,b) = (c,d)(d+(a-c),b)$$

so that

$$\big((0,0),(d+(a-c),b)\big)B \subseteq R.$$

Conversely, suppose that $\big((x,y),(u,v)\big) \in R$. Then

$$(a,b)(x,y) = (c,d)(u,v),$$

telling us that

$$a-b+\max\{b,x\} = c-d+\max\{d,u\} \text{ and } y-x+\max\{b,x\} = v-u+\max\{d,u\}.$$

We therefore have that

$$c-d+\max\{d,u\} \geq a > c$$

whence $u > d$, so that

$$a-b+\max\{b,x\} = c-d+u \text{ and } y-x+\max\{b,x\} = v.$$

Clearly $(x,y) = (0,0)(x,y)$, and

$$(d+(a-c),b)(x,y) = (d+(a-c)-b+\max\{b,x\}, y-x+\max\{b,x\}) = (u,v)$$

so that

$$\big((x,y),(u,v)\big) \in \big((0,0),(d+(a-c),b)\big)B$$

and we deduce that in this case,

$$R = \big((0,0),(d+(a-c),b)\big)B$$

is finitely generated, indeed monogenic.

We now turn our attention to the case where $a = c$. For any $s \leq b$ and $s' \leq d$ we define $w_{s,s'}$ =max$\{b - s, d - s'\}$. Notice that

$$(a, b)(s, w_{s,s'} - (b - s)) = (a, d)(s', w_{s,s'} - (d - s'))$$

so that putting

$$U = \{\big((s, w_{s,s'} - (b - s)), (s', w_{s,s'} - (d - s'))\big) \mid s \leq b, s' \leq d\}$$

we have that $UB \subseteq R$.

For the converse, suppose that $\big((x, y), (u, v)\big) \in R$. Then $(a, b)(x, y) = (a, d)(u, v)$ and so

$$\max\{b, x\} - b = \max\{d, u\} - d \text{ and } y - x + \max\{b, x\} = v - u + \max\{d, u\}.$$

The first equation tells us that $b \geq x$ if and only if $d \geq u$.

Suppose first that $b < x$ and $d < u$. Then $x - b = u - d$ and $y = v$. With $s = s' = 0$ we have that $w_{0,0} = \max\{b, d\}$; without loss of generality we assume that $w_{0,0} = b$. Then $\big((0, 0), (0, b - d)\big) \in U$ and in this case,

$$\big((x, y), (u, v)\big) = \big((0, 0), (0, b - d)\big)(x, y) \in UB.$$

On the other hand, if $x \leq b$ and $u \leq d$, then $y - x + b = v - u + d$ and

$$\big((x, w_{x,u} - (b - x)\big), (u, w_{x,u} - (d - u)\big) \in U.$$

Without loss of generality suppose that $w_{x,u} = b - x$, so that $b - x \geq d - u$. Then $\big((x, 0), (u, (b - x) - (d - u)\big) \in U$, that is, $\big((x, 0), (u, v - y)\big) \in U$ and

$$\big((x, y), (u, v)\big) = \big((x, 0), (u, v - y)\big)(0, y) \in UB.$$

We conclude that $UB = R$ and that R is finitely generated as required.

□

We promised at the end of the introduction to make clear the connections between the five conditions FGR + FGr, left perfection, (F), H_1 having finite right index and locality. This is best achieved in tabular form. The final column contains a reference to a result or a counterexample. One open question remains: if H_1 has finite right index, is S left perfect?

FGR + FGr	$\not\Rightarrow$	left perfect	B
FGR + FGr	$\not\Leftarrow$	left perfect	S_3 with G_1 infinite
FGR + FGr	$\not\Rightarrow$	Condition (F)	S_1 with G infinite
FGR + FGr	$\not\Leftarrow$	Condition (F)	D^1
FGR + FGr	$\not\Rightarrow$	H_1 has finite right index	S_1 with G infinite
FGR + FGr	$\not\Leftarrow$	H_1 has finite right index	S_3 with G_1 infinite and G_0 finite
FGR + FGr	$\not\Rightarrow$	local	B
FGR + FGr	$\not\Leftarrow$	local	S_3 with G_1 infinite
left perfect	$\not\Rightarrow$	Condition (F)	S_1 with G infinite
left perfect	$\not\Leftarrow$	Condition (F)	D^1
left perfect	$\not\Rightarrow$	H_1 has finite right index	S_1 with G infinite
left perfect	$\overset{?}{\Leftarrow}$	H_1 has finite right index	
left perfect	$\Rightarrow$	local	Lemma 4.5
left perfect	$\not\Leftarrow$	local	D^1
Condition (F)	$\not\Rightarrow$	H_1 has finite right index	D^1
Condition (F)	$\Leftarrow$	H_1 has finite right index	Lemma 5.5
Condition (F)	$\Rightarrow$	local	Lemma 5.2
Condition (F)	$\not\Leftarrow$	local	S_1 with G infinite
H_1 has finite right index	$\Rightarrow$	local	Corollary 5.4
H_1 has finite right index	$\not\Leftarrow$	local	S_1 with G infinite

The condition that H_1 has finite right index restricts severely the right ideal structure of the monoid. We have observed in Lemma 5.4 that it implies (M_R), so in order to show that it implies left perfection we are left with the question of whether or not it implies Condition (A). As a first attempt in this direction we present our last result.

LEMMA 6.3. *Let S be a monoid such that for any $a \in S$ there exist $n, k \in \mathbb{N}$ with $a^n \,\mathcal{L}\, a^{n+k}$. If H_1 has finite right index, then S satisfies (M^L).*

PROOF. Suppose that H_1 has finite right index in S, so that certainly S has only finitely many $\mathcal{R}$-classes. If we have an ascending chain of principal left ideals

$$Sa_1 \subseteq Sa_2 \subseteq \dots$$

then there must be an $\mathcal{R}$-class R containing infinitely many elements a_i. Pick m such that $a_m \in R$ and let $\ell \geq m$. Then there exists $h \geq \ell$ with $a_h \in R$, so that from

$$Sa_m \subseteq Sa_\ell \subseteq Sa_h$$

we obtain

$$Sa_mS \subseteq Sa_\ell S \subseteq Sa_hS$$

whence $Sa_mS = Sa_\ell S$.

From $Sa_m \subseteq Sa_\ell$ we have that $a_m = ua_\ell$ for some $u \in S$, and from $Sa_mS = Sa_\ell S$ there are elements $p, q \in S$ with $a_\ell = pa_mq$. It follows that $a_\ell = pua_\ell q = (pu)^r a_\ell q^r$ for all $r \in \mathbb{N}$.

By hypothesis there exist $n, k \in \mathbb{N}$ with $(pu)^n \,\mathcal{L}\, (pu)^{n+k}$. Calculating, we have that

$$a_\ell = (pu)^n a_\ell q^n \,\mathcal{L}\, (pu)^{n+k} a_\ell q^n = (pu)^k (pu)^n a_\ell q^n = (pu)^k a_\ell = (pu)^{k-1} pua_\ell = (pu)^{k-1} pa_m,$$

whence $Sa_m = Sa_\ell$ and the chain terminates at Sa_m.

□

References

[1] S. Bulman-Fleming, 'Pullback-flat acts are strongly flat', *Canadian Math. Bull.* **34** (1991), 456–461.

[2] S. Bulman-Fleming and V. Gould, 'Axiomatisability of weakly flat, flat and projective acts', *Communications in Algebra* **30** (2002), 5575–5593.

[3] S. Bulman-Fleming and K. McDowell, 'Monoids over which all weakly flat acts are flat', *Proc. Edinburgh Math. Soc.* **33** (1990), 287–298.

[4] C.C. Chang and H.J. Keisler, *Model Theory*, North-Holland, Amsterdam (1973).

[5] M.P. Dorofeeva, 'Hereditary and semi-hereditary monoids', *Semigroup Forum* **4** (1972), 301–311.

[6] J.B. Fountain, 'Perfect semigroups', *Proc. Edinburgh Math. Soc.* **20** (1976), 87–93.

[7] V. Gould, 'Axiomatisability problems for S-systems', *J. London Math. Soc.* **35** (1987), 193–201.

[8] V. Gould, E. Palyutin, A. Mikhalev and A. Stepanova, 'Model theoretic properties of free, projective and flat acts', *Fund. Appl. Math.*, to appear.

[9] J.M. Howie, *Fundamentals of Semigroup Theory*, Clarendon Press, Oxford (1995).

[10] J.R. Isbell, 'Perfect monoids', *Semigroup Forum* **2** (1971), 95–118.

[11] M. Kilp, 'On homological classification of monoids', *Siberian Math. J.* **13** (1972), 578–586.

[12] M. Kilp, 'Characterisation of monoids by properties of their left Rees factors', *Tartu Ül. Toimetised* **640** (1983), 29–37.

[13] M. Kilp, 'Perfect monoids revisited', *Semigroup Forum* **53** (196), 225–229.

[14] M. Kilp, U. Knauer and A.V. Mikhalev, *Monoids, Acts and Categories*, Walter De Gruyter, Berlin New York (2000).

[15] U. Knauer, ' Projectivity of acts and Morita equivalence of monoids', *Semigroup Forum* **3** (1972), 359–370.

[16] P. Normak, 'On equalizer flat acts and pullback flat acts', *Semigroup Forum* **36** (1987), 293–313.

[17] L.A. Skornjakov, 'Characterization of the category of polygons', *Math. USSR Sb.* **9** (1969), 455–465.

[18] L.A.Skornjakov, 'On the homological classification of monoids', *Sibirsk Mat. Z.* **10** (1969), 1139–1143.

[19] B. Stenström, 'Flatness and localization over monoids' *Math. Nach.* **48** (1971), 315–334.

[20] A.A. Stepanova, 'Axiomatisability and completeness in some classes of S-polygons' *Algebra and Logic* **30** (1991/92), 379–388.

[21] R.J. Warne, 'I-bisimple semigroups', *Trans. American Math. Soc.* **130** (1968), 367–386.

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