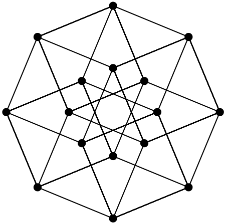
Congruences on ideals of semigroups and categories



WESTERN SYDNEY UNIVERSITY

James East

Centre for Research in Mathematics

Semigroup afternoon
University of York
3 July 2019

Joint work with Nik Ruškuc

Groups

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Normal subgroups of the symmetric group $\mathcal{S}_n$

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- $\{\mathrm{id}_4\} \trianglelefteq K \trianglelefteq \mathcal{A}_4 \trianglelefteq \mathcal{S}_4$.

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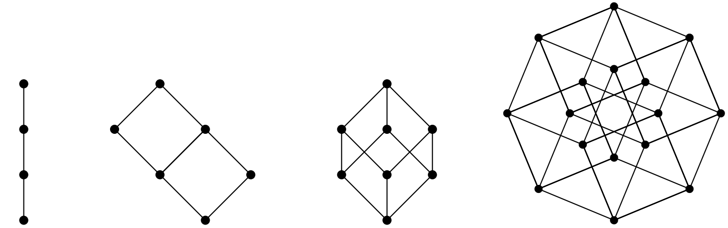
- $\{\mathrm{id}_4\} \trianglelefteq K \trianglelefteq \mathcal{A}_4 \trianglelefteq \mathcal{S}_4$.

General shape of $\mathcal{N}(\mathcal{S}_n)$:

Groups

(Normal) subgroups of the cyclic group C_n

They correspond to the divisors of n.

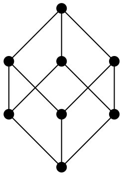


$\mathcal{N}(C_8)$ $\mathcal{N}(C_{12})$ $\mathcal{N}(C_{30})$ $\mathcal{N}(C_{210})$

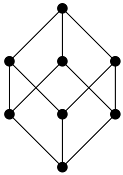
Groups

Normal subgroups of the dihedral group D_n

They correspond to the divisors of n (sort of); also depends on parity of n.



$\mathcal{N}(C_{30})$



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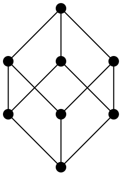

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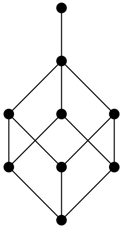

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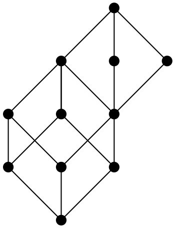

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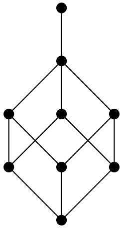

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An equivalence on S compatible with its operation(s).

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Definition (congruence on a semigroup S)

An equivalence σ on S such that:

- $(x, y) \in \sigma \;\Rightarrow\; (ax, ay), (xa, ya) \in \sigma$ for all $a \in S$.

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Definition (congruence on a structure S)

An equivalence on S compatible with its operation(s).

Definition (congruence on a category S)

An equivalence σ on (morphisms of) S such that:

- $(x, y) \in \sigma \;\Rightarrow\; (ax, ay), (xa, ya) \in \sigma$ when products defined,
- $(x, y) \in \sigma \;\Rightarrow\; \mathbf{d}(x) = \mathbf{d}(y)$ and $\mathbf{r}(x) = \mathbf{r}(y)$.

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Natural problem

Given S, find $\mathrm{Cong}(S)$. Today S will usually be a semigroup.

Congruences on $\mathcal{T}_n$

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- Let $\mathcal{T}_n$ = full transformation semigroup on $\mathbf{n} = \{1, \dots, n\}$

  $= \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

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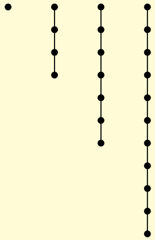
Theorem (Mal'cev, 1952)

n	1	2	3	4	5	6	7	8	9	10
Cong($\mathcal{T}_n$)										



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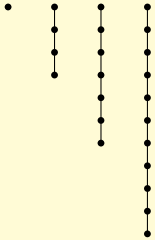

- What are these congruences?

Ideals and Rees congruences

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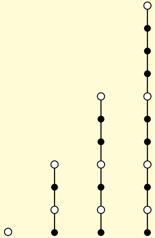

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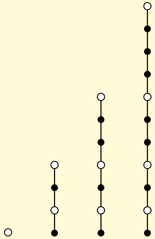

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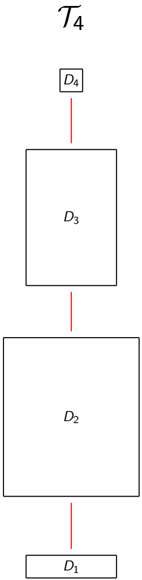

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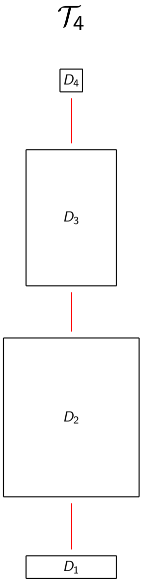

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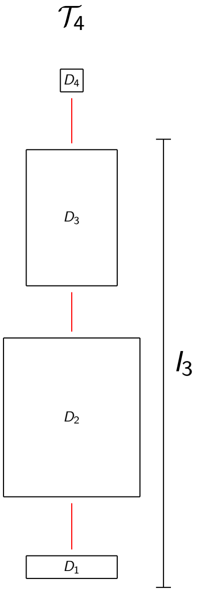

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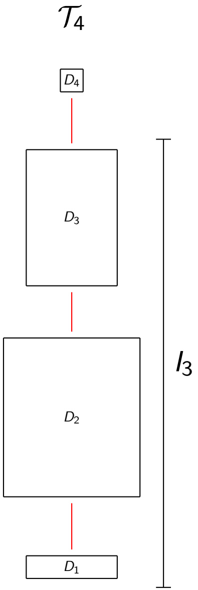

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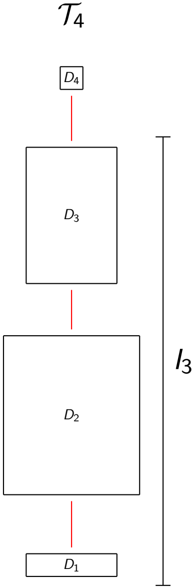

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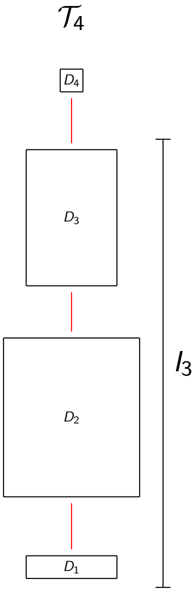

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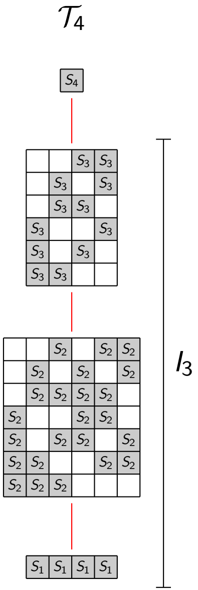

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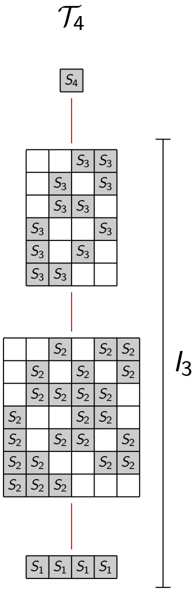

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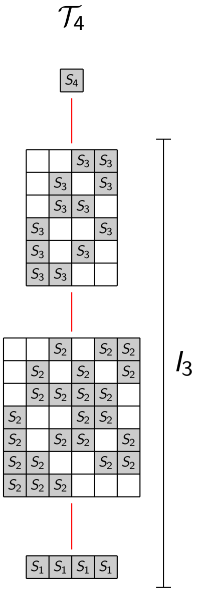

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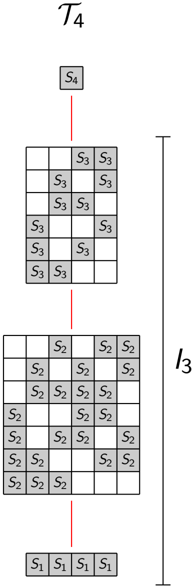
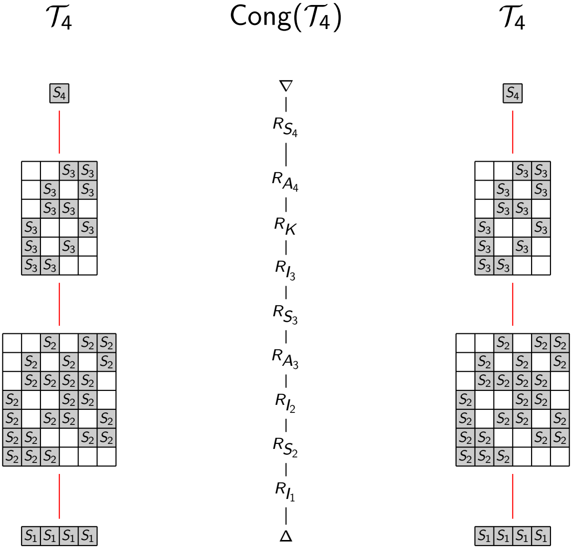
  - all of I_{r-1} collapses to a point.

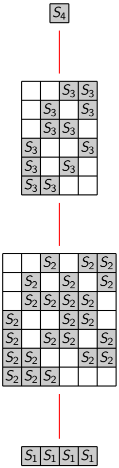


Congruences on $\mathcal{T}_n$



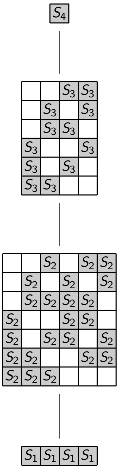
Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$

$\mathcal{T}_4/\Delta$



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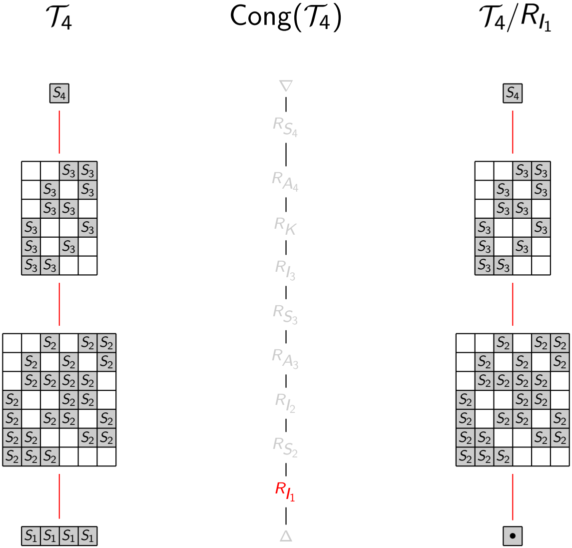

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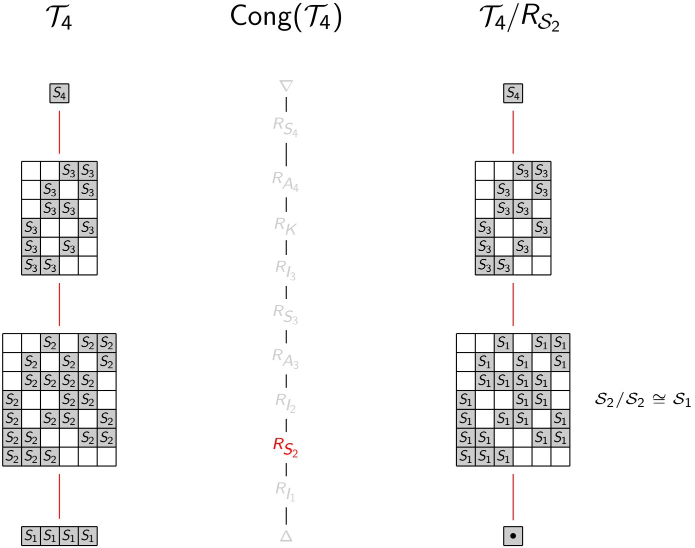
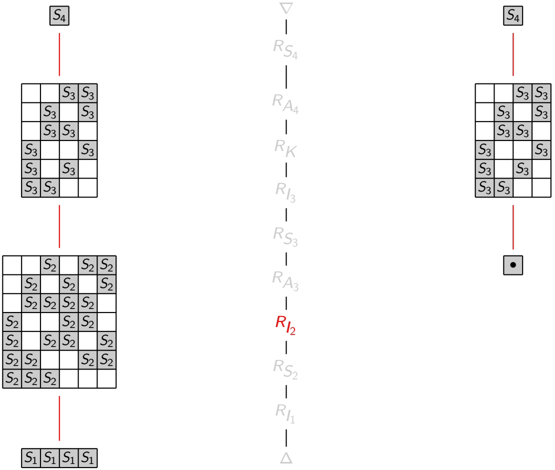


Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\mathrm{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/R_{I_2}$



Congruences on $\mathcal{T}_n$

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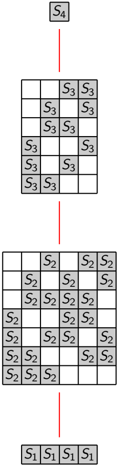
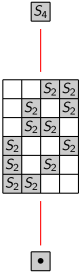


Cong($\mathcal{T}_4$)

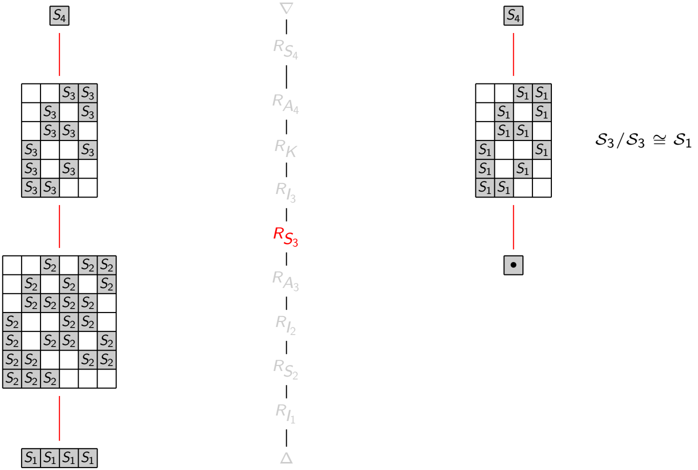
$\mathcal{T}_4/R_{\mathcal{A}_3}$



$\mathcal{S}_3/\mathcal{A}_3 \cong \mathcal{S}_2$

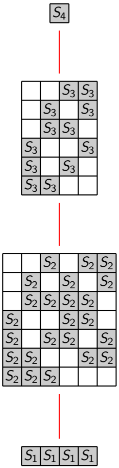
Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\text{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/R_{\mathcal{S}_3}$



Congruences on $\mathcal{T}_n$

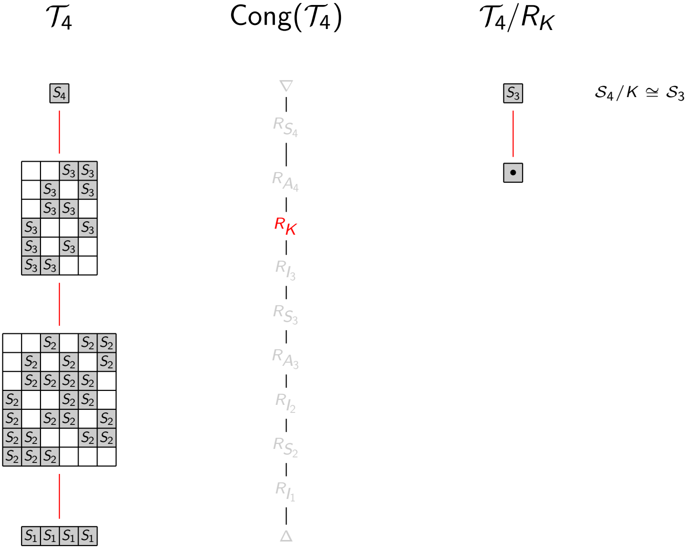
$\mathcal{T}_4$



$\mathrm{Cong}(\mathcal{T}_4)$

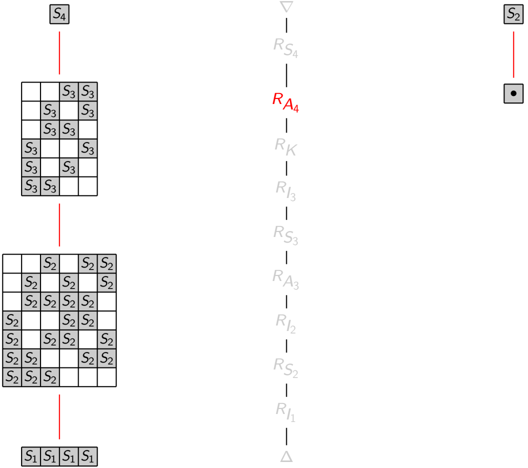
$\mathcal{T}_4/R_{I_3}$

Congruences on $\mathcal{T}_n$



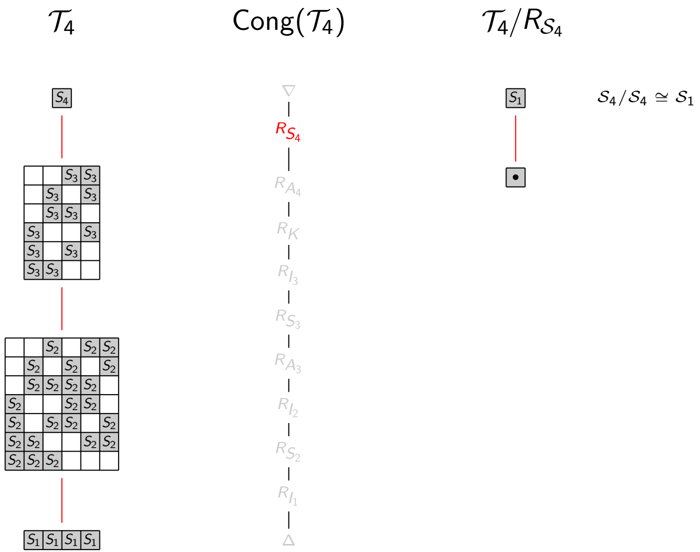
Congruences on $\mathcal{T}_n$

$\mathcal{T}_4$ | $\text{Cong}(\mathcal{T}_4)$ | $\mathcal{T}_4/R_{\mathcal{A}_4}$



$\mathcal{S}_4/\mathcal{A}_4 \cong \mathcal{S}_2$

Congruences on $\mathcal{T}_n$



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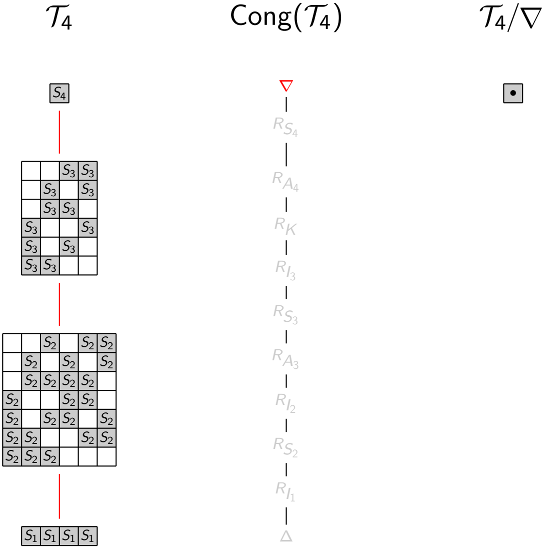

Congruences on ideals

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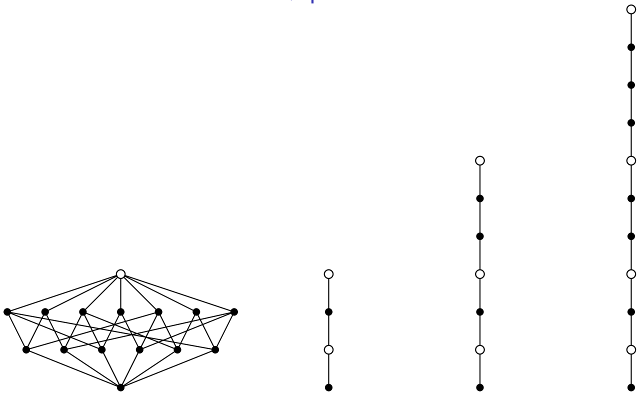
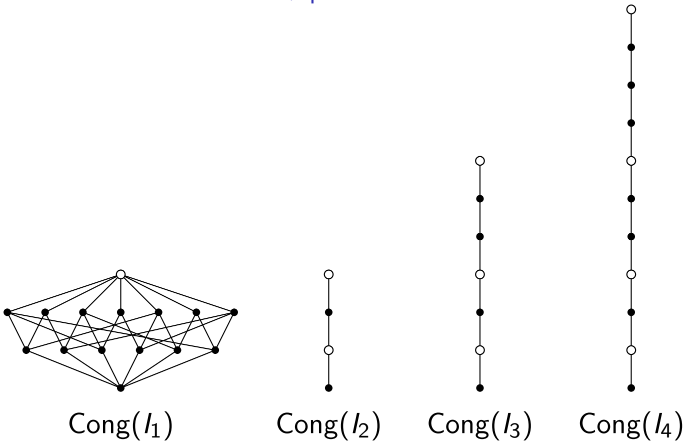
- Let's ask GAP!

Congruences on ideals of $\mathcal{T}_4$



$\mathrm{Cong}(I_1)$ $\mathrm{Cong}(I_2)$ $\mathrm{Cong}(I_3)$ $\mathrm{Cong}(I_4)$

Congruences on ideals of $\mathcal{T}_4$



- I_1 is an n-element right-zero semigroup: $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.

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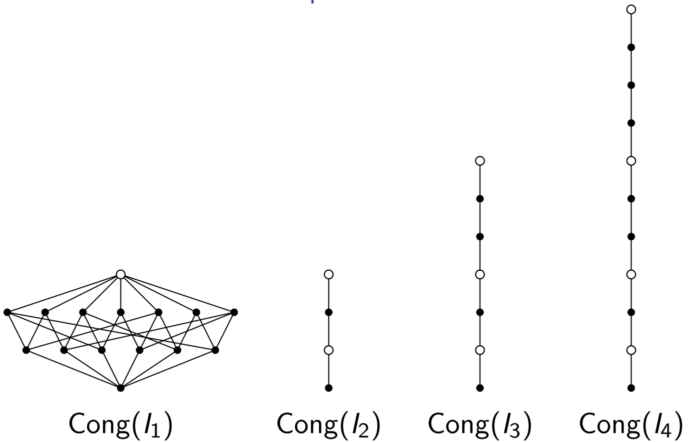

- I_1 is an n-element right-zero semigroup: $\mathrm{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- For $r \geq 2$, is $\mathrm{Cong}(I_r)$ just $\mathrm{Cong}(\mathcal{T}_n)$ chopped off?

Congruences on ideals of $\mathcal{T}_n$

Theorem

Yes!

Congruences on ideals of $\mathcal{T}_n$

Theorem

- $\text{Cong}(I_1) \cong \mathfrak{Eq}_n$.
- $\text{Cong}(I_r) = \left\{ R_N^{I_r} : N \trianglelefteq \mathcal{S}_q,\ q \leq n \right\} \cup \left\{ \nabla_{I_r} \right\}$ for $2 \leq r \leq n$.

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- Later: general machinery that works for many other semigroups and categories...
  - transformations, linear transformations, diagrams, braids...
  - Treat smallest ideal(s) of S, then "lift" from one to the next.
  - No need to know $\text{Cong}(S)$ in advance.

Congruences on ideal extensions

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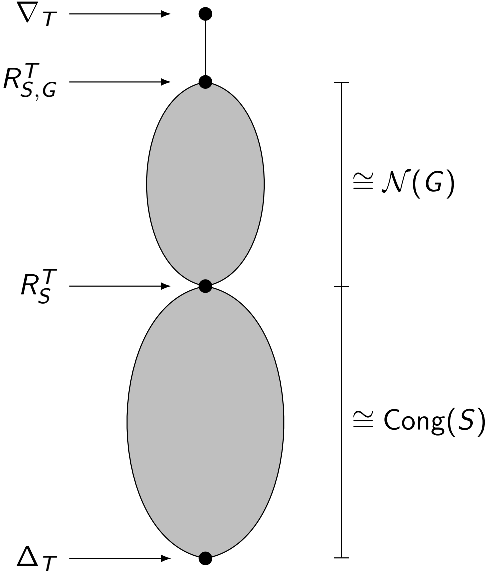
Theorem

- Suppose T is a semigroup with a stable, regular maximum $\mathscr{J}$-class D_T.
- Suppose the ideal $S = T \setminus D_T$ has a stable, regular maximum $\mathscr{J}$-class D_S.
- Suppose $(x, y)^\sharp = \nabla_S$ for all $x \in D_S$ and $y \in S \setminus H_x$.
- Suppose every congruence on S is liftable to T.
- One more technical assumption.
- Let G be a group $\mathscr{H}$-class contained in D_T.

Then

$$\mathrm{Cong}(T) = \left\{ \Delta_{D_T} \cup \sigma : \sigma \in \mathrm{Cong}(S) \right\} \cup \left\{ R^T_{S,N} : N \trianglelefteq G \right\} \cup \left\{ \nabla_T \right\}.$$

Congruences on ideal extensions



Congruences on ideal extensions

Congruences on ideal extensions

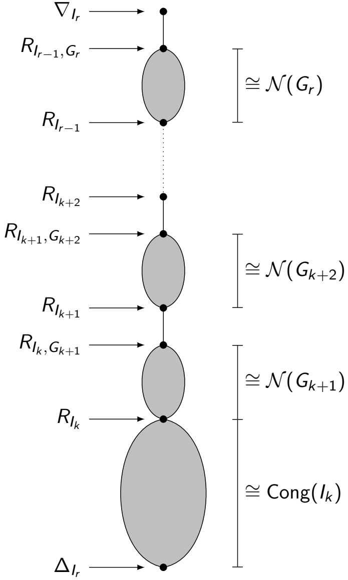
Theorem

- Let S be a stable, regular partial semigroup with a chain of $\mathscr{J}$-classes $D_0 < D_1 < \cdots$.
- The ideals of S are $I_r = D_0 \cup \cdots \cup D_r$ (and $I_\omega = S$ if the chain is infinite).
- Let G_q be a group $\mathscr{H}$-class in D_q.
- Suppose for some k every congruence on I_k is liftable to S.
- A technical property on I_k, and another on $I_{k+1}, I_{k+2}, \ldots$

Then for any $r \geq k$ (including $r = \omega$),

$$\begin{aligned}\mathrm{Cong}(I_r) = \big\{\Delta_{I_r} \cup \sigma : \sigma \in \mathrm{Cong}(I_k)\big\} \\ \cup \big\{R^{I_r}_{I_q,N} : k \leq q < r,\ N \trianglelefteq G_{q+1}\big\} \cup \big\{\nabla_{I_r}\big\}.\end{aligned}$$

Congruences on ideal extensions



More applications

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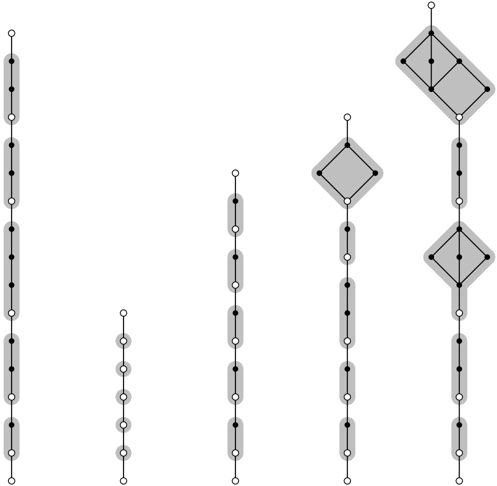
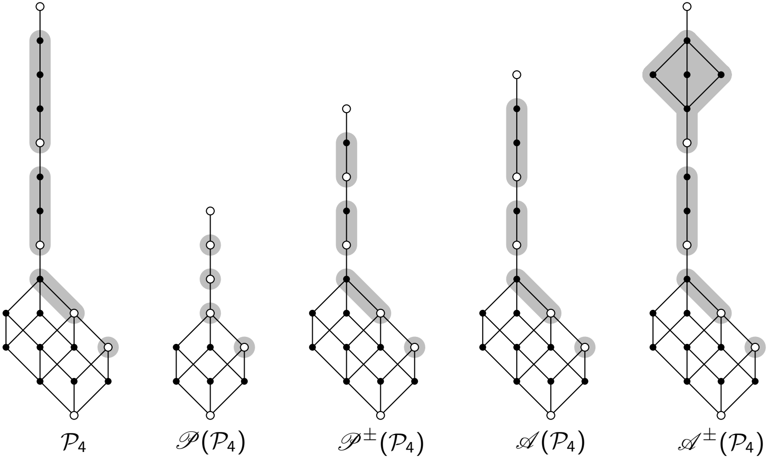


More applications

- Partition categories $\mathcal{P} = \mathcal{P}(\mathscr{C})$
  - Planar, anti-planar, annular, anti-annular subcategories.



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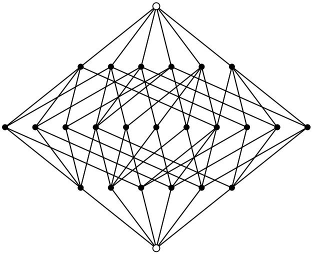
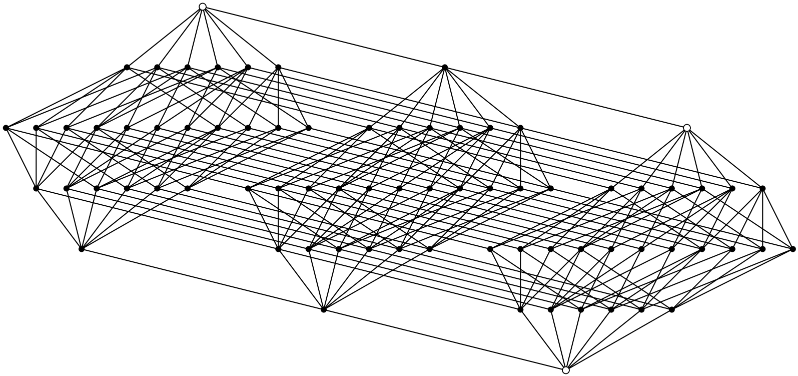
- $I_0(\mathcal{B}_4)$: $\mathfrak{Eq}_3 \times \mathfrak{Eq}_3$



More applications

- Brauer categories $\mathcal{B}$
- $I_2(\mathcal{B}_4)$: $(\mathfrak{Eq}_3 \times \mathfrak{Eq}_3) \times \mathbf{3}$



More applications

- Brauer categories $\mathcal{B}$
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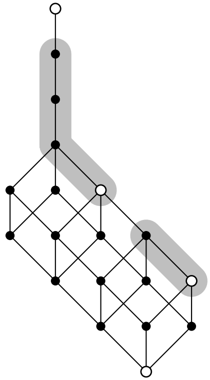
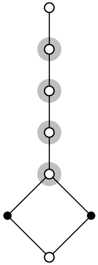


More applications

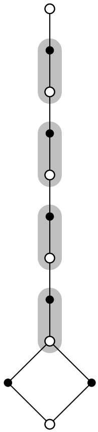
- Brauer categories $\mathcal{B}$
- (Anti-)planar/annular subcategories: Temperley-Lieb, Jones...

More applications

- (Anti-)Temperley-Lieb categories $\mathcal{TL}$ and $\mathcal{TL}^{\pm}$



$I_9(\mathcal{TL})$



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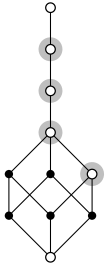
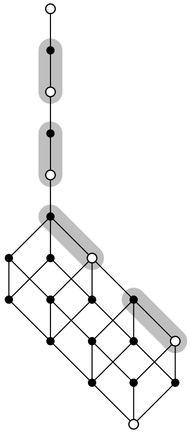


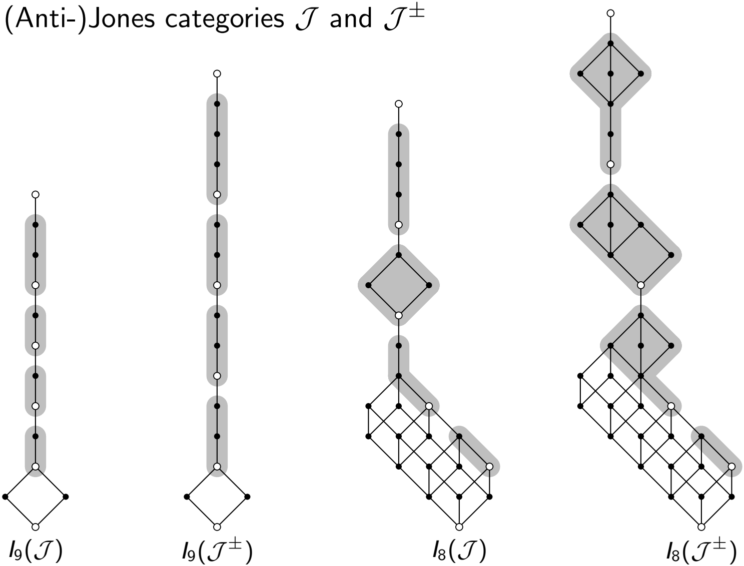
$I_8(\mathcal{TL})$



$I_8(\mathcal{TL}^{\pm})$

More applications

- (Anti-)Jones categories $\mathcal{J}$ and $\mathcal{J}^{\pm}$



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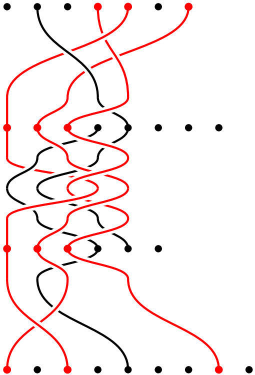

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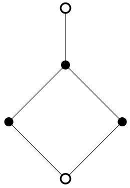
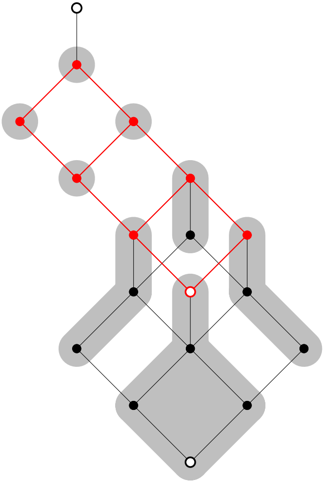
- Congruences of form $R^{I_r}_{I_q, N_{q+1}, N_{q+2}, \ldots}$, with $N_{q+1} \succeq N_{q+2} \succeq \cdots$
- Can still build $\mathrm{Cong}(I_{r+1})$ from $\mathrm{Cong}(I_r)$.
  - It's just more complicated...

More applications

- Linear category $\mathcal{L} = \mathcal{L}(\mathbb{F}_7)$



$I_1(\mathcal{L})$



$I_2(\mathcal{L})$

More applications

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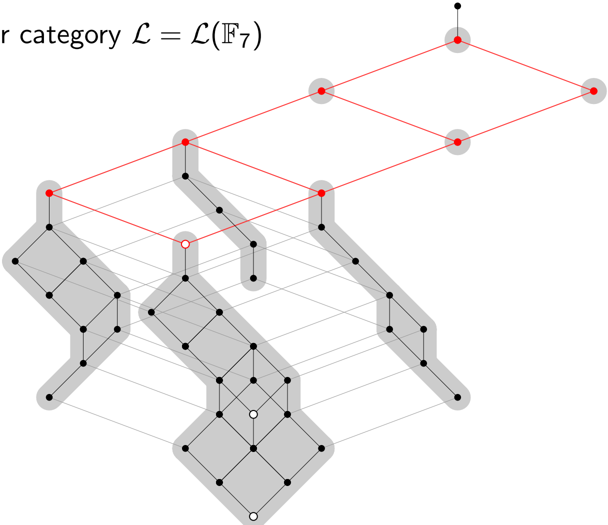

$I_3(\mathcal{L})$

Current/future work

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- Other categories:

Current/future work

- Other categories:
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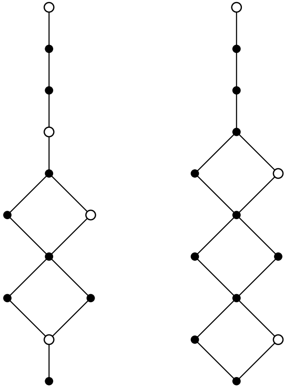

$\mathcal{P}_3^0$ $\mathcal{B}_4^0$

Current/future work

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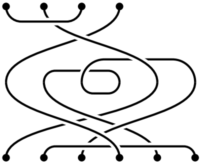

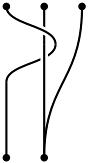

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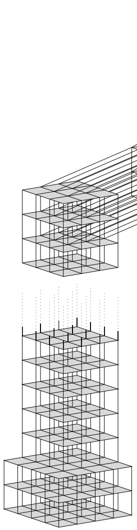

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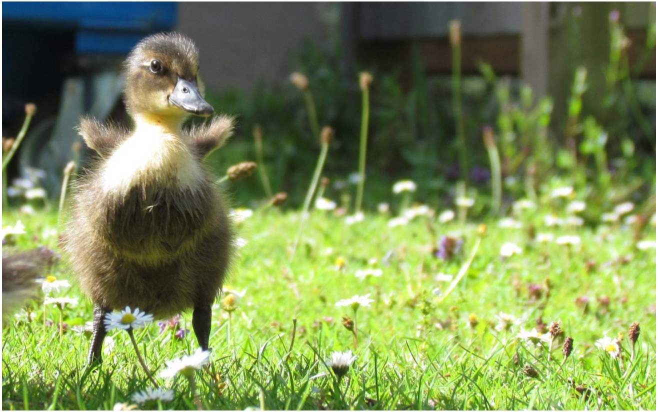
- Variants/sandwich semigroups

Thank you :-)



Congruences lattices of ideals in categories and (partial) semigroups

- James East and Nik Ruškuc
- Coming soon to arXiv...