Congruences and quotients of inverse semigroups

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People with one idea

- William Hazlitt (1778-1830)
- *On people with one idea* in Table Talk (1822)
- "There are people who have but one idea: at least, if they have more, they keep it a secret . . ."

Origins

- Idea arose in PhD work of Nouf AlYamani
- To construct a quotient of an ordered groupoid
- Some cases already considered:
  - Higgins 1971: *Categories and groupoids*. Normal subgroupoids and quotients
  - Matthews 2004: Bangor PhD thesis. Quotient of an ordered groupoid by a union of subgroups
- No general construction?

Two scenarios

- Optimism
- A new, useful, and interesting construction
- Applications to inverse semigroups . . .
- . . . and other classes such as restriction semigroups?
- Another ordered groupoid construction with a home in semigroup theory

- Pessimism
- An obscure idea that has been around a while
- Already known about in semigroups
- Idea doesn't fulfill its promise
- Think about something else

Two scenarios

- A new, useful, and interesting construction
- Applications to inverse semigroups . . .
- . . . and other classes such as restriction semigroups?
- Another ordered groupoid construction with a home in semigroup theory
- 55%

- An obscure idea that has been around a while
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- Idea doesn't fulfill its promise
- Think about something else
- 45%

Normal inverse subsemigroups

Inverse subsemigroup N of inverse semigroup S is *normal* if

- $E(N) = E(S)$. This says N is *full*
- for all $s \in S, n \in N$, $s^{-1}ns \in N$. This say N is *self-conjugate.*

$E(S)$ itself is normal, and defines the *natural partial order*:

$$s \leq t \iff \exists e \in E(S) \text{ with } s = et \iff s = ss^{-1}t \, .$$

The N–preorder

Normal N now gives:

$$s \leq_N t \iff \exists\, a, b \in N \text{ with } a \cdot s \cdot b \leq t$$

where $x \cdot y$ is a *trace product*:

$$x \cdot y = xy \text{ and defined when } x^{-1}x = yy^{-1}\,.$$

Symmetrize $\leq_N$ to get $\simeq_N$:

$$s \simeq_N t \iff s \leq_N t \text{ and } t \leq_N s\,.$$

Properties of $\leq_N$

- $\leq_{E(S)}$ is the natural partial order $\leq$,
- $s \leq t \implies s \leq_N t$,
- $s \leq_N e \implies s \in N$,
- $s \leq_N t \implies st^{-1} \in N$,
- $\leq_N$ is a preorder.

Properties of $\simeq_N$

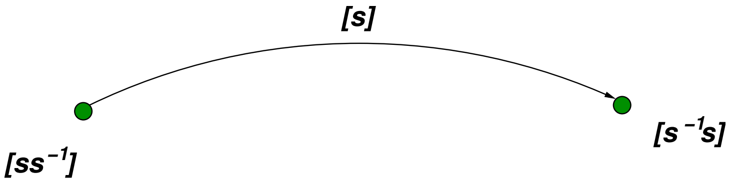
- If $n \in N$ then $nn^{-1} \simeq_N n \simeq_N n^{-1} \simeq_N n^{-1}n$,
- If $s \simeq_N t$ then $ss^{-1} \simeq_N tt^{-1}, s^{-1}s \simeq_N t^{-1}t$ and $s^{-1} \simeq_N t^{-1}$,
- $\simeq_N$ restricted to $E(S)$ is Green's $\mathcal{J}$–relation on $E(N) = E(S)$ in N,
- if $N = S$ then $\simeq_S$ is $\mathcal{J}$,
- if $N = E(S)$ then $\simeq_{E(S)}$ is equality,
- $\simeq_N$ is an equivalence relation that saturates N.

Only get an ordered groupoid

$\simeq_N$ need not be a congruence on S so quotient set $S /\!\!/ N$ need not be an inverse semigroup, but:

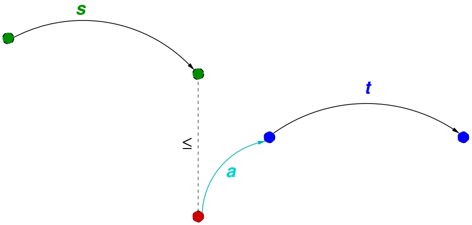
Theorem

$S /\!\!/ N$ is an ordered groupoid.



Composition in $S/\!\!/N$

If $s^{-1}s \simeq_N tt^{-1}$:



then

$$[s]_N \cdot [t]_N = [sat]_N \,.$$

Polycyclic monoids

Nivat-Perrot (1970): for $A = \{a_1, \ldots, a_n\}$,

$$P_n = \langle A : a_i a_i^{-1} = 1, a_i^{-1} a_j = 0 \; (i \neq j) \rangle \, .$$

Non-zero elements represented by $A^* \times A^*$:

$$\begin{aligned}
(r,s)(s,u) &= (r,u) \\
(r,s)(ps,u) &= (pr,ps)(ps,u) = (pr,u) \\
(r,pt)(t,u) &= (r,pt)(pt,pu) = (r,pu) \\
(r,s)(t,u) &= 0 \text{ otherwise}
\end{aligned}$$

Full inverse subsemigroups of P_n

Non-zero elements of a submonoid $\leftrightarrow$ subset of $A^* \times A^*$:

- Meakin-Sapir (1993): positively self-conjugate submonoids of P_n correspond to congruences on A^*,
- Lawson (2009): full inverse submonoids correspond to left congruences on A^*,
- normal inverse submonoids correspond to right-cancellative congruences on A^*.

Gauge inverse monoids

Jones and Lawson (2012): *gauge inverse monoid*

$$G_n = \{(s,t) : |s| = |t|\} \cup \{0\} \subset P_n\,.$$

- Corresponds to length relation on A^*,
- Normal inverse submonoid of P_n,
- $\mathcal{D} = \mathcal{J}$
- $\mathcal{J}$–classes indexed by word-length
- $P_n /\!\!/ G_n$ is the Brandt semigroup on the non-negative integers.

Congruence pairs

For a relation ρ on S, its *trace* is its restriction to $E(S)$ and its kernel is

$$\ker \rho = \{s \in S : s \rho e \text{ for some } e \in E(S)\} .$$

A congruence on $E(S)$ is normal if

$$\forall s \in S : e \; \rho \; f \Longrightarrow s^{-1} es \; \rho \; s^{-1} fs .$$

A *congruence pair* is a normal inverse semigroup K and a normal congruence ν on $E(S)$ such that

- $se \in K$ and $s^{-1}s \; \nu \; e \Longrightarrow s \in K$,
- $u \in K \Longrightarrow uu^{-1} \; \nu \; u^{-1}u$.

Congruences vs Pairs

Reilly-Scheiblich (1967), Scheiblich (1974), D.G. Green (1975), M. Petrich (1978): congruences correspond to *congruence pairs*:

$$\rho \to (\ker \rho, \operatorname{trace} \rho)$$

$$st^{-1} \in K, s^{-1}s \; \nu \; t^{-1}t \leftarrow (K, \nu) \, .$$

Congruences vs Quotients

Theorem

If K is the kernel of a congruence ρ then $s \simeq_K t \implies s \;\rho\; t$ and

$$\kappa : S /\!\!/ K \to S/\rho$$

is a functor.

Theorem

If ρ is an idempotent separating congruence with kernel K then the relations $\simeq_K$ and ρ are equal and κ is an isomorphism.

The kernel property

Howie (*Fundamentals ...*): full inverse subsemigroup N of S has the *kernel property* if

$$st \in N \text{ and } n \in N \implies snt \in N\,.$$

Theorem

- *kernel property implies normality,*
- *(D G Green 1975) N is the kernel of a congruence iff it has the kernel property.*

So if $\simeq_N$ is a congruence, N has the kernel property.

Minimality

Theorem

If N has the kernel property then $\simeq_N$ is a congruence if and only if $\mathcal{J}_N$ is a normal congruence on $E(S)$, and then $\simeq_N$ is the minimal congruence on S with kernel N.

But:

- When is $\mathcal{J}_N$ a normal congruence on $E(S)$?
- When is $\mathcal{J}_N$ a congruence on $E(S)$?